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THE PRAGMATICS OF MONADIC QUANTIFICATION

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I. *Pragmatics of Truth Functions*

1. *Sentential Interpretations* In [2] the truth functional logic of a set **S** of sentences was characterized pragmatically with respect to a set **U** of users of **S**, a set **T** of times of valuation of sentences of **S**, a set **C** of conditions of such valuations, and a set **V** of pragmatic values. Pragmatic interpreting functions were defined with domain $\mathbf{U} \times \mathbf{T} \times \mathbf{C} \times \mathbf{S}$ and range **V**, so that these functions induced a Boolean structure on **S**. In the present paper, pragmatic interpreting functions are defined whose applicability is more general and whose structure is more fundamental from a pragmatic point of view than those of [2].

Let **L** be the set of all expressions of some language. More precisely, let **L** be the set of all concatenates of a set of expressions which may be regarded as the alphabet of some language. Abstractly, **L** is any set of objects generated from a given finite set **L**, by a binary operation satisfying the properties of concatenation.

Let **C** be a set of conditions of valuation of the expressions of **L**. No assumptions about the nature or structure of the elements of **C** are required in the definition of pragmatic interpreting functions; abstractly, **C** is any set. Intuitively, **C** may be regarded as a set of conditions which may be conjointly realized as an experimental state, or partial state, of the world, identifiable by the users of **L**. On the basis of sets of such conditions, the expressions of **L** assume pragmatic values for the users of **L**. In being so valued, the expressions of **L** are "confronted with the world". The set **C** may be regarded as the total evidence available to the users of **L**, and subsets of **C** as partial evidence, for valuing the expressions of **L**. Let $\mathfrak{C}$ be the set of all subsets of **C**.

Let $\mathbf{V} = \{0, 1, 2\}$ be the set of pragmatic values which may be assumed by the expressions of **L**. No further structure on **V** need be assumed in the definition of pragmatic interpreting functions; abstractly, **V** is any three-membered set. Intuitively, the element 0 may be regarded as the value assigned to an expression of **L** under a given set of conditions, when the expression is *rejected* under that set of conditions. The value 1 is assigned

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to an expression under a set of conditions when the expression is *accepted* under that set of conditions. Finally, the value 2 is assigned to an expression under a set of conditions when the set of conditions is not germane to valuing the expression, so that the expression is *neither accepted nor rejected* under that set of conditions.

On the basis of the meanings we have assigned to **L**, **C**, and **V**, expressions of **L** are valued under subsets of **C** according to empirical conventions, or according to logical (deductive or inductive) conventions as well. Whatever conventions or rules are employed to decide finally whether to accept or reject or suspend judgment about an expression e of **L** under a set of conditions c of $\mathfrak{C}$, the decision may be expressed as an assignment of an element v of **V** to the pair $< e, c >$ of $\mathbf{L} \times \mathfrak{C}$. Thus we are led to consider mappings from $\mathbf{L} \times \mathfrak{C}$ onto **V**, which we shall call *pragmatic interpreting functions*, since they will be shown to be definable by functions whose field also includes the set **U** of users of **L** and the set **T** of their times of valuation of the expressions of **L**.

Ideally, it should be possible to characterize the syntax and semantics of **L** by means of pragmatic interpreting functions from $\mathbf{L} \times \mathfrak{C}$ onto **V**. We begin by studying interpreting functions which determine the truth-functional structure of **L**; such functions we shall call *sentential interpretations* of **L**. We should expect that a sentential interpretation of **L** determines a subset **S** of **L**, whose elements may be regarded as *sentences*. We shall introduce the concept of a sentential interpretation of **L**, relativized to a subset **S** of **L**, whose elements we shall subsequently show may justly be regarded as sentences. Such an interpretation we shall call an **S**-*interpretation* of **L**. It is convenient to begin by defining the concept of a *quasi*-**S**-*interpretation* of **L**.

We shall refer to concatenates of expressions of **L** by juxtaposing the terms which refer to the concatenated expressions. We shall also employ symbols autonymously, when the context makes this convenience clear.

D1. *π is a quasi-*__S__*-interpretation of* **L** *iff π is a mapping from* $\mathbf{L} \times \mathfrak{C}$ *onto* **V**, **S** *is a subset of* **L**, *and there are unique expressions* ., $\sim$, (,) *in* **L** *such that for all* s, s' *in* **L**:

I. *If* $\pi(s,c) \neq 2$ *for some* $c \varepsilon \mathfrak{C}$, *then* $s \varepsilon \mathbf{S}$.

II. *If* $s, s' \varepsilon \mathbf{S}$, *then* $\sim(s)$, $(s) \cdot (s') \varepsilon \mathbf{S}$ *and the π-values of* $\sim(s)$ *and* $(s) \cdot (s')$, *under each* $c \varepsilon \mathfrak{C}$, *are (partially) determined by the tables:*

	$\sim$
0	1
1	0
2	2

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	$\neq 1$

If $\pi(s,c) = 2 = \pi(s',c)$, then the value of $\pi((s) \cdot (s'),c)$ is only partially determined by the table for $(s) \cdot (s')$, as either 0 or 2. This ambiguity may be diminished by further conditions on π, as in the subsequent definition D2.

From the table for $\sim s$ in D1, the following consequences are immediate. We shall often omit parentheses in referring to expressions of **L**, when the context makes this simplification unambiguous.

(1.1) $\pi(s,c) = \pi(\sim\sim s,c)$.
(1.2) *If* $\pi(s,c) = \pi(s',c)$, *then* $\pi(\sim s,c) = \pi(\sim s',c)$.

An analogous substitutivity property for the expression $\cdot$ is not derivable, however, because of the ambiguity of the value of $s \cdot s'$ when s and s' are both valued 2 under c.

In subsequent definitions it is useful to employ the following substitutivity notation. Let $e(e')$ represent an expression of **L** containing any number of occurrences of e'.

(1.3) *$e(e'')$ represents ambiguously any expression obtained from $e(e')$ by substituting e'' for any of the occurrences of e' in $e(e')$.*

D2. *π is an **S**-interpretation of **L** iff π is a quasi-**S**-interpretation of **L** and for all $e \varepsilon$ **L**; $s, s', s'' \varepsilon$ **S**; $c \varepsilon \mathfrak{C}$:*

I. $\pi(s \cdot \sim s,c) = 0$
II. *If* $\pi(s \cdot \sim s',c) = 0 = \pi(s' \cdot \sim s'',c)$, *then* $\pi(s \cdot \sim s'',c) = 0$.
III. *If* $\pi(s \cdot \sim s',c) = 0$, *then* $\pi((s \cdot s'') \cdot \sim(s' \cdot s''),c) = 0$.
IV. $\pi(e(s),c) = \pi(e(\sim\sim s),c)$.
V. $\pi(e(s \cdot s'),c) = \pi(e(s' \cdot s),c)$.
VI. $\pi(e(s \cdot (s' \cdot s'')),c) = \pi(e((s \cdot s') \cdot s'')),c)$.
VII. $\pi(e(s),c) = \pi(e(s \cdot s),c)$.

In the following consequences of D2, π is an **S**-interpretation of **L**. Roman numerals refer to conditions of D2, unless otherwise indicated.

(1.4) *If* $\pi(s,c) = 2 = \pi(s',c)$, *then* $\pi(s \cdot s',c) = 2$ *or* $\pi(s \cdot \sim s',c) = 2$.

Proof. On the hypothesis of (1.4), if $\pi(s \cdot s',c) \neq 2$, then by D1, $\pi(s \cdot s',c) = 0)$. Therefore $\pi(\sim(s \cdot s'),c) = 1$. Now suppose $\pi(s \cdot \sim s',c) \neq 2$. Then by D1, $\pi(s \cdot \sim s',c) = 0$. Hence $\pi((s \cdot s) \cdot \sim(s' \cdot s),c) = 0$, by III. Hence by V, $\pi((s \cdot s) \cdot \sim(s \cdot s'),c) = 0$. Since $\pi(\sim(s \cdot s'),c) = 1$, $\pi(s \cdot s,c) = 0$, by D1. Hence $\pi(s,c) = 0$, by VII, against the hypothesis of (1.4). Therefore $\pi(s \cdot \sim s',c) = 2$, and (1.4) is shown.

(1.5) *If* $\pi(s \cdot \sim s',c) = 0$, *then* $\pi(s,c) = \pi(s \cdot s',c)$.

Proof. We distinguish three cases for $\pi(s,c)$.
Case 1. $\pi(s,c) = 0 = \pi(s \cdot s',c)$.
Case 2. $\pi(s,c) = 1$. Then by hypothesis of (1.5) and D1, $\pi(\sim s',c) = 0$. Hence $\pi(s',c) = 1 = \pi(s \cdot s',c)$.
Case 3. $\pi(s,c) = 2$. Then by hypothesis of (1.5), either $\pi(\sim s,c) = 0$ or $\pi(\sim s',c) = 2$. If $\pi(\sim s',c) = 0$, then $\pi(s',c) = 1$ and $\pi(s \cdot s',c) = 2$. If $\pi(\sim s',c) = 2$, then $\pi(s',c) = 2 = \pi(s \cdot s',c)$. For suppose $\pi(s \cdot s',c) \neq 2$. Then by (1.4), $\pi(s \cdot \sim s',c) = 2$, against the hypothesis of (1.5). Thus (1.5) is shown.

(1.6) *If* $\pi(s \cdot \sim(s \cdot s'),c) = 0$, *then* $\pi(s \cdot \sim s',c) = 0$.

Proof. By I, D1, V, and VI,
$\pi(s' \cdot \sim s', c) = 0 = \pi((s' \cdot \sim s') \cdot s, c) = \pi(s \cdot (s' \cdot \sim s'), c) = \pi((s \cdot s') \cdot \sim s', c)$.
Thus if $\pi(s \cdot \sim (s \cdot s'), c) = 0$, then by II, $\pi(s \cdot \sim s', c) = 0$.

(1.7) *If* $\pi(s \cdot \sim s', c) = 0 = \pi(s' \cdot \sim s, c)$, *then* $\pi(s,c) = \pi(s',c)$.

Proof. (1.7) follows immediately from (1.5) and V.

If π is an **S**-interpretation of **L**, we may define a subset **Q** of **S** whose elements may be referred to as "counter-tantologies", a name which will be justified in section 2.

(i) If $s \varepsilon \mathbf{S}$, then $s \cdot \sim s \varepsilon \mathbf{Q}$.
(ii) If $s, s' \varepsilon \mathbf{Q}$, then $\sim(\sim s \cdot \sim s') \varepsilon \mathbf{Q}$.
(iii) If $s \varepsilon \mathbf{Q}$ and $s' \varepsilon \mathbf{S}$, then $s \cdot s' \varepsilon \mathbf{Q}$.
(iv) Only by (i)-(iii) may $s \varepsilon \mathbf{Q}$.

If s is a counter-tantology, $\sim s$ may be referred to as a "tantology".

(1.8) *All tantologies are valued always* 1; *all counter-tantologies are valued always* 0.

Proof. All counter-tantologies established by (i) are valued always 0, by D2 (I). All counter-tantologies established by (ii) are valued always 0, by D1; and this holds also for counter-tantologies established by (iii). Thus by (iv), (1.8) is shown for counter-tantologies. Hence by D1, (1.8) is shown for tantologies also.

2. *Boolean Algebras of Expressions* If π is an **S**-interpretation of **L**, there is an equivalence relation on **S** with respect to which **S** has a Boolean structure, under the operations of infixing $\cdot$ and prefixing $\sim$. Let $s, s' \varepsilon \mathbf{S}$.

D3. $\mathbf{R}_\pi(s, s')$ *iff* $\pi(s \cdot \sim s', c) = 0 = \pi(s' \cdot \sim s, c)$.

$\mathbf{R}_\pi$ is reflexive and symmetric, by D2(I). $\mathbf{R}_\pi$ is transitive by D2(II). Thus $\mathbf{R}_\pi$ is an equivalence relation on **S**. We may now prove that π induces the appropriate Boolean structure on **S**. Roman numerals refer to conditions of D2, unless otherwise indicated.

Theorem 1. *If* π *is an* **S**-*interpretation of* **L**, *then* $\langle \mathbf{S}, \cdot, \sim \rangle$ *is a Boolean algebra with respect to* $\mathbf{R}_\pi$.

Proof. **S** is closed under the operations of infixing $\cdot$ and prefixing $\sim$, by D1(II). By I, $\pi((s \cdot s') \cdot \sim (s \cdot s'), c) = 0$. Hence by V, $\pi((s \cdot s') \cdot \sim (s' \cdot s) c) = 0 = \pi((s' \cdot s) \cdot \sim (s \cdot s'), c)$.
Thus we have shown (i) $\mathbf{R}_\pi(s \cdot s', s' \cdot s)$. In the same way, by VI, one may show (ii) $\mathbf{R}_\pi(s \cdot (s' \cdot s''), (s \cdot s') \cdot s'')$.

We next show (iii) if $\mathbf{R}_\pi(s, s')$, then $\mathbf{R}_\pi(\sim s, \sim s')$. By hypothesis, $\pi(s \cdot \sim s', c) = 0 = \pi(s' \cdot \sim s, c)$. Thus by IV and V, $\pi(\sim s' \cdot \sim \sim s, c) = 0 = \pi(\sim s \cdot \sim \sim s', c)$. From III there follows immediately (iv) if $\mathbf{R}_\pi(s \cdot s')$, then $\mathbf{R}_\pi(s \cdot s'', s' \cdot s'')$.

We next show (v) if $\mathbf{R}_\pi(s \cdot \sim s', s'' \cdot \sim s'')$, then $\mathbf{R}_\pi(s, s \cdot s')$. Since

$\pi((s \cdot s') \cdot \sim s, c) = 0 = \pi((\sim s \cdot s) \cdot s', c)$ is true for all $s, s' \varepsilon \mathbf{S}$, it is sufficient to show that if $\pi((s \cdot \sim s') \cdot \sim (s'' \cdot s''), c) = 0$, then $\pi(s \cdot \sim (s \cdot s'), c) = 0$. By hypothesis, $\pi(s \cdot \sim s', c) = 0$. Hence $\pi((s \cdot s) \cdot \sim (s' \cdot s), c) = 0$, by III. Now by I and VII, $\pi(s \cdot \sim (s \cdot s), c) = 0$. Hence by II and V, $\pi(s \cdot \sim (s \cdot s'), c) = 0$.

Finally, we show (vi) if $\mathbf{R}_\pi(s, s \cdot s')$, then $\mathbf{R}_\pi(s \cdot \sim s', s'' \cdot \sim s'')$. Since $\pi((s'' \cdot \sim s'') \cdot \sim (s \cdot \sim s'), c) = 0$ is true for all $s, s', s'' \varepsilon \mathbf{S}$, it is sufficient to show that if $\pi(s \cdot \sim (s \cdot s'), c) = 0$, then $\pi(s \cdot \sim s', c) = 0 = \pi((s \cdot \sim s') \cdot \sim (s'' \cdot s''), c)$. But this is (1.6). The proof of Theorem 1 is complete.

From Theorem 1 it does not follow that the pair $<\mathbf{S}, \mathbf{R}_\pi>$ is a sentential calculus, since $\mathbf{R}_\pi$ may be too large. A sentential calculus for **S** may be obtained by considering all possible **S**-interpretations of **L**. Let Π be the set of all **S**-interpretations of **L** (relative to the conditions **C**). Define

(2.1) $\mathbf{R} = \bigcap_{\pi \in \Pi} \mathbf{R}_\pi$.

Theorem 2. $< \mathbf{S}, \mathbf{R} >$ *is a sentential calculus.*

Proof. **R** is the smallest equivalence relation with respect to which $< \mathbf{S}, \cdot, \sim >$ is a Boolean algebra, by Theorem 1 and (2.1). Thus **R** relates only pairs of expressions of **S** which satisfy the conditions (i)-(vi) in the proof of Theorem 1; hence $<\mathbf{S}, \mathbf{R}>$ is a sentential calculus.

Let π be an **S**-interpretation of **L**. Define

(2.2) $\mathfrak{T} = \hat{s}\{\pi(s, c) = 1\}$.

Then $\mathfrak{T}$ is a sum ideal, or filter, of the Boolean algebra $<\mathbf{S}, \cdot, \sim>$, with respect to $\mathbf{R}_\pi$. For if $s \varepsilon \mathbf{S}$ and $t = \sim(s' \cdot \sim s')$, then

(2.3) $s \varepsilon \mathfrak{T}$ *iff* $\mathbf{R}_\pi(s, t)$,

which is an abbreviation of

(2.4) $\pi(s, c) = 1$ *iff* $\pi(s \cdot \sim t, c) = 0 = \pi(t \cdot \sim s, c)$, which is obviously true.

$\mathfrak{T}$ is the set of expressions which are $\mathbf{R}_\pi$-equivalent to the unit expression t of the Boolean algebra $< \mathbf{S}, \cdot, \sim >$, with respect to $\mathbf{R}_\pi$.

Theorem 3. $<\mathbf{S}, \mathfrak{T}, \cdot, \sim >$ *is a consistent Boolean logic, with respect to* $\mathbf{R}_\pi$.

Proof. $\mathfrak{T}$ is a sum ideal of **S**, by (2.3). Moreover, if $s \varepsilon \mathfrak{T}$, then $\sim s \not\varepsilon \mathfrak{T}$. For suppose $s, \sim s \varepsilon \mathfrak{T}$. Then $\mathbf{R}_\pi(s, \sim s)$, by (2.3). Then $\pi(s \cdot \sim \sim s, c) = 0 = \pi(\sim s \cdot \sim s, c)$. Then by IV, VII, $\pi(s, c) = 0 = \pi(\sim s, c)$, against D1 (II).

The logic $< \mathbf{S}, \mathfrak{T}, \cdot, \sim >$ is *complete* if, for each $s \varepsilon \mathbf{S}$, either $s \varepsilon \mathfrak{T}$ or $\sim s \varepsilon \mathfrak{T}$. In terms of sentential interpretations π, this is to say that, for each $s \varepsilon \mathbf{S}$, $\pi(s, c) \neq 2$. For if $\pi(s, c) = 1$, then $s \varepsilon \mathfrak{T}$, and conversely; if $\pi(s, c) = 0$, then $\pi(\sim s, c) = 1$, so that $\sim s \varepsilon \mathfrak{T}$, and conversely. The logic $< \mathbf{S}, \mathfrak{T}, \cdot, \sim >$ is complete in the above sense of being *categorical*, if and only if it is pragmatically complete in the sense that *the set* **C** *of conditions is germane for all* $s \varepsilon \mathbf{S}$.

In terms of the intended meanings of **C** and **V**, it is now possible to assign a meaning to the relation $\mathbf{R}_\pi$ on **S**. Let $\mathbf{R}_\pi(s, s')$. Then by (1.7),

$s\varepsilon\mathfrak{T}$ iff $s'\varepsilon\mathfrak{T}$. Since $< \mathbf{S}, \mathfrak{T} >$ is a logic, it is natural to regard $\mathfrak{T}$ as the set of sentences *asserted* by π (relative to the set $\mathbf{C}$ of conditions). Thus $\mathbf{R}_\pi(s,s')$ means that s and s' are *co-assertible*. Theorem 1 then states that if π is an $\mathbf{S}$-interpretation of $\mathbf{L}$, then $< \mathbf{S}, \cdot, \sim >$ is a Boolean algebra with respect to the relation of *co-assertibility* on the expressions of $\mathbf{S}$. This justifies our considering $\mathbf{S}$ to be a set of *sentences* and calling π a *sentential interpretation*. From a purely syntactic point of view, $\mathbf{R}_\pi$ is simply the relation of (sentential) deductive equivalence on $\mathbf{S}$.

3. *Sentential Interpretants* In this section a pragmatic foundation for the theory of sentential interpretations of $\mathbf{L}$ is established, by relating such interpretations to the set $\mathbf{U}$ of users of $\mathbf{L}$ and the set $\mathbf{T}$ of the times of their valuations of the expressions of $\mathbf{L}$. Let $\mathbf{D}$ be the set of all mappings d from $\mathbf{U} \times \mathbf{T} \times \mathbf{C}$ onto $\mathbf{V}$ such that, for all $u,u'\varepsilon\mathbf{U}$; $t,t'\varepsilon\mathbf{T}$; $c\varepsilon\mathfrak{C}$,

(3.1) $d(u,t,c) = d(u',t',c)$ *if* $d(u,t,c) \neq 2 \neq d(u',t',c)$.

An element $< u,t,c,v >$ of a mapping $d\varepsilon\mathbf{D}$ may be regarded as a disposition of a user u, at a time t, under a set of conditions c, to perform the valuation v. Such a disposition we shall call an *interpreting disposition*. We wish to assign to each expression of $\mathbf{L}$ an appropriate set of interpreting dispositions. (3.1) is a requirement of uniformity on the sets of interpreting dispositions which may be assigned to expressions of $\mathbf{L}$. If $d\varepsilon\mathbf{D}$, then d contains no dispositions to perform distinct valuations under any set of conditions germane for those valuations. Each $d\varepsilon\mathbf{D}$ is a set of *uniform* interpreting dispositions, in the sense of (3.1). Given (3.1), it is natural to ask whether the assignment of sets of uniform interpreting dispositions to expressions of $\mathbf{L}$, may determine definite meanings of those expressions.

The first step in answering this question is to define a mapping from $\mathbf{L}$ to $\mathbf{D}$, such that a Boolean structure is induced on a subset $\mathbf{S}$ of $\mathbf{L}$. By Theorem 1, it is sufficient to find a mapping which determines an $\mathbf{S}$-interpretation of $\mathbf{L}$. We shall call the range of such a mapping, a *system of* $\mathbf{S}$*-interpretants of* $\mathbf{L}$, following the terminology of Peirce. It is convenient to begin by defining the concept of a *system of quasi-*$\mathbf{S}$*-interpretants of* $\mathbf{L}$. In the following definitions, $g(s)\,(u,t,c)$ is the value of the function $d\varepsilon\mathbf{D}$ which is assigned to $\mathbf{S}$ by g, for the argument (u,t,c). The set of interpreting dispositions which uniformly map all (u,t,c) to 0, we shall call d_0.

D4. $g(\mathbf{L})$ *is a system of quasi-*$\mathbf{S}$*-interpretants iff* g *is a mapping of* $\mathbf{L}$ *into* $\mathbf{D}$, $\mathbf{S}$ *is a subset of* $\mathbf{L}$, *and there are unique expressions* $\cdot, \sim, (,)$ *in* $\mathbf{L}$ *such that for all* s,s' *in* $\mathbf{L}$:

I. *If* $g(s)\,(u,t,c) \neq 2$ *for some* $u\varepsilon\mathbf{U}$, $t\varepsilon\mathbf{T}$, $c\varepsilon\mathfrak{C}$, *then* $s\varepsilon\mathbf{S}$.
II. *If* $s,s'\varepsilon\mathbf{S}$, *then* $\sim(s)$, $(s)\cdot(s')\varepsilon\mathbf{S}$ *and* $g(\sim s)\,(u,t,c)$ *and* $g((s)\cdot(s'))\,(u,t,c)$ *are (partially) determined by* $g(s)\,(u,t,c)$ *and* $g(s')\,(u,t,c)$, *for all* (u,t,c) *in the domain of* $g(s)$ *according to the tables:*

	$\sim$
0	1
1	0
2	2

	0	1	2
0	0	0	0
1	0	1	2
2	0	2	$\neq 1$

Theorem 4. *If $g(\mathsf{L})$ is a system of quasi-S-interpretants, then there is a quasi-S-interpretation of L.*

Proof. By hypothesis, g is a mapping from L to D. Define the mapping π from $\mathsf{L} \times \mathfrak{C}$ onto V such that, for all $e \varepsilon \mathsf{L}$, $c \varepsilon \mathfrak{C}$,

(3.2) $\pi(e,c) = \begin{cases} v, \textit{ if } g(e)\ (u,t,c) = v \neq 2 \textit{ for some } u \varepsilon \mathsf{U},\ t \varepsilon \mathsf{T}. \\ 2, \textit{ otherwise.} \end{cases}$

By hypothesis, there is a subset S of L and unique $\sim$, $\cdot$, (,) in L, such that g satisfies D4. It is sufficient to show that, for these expressions, and for all s,s' in L, π satisfies D1. Now if $\pi(s,c) \neq 2$ for some $c \varepsilon \mathfrak{C}$, then by (3.2), $g(s)\ (u,t,c) \neq 2$ for some $u \varepsilon \mathsf{U}$, $t \varepsilon \mathsf{T}$, $c \varepsilon \mathfrak{C}$; so that $s \varepsilon \mathsf{S}$, by D4 (I). Thus π satisfies D1 (I).

By D4 (II), if $s,s' \varepsilon \mathsf{S}$, then $\sim s, s \cdot s' \varepsilon \mathsf{S}$, and it remains to show that π satisfies the tables of D1 (II). If $\pi(s,c) = 0$, then $g(s)\ (u_0,t_0,c) = 0$ for fixed $u_0 \varepsilon \mathsf{U}$, $t_0 \varepsilon \mathsf{T}$, by (3.2); then by D4 (II), $g(\sim s)\ (u_0,t_0,c) = 1$, and by (3.2) again, $\pi(\sim s,c) = 1$. The reasoning is the same for $\pi(s,c) = 1$. If $\pi(s,c) = 2$, then for all $u \varepsilon \mathsf{U}$, $t \varepsilon \mathsf{T}$, $g(s)\ (u,t,c) = 2 = g(\sim s)\ (u,t,c) = \pi(\sim s,c)$. Thus π satisfies the *tilde* table of D1. The reasoning is the same for the remaining table. Thus π is a quasi-S-interpretation of L.

D5. *$g(\mathsf{L})$ is a system of S-interpretants iff $g(\mathsf{L})$ is a system of quasi-S-interpretants and for all s, s', s'' $\varepsilon \mathsf{S}$; $e \varepsilon \mathsf{L}$; $u \varepsilon \mathsf{U}$; $t \varepsilon \mathsf{T}$; $c \varepsilon \mathfrak{C}$:*

I. $g(s \cdot \sim s) = d_0$
II. *If* $g(s \cdot \sim s')\ (u,t,c) = 0 = g(s' \cdot \sim s'')\ (u,t,c)$, *then* $g(s \cdot \sim s'')\ (u,t,c) = 0$.
III. *If* $g(s \cdot \sim s')\ (u,t,c) = 0$, *then* $g((s \cdot s'') \cdot \sim (s' \cdot s''))\ (u,t,c) = 0$.
IV. $g(e(s)) = g\ (e(\sim \sim s))$
V. $g(e(s \cdot s')) = g(e(s' \cdot s))$
VI. $g(e(s \cdot (s' \cdot s''))) = g(e((s \cdot s') \cdot s''))$
VII. $g(e(s)) = g(e(s \cdot s))$

If $g(\mathsf{L})$ is a system of S-interpretants, then a subset $\mathfrak{T}g$ of S is determined by

(3.3) $\mathfrak{T}g = \hat{s}\{g(s)\ (u,t,c) = 1 \textit{ for some } u \varepsilon \mathsf{U},\ t \varepsilon \mathsf{T}\}$.

The following theorem establishes a relation between systems of sentential interpretants g and sentential interpretations π, in virtue of which the desired Boolean structure is induced on S by g. If π is a sentential interpretation of L, let $\mathfrak{T}_\pi$ be defined by (2.2).

Theorem 5. *If $g(\mathsf{L})$ is a system of S-interpretants, then there is an S-interpretation π of L such that $\mathfrak{T}_g = \mathfrak{T}_\pi$.*

Proof. On the hypothesis of Theorem 5, there is a quasi-**S**-interpretation π of **L**, defined by (3.2). π satisfies D2 (I)-(III) by reasoning as in the proof of Theorem 4, by way of D5 (I)-(III). We have also:

(3.4) $g(e) = g(e')$ *iff* $g(e)\ (u,t,c) = g(e')\ (u,t,c)$.
(3.5) *If* $g(e) = g(e')$, *then* $\pi(e,c) = \pi(e',c)$.

(3.5) follows from (3.4) and (3.2). For if $g(e)\ (u,t,c) = v \neq 2$, for some $u \varepsilon \mathbf{U}$, $t \varepsilon \mathbf{T}$, then $\pi(e,c) = v = \pi(e',c)$; and otherwise, $\pi(e,c) = 2 = \pi(e',c)$. Thus by D5 (IV)-(VII), π satisfies D2 (IV)-(VII). Thus π is an **S**-interpretation of **L**. Finally, $\mathfrak{T}_g = \mathfrak{T}_\pi$, since by (3.2), $g(s)\ (u,t,c) = 1$ for some $u \varepsilon \mathbf{U}$, $t \varepsilon \mathbf{T}$, if and only if $\pi(s,c) = 1$. The proof of Theorem 5 is complete.

II. *Pragmatics of Monadic Quantification*

4. *Monadic Interpretations* In this section we shall study pragmatic interpreting functions which determine a monadic predicate structure on a subset of **L**. Such functions we shall call *monadic interpretations* of **L**. We begin by defining the concept of a *quasi-monadic interpretation* of **L**.

D1. *π is a quasi-monadic interpretation of* **L** *iff π is a quasi-*S*-interpretation of* **L**, *for some* $\mathbf{S} \subset \mathbf{L}$, *and there are disjoint subsets* **M** *and* **K** *of* **L**, *and unique expressions* $\exists$ *and* x *of* **L**, *such that for all* $Q,a,s \varepsilon \mathbf{L}$*:*

I. *If* $Q \varepsilon \mathbf{M}$, *then* $\pi((\exists x)\ (Q(x)),c) \neq 2$ *for some* $c \varepsilon \mathfrak{C}$.
II. *If* $a \varepsilon \mathbf{K}$, *then* $\pi(Q(a),c) \neq 2$ *for some* $Q \varepsilon \mathbf{M}$, $c \varepsilon \mathfrak{C}$.
III. $s \varepsilon \mathbf{S}$ *iff* s *is constructed from* $Q_1\ (t_1), \ldots, Q_k\ (t_k)$, *where* $Q_1, \ldots, Q_k \varepsilon \mathbf{M}$ *and* $t_1, \ldots, t_k \varepsilon \mathbf{K} \cup \{x\}$, *by infixing* $\cdot$ *or prefixing* $\sim$ *or* $(\exists x)$, *and if* $t_i \varepsilon \{x\}$ $(1 \leq i \leq k)$, *then* Q_i *is in a part of* s *of the form* $(\exists t_i)\ (A)$.
IV. $\sim, \cdot, \exists, x \not\varepsilon \mathbf{M}, \mathbf{K}$.

The following consequences of D1 concern the parsing of **L** into syntactic categories by a quasi-monadic interpretation π of **L**.

(4.1) **M** *is unique.*

For suppose there is some $\mathbf{M}' \neq \mathbf{M}$ which satisfies D1. Then there is some $Q \varepsilon \mathbf{M}$ such that $\pi((\exists x)\ (Q(x)),c) \neq 2$ for some $c \varepsilon \mathfrak{C}$. Then $(\exists x)\ (Q(x)) \varepsilon \mathbf{S}$, by D1 (I) of Part I. Then by D1 (III), with $k = 1$, $Q \varepsilon \mathbf{M}$. Therefore **M** is unique, and may be defined

$$\mathbf{M} = \hat{Q}\{\pi((\exists x)\ (Q(x)),c) \neq 2, \textit{for some } c \varepsilon \mathfrak{C}\}.$$

(4.2) **K** *is unique.*

For suppose there is some $a \not\varepsilon \mathbf{K}$ such that $\pi(Q(a),c) \neq 2$ for some $c \varepsilon \mathfrak{C}$, $Q \varepsilon \mathbf{M}$. Then $Q(a) \varepsilon \mathbf{S}$. Then by D1 (III) with $k = 1$, $a \varepsilon \mathbf{K}$. Therefore **K** is unique, and may be defined

$$\mathbf{K} = \hat{a}\{\pi(Q(a),c) \neq 2, \textit{for some } Q \varepsilon \mathbf{M}, c \varepsilon \mathfrak{C}\}.$$

The algebraic justification of the following parsing results depends upon theorems to be proved subsequently. Such theorems are indicated as the corresponding parsing results are stated.

D1 is so constructed that, intuitively, **M** is a set of monadic predicates and x is a variable. We may show that **M** is representable as the set of monadic predicates, and that $\{x\}$ is the set of variables, of a monadic algebra. For we shall show that there is a subset **B** of **L** which is a monadic Boolean algebra of expressions (Theorem 2). We shall also show that to each $Q \varepsilon \mathbf{M}$ there corresponds $Q(x) \varepsilon \mathbf{B}$ (Theorem 1). Thus **M** may be represented as a set of mappings from $\{x\}$ into **B**. It is also possible to show that, from the point of view of polyadic Boolean algebras, **B** is an $\{x\}$-algebra, with $\{x\}$ the set of variables. Now a predicate Q of a monadic algebra **B** is a mapping from the set of variables into **B**. (The general condition on substitution of variables in such mappings [Halmos, 173] is trivial in the monadic case: $Q(x) = Q(x)$.) Thus **M** is representable as a set of monadic predicates.

We shall show that $(\exists x)$ is representable as an existential quantifier (Theorem 2). Thus if $Q \varepsilon \mathbf{M}$, then $Q(x)$ may be regarded as an atomic open formula, and $(\exists x)\ (Q(x))$ as its quantification.

The set **K** is representable as a set of individual constants (Theorem 3). Thus if $a \varepsilon \mathbf{K}$ and $Q \varepsilon \mathbf{M}$, then $Q(a)$ may be regarded as an instance of $Q(x)$. Let $\mathbf{A}_0$ be the set of all atomic open (monadic) formulas of **L**. Let $\mathbf{S}_0$ be the set of instances $Q(a)$, for all $Q \varepsilon \mathbf{M}, a \varepsilon \mathbf{K}$. Let **B** be the set of all expressions generated from $\mathbf{A}_0 \cup \mathbf{S}_0$ by infixing $\cdot$, and prefixing $(\exists x)$ and $\sim$. Then **B** may be regarded as the set of monadic formulas of **L** (Theorem 2). **B** is clearly unique. Let **A** be the set of expressions generated from $\mathbf{A}_0$ by infixing $\cdot$, and prefixing $(\exists x)$ and $\sim$. Then **A** may be regarded as the set of monadic formulas without constants of **L** in the variable x (Theorem 2 Corollary 1). **A** is clearly unique.

Free and bound variables may be defined in the usual way. If $p \varepsilon \mathbf{B}$, then in the formula $(\exists x)\ (p)$, we say that p is the *scope* of $(\exists x)$. If x occurs in p, then we shall say that this occurrence of x is *bound* if it is in the scope of some $(\exists x)$. Otherwise it is free. If x is not free in p, we say that p is *closed*; otherwise p is *open*.

Let **S** be the subset of **L** with respect to which the quasi-monadic interpretation π is an **S**-interpretation. Then by D1 (IV) and the preceding paragraph, **S** is a subset of **B** and may be regarded as the set of syntactically closed formulas of **B** (Theorem 2 Corollary 2). **S** is clearly unique. The above parsing results may be summarized as follows.

Theorem 1. *If π is a quasi-monadic interpretation of* **L**, *then there are unique subsets* **B**, **A**, *and* **S** *of* **L**, *whose elements may be regarded, from the manner of their construction, as, respectively: monadic formulas, monadic formulas without constants, and closed monadic formulas.*

In the next section we shall show that **B**, **A**, and **S** also have the algebraic structure which justifies their being regarded as formulas. For this purpose we must introduce the concept of a *monadic interpretation*. This task is facilitated by the following notational simplification.

Since **B** is a set of formulas which contain x as their only variable, we may conveniently refer to such formulas by deleting all x's from the terms we use for such reference, along with unnecessary parentheses. Let $p(x)$ be any formula of **B** containing x. Let $p(a)$ be the formula obtained from $p(x)$ by replacing each occurrence of x with $a \varepsilon$ **K**. Then we abbreviate, for example,

$$(\exists x)((\exists x)(p(x))) \text{ as } \exists\exists p$$
$$(\exists x)(p(x) \cdot q(x)) \text{ as } \exists(p \cdot q)$$
$$(\exists x)(p(a) \cdot q(x)) \text{ as } \exists(p(a) \cdot q).$$

It is sufficient to consider at the present time only formulas in the one variable x, since every closed monadic formula is equivalent to a formula in the one variable x. We shall not study the pragmatics of this equivalence, since the substitution properties of open monadic formulas with distinct free variables is best studied in the context of the polyadic predicate calculus.

It is useful to extend the notation "$p(a)$" to each $p \varepsilon$ **B**. If $a \varepsilon$ **K** and $p \varepsilon$ **B**, then $p(a)$ is the formula obtained from p by replacing all free occurrences of x with a. Thus if x is not free in p, then $p(a) = p$. This convention determines, for each $a \varepsilon$ **K**, a mapping $a{:}p \rightarrow p(a)$, from **B** into **S**, such that:

(4.3) $(\exists p)(a) = p$.

(4.4) $(p \cdot q)(a) = p(a) \cdot q(a)$.

(4.5) $(\sim p)(a) = \sim(p(a))$.

Conditions IV, VII, and X of the following definition employ the substitutivity notation (1.3).

D2. *π is a monadic interpretation of* **L** *iff π is a quasi-monadic interpretation of* **L** *and for all p, q, $r \varepsilon$* **B**; *$e \varepsilon$* **L**; *$c \varepsilon \mathfrak{C}$:*

I. $\pi(\exists(p \cdot \sim p), c) = 0$.
II. *If* $\pi(\exists(p \cdot \sim q), c) = 0 = \pi(\exists(q \cdot \sim r), c)$, *then* $\pi(\exists(p \cdot \sim r), c) = 0$.
III. *If* $\pi(\exists(p \cdot \sim q), c) = 0$, *then* $\pi(\exists((p \cdot r) \cdot \sim (q \cdot r)), c) = 0$.
IV. *If p and q are tantalogically equivalent, then* $\pi(e(p), c) = \pi(e(q), c)$.
V. *If* $\pi(\exists p, c) = 0$, *then* $\pi(\exists(p \cdot q), c) = 0$.
VI. *If* $\pi(\exists(p \cdot \sim q), c) = 0$, *then* $\pi(\exists p, c) = \pi(\exists(p \cdot q), c)$.
VII. $\pi(e(\exists(p \cdot \exists q)), c) = \pi(e(\exists p \cdot \exists q), c)$.
VIII. $\pi(\exists(p \cdot \sim \exists p), c) = 0$.
IX. *If* $\pi(\exists(p \cdot \sim q), c) = 0 = \pi(\exists(q \cdot \sim p), c)$, *then* $\pi(\exists p \cdot \sim \exists q, c) = 0 = \pi(\exists q \cdot \sim \exists p, c)$.
X. *If p is closed,* $\pi(e(p), c) = \pi(e(\exists p), c)$.

The theory of monadic interpretations includes the theory of **S**-interpretations. For if π is a monadic interpretation of **L**, then by D1 and T1, π is a quasi-interpretation of **L**, with **S** the set of closed monadic formulas of **B**. Then by D2 (I-IV, X), π is an **S**-interpretation of **L**. Then by Theorem 1 of Part 1, $<$**S**, $\cdot$, $\sim>$ is a Boolean algebra with respect to $\mathbf{R}_\pi$, defined as in D3 of Part 1, and all the properties of **S** which are proved in Part I are also forthcoming.

In the following consequences of D2, π is a monadic interpretation of **L**. Roman numerals refer to conditions of D2, unless indicated otherwise, in this and the following section.

(4.6) $\pi(\exists((p\cdot q)\cdot \sim p),c) = 0$.

Proof. By I, IV, V, $\pi(\exists((\sim p\cdot p)\cdot q),c) = 0 = \pi(\exists(\sim p\cdot(p\cdot q)),c) = \pi(\exists((p\cdot q)\cdot \sim p),c)$.

(4.7) *If* $\pi(\exists(p\cdot \sim q),c) = 0 = \pi(\exists(q\cdot \sim p),c)$, *then* $\pi(\exists p,c) = \pi(\exists q,c)$.

Proof. By VI, IV, one proves (4.8). In the same way, one proves:

(4.8) *If* $\pi(\exists(p\cdot \sim q),c) = 0 = \pi(\exists(q\cdot \sim p),c)$, *then* $\pi(\exists \sim p,c) = \pi(\exists \sim q,c)$.

5. *Monadic Boolean Algebras of Expressions* If π is a monadic interpretation of **L**, there is an equivalence relation on the set of formulas **B**, with respect to which **B** is a monadic Boolean algebra, under the operations of infixing $\cdot$ and prefixing $\sim$ and $(\exists x)$.

D3. $\mathbf{E}_\pi(p,q)$ *iff* $\pi(\exists(p\cdot \sim q),c) = 0 = \pi(\exists(q\cdot \sim p),c)$, *where* π *is a monadic interpretation of* **L**, *and* $p, q \varepsilon$ **B**.

E is clearly an equivalence relation on **B**, by I and II.

Theorem 2. *If* π *is a monadic interpretation of* **L**, *and* $\exists$ *is the mapping from* $p \varepsilon$ **B** *to* $(\exists x)(p) \varepsilon$ **B**, *then* $< \mathbf{B}, \exists, \cdot, \sim >$ *is a monadic Boolean algebra with respect to* $\mathbf{E}_\pi$.

Proof. **B** is closed under the operations of infixing $\cdot$ and prefixing $\sim$ and $(\exists x)$. By I, $\pi(\exists((p\cdot q)\cdot \sim(p\cdot q),c) = 0$. Hence by IV, $\pi(\exists((p\cdot q)\cdot \sim(q\cdot p)),c) = \pi(\exists((q\cdot p)\cdot \sim(p\cdot q)),c) = 0$. Thus we have shown (i) $\mathbf{E}_\pi(p\cdot q, q\cdot p)$. In the same way, one may show (ii) $\mathbf{E}_\pi(p\cdot(q\cdot r), (p\cdot q)\cdot r)$.

We next show (iii) if $\mathbf{E}_\pi(p,q)$, then $\mathbf{E}_\pi(\sim p, \sim q)$. By hypothesis, $\pi(\exists(p\cdot \sim q),c) = 0 = \pi(\exists(q\cdot \sim p),c)$. Thus by IV, $\pi(\exists(\sim p\cdot \sim\sim q),c) = 0 = \pi(\exists(\sim q\cdot \sim\sim p),c)$. From III there follows immediately (iv) if $\mathbf{E}_\pi(p,q)$, then $\mathbf{E}_\pi(p\cdot r, q\cdot r)$.

We next show (v) if $\mathbf{E}_\pi(p\cdot \sim q, r\cdot \sim r)$, then $\mathbf{E}_\pi(p, p\cdot q)$. By hypothesis, $\pi(\exists((p\cdot \sim q)\cdot \sim(r\cdot \sim r)),c) = 0$. Then by VI, I, V, IV, $\pi(\exists(p\cdot \sim q),c) = \pi(\exists((p\cdot \sim q)\cdot(r\cdot \sim r)),c) = 0$. Then by III, $\pi(\exists((p\cdot p)\cdot \sim(q\cdot p)),c) = 0$. Hence by IV, $\pi(\exists(p\cdot \sim(p\cdot q)),c) = 0$. This with (4.6) gives (v).

We next show (vi) if $\mathbf{E}_\pi(p, p\cdot q)$, then $\mathbf{E}_\pi(p\cdot \sim q, r\cdot \sim r)$. By hypothesis, $\pi(\exists(p\cdot \sim(p\cdot q)),c) = 0$. Then by IV, V, $\pi(\exists(p\cdot \sim q),c) = 0 = \pi(\exists((p\cdot \sim q)\cdot \sim(r\cdot \sim r),c)$.

It remains to show $\pi(\exists((r\cdot \sim r)\cdot \sim(p\cdot \sim q)),c) = 0$, which is true in general.

By (i)-(vi), $< \mathbf{B}, \cdot, \sim >$ is a Boolean algebra with respect to $\mathbf{E}_\pi$. It remains to show that $\exists$ is a quantifier. We first show, where p_0 is the zero element of **B** with respect to $\mathbf{E}_\pi$ (i.e. $\mathbf{E}_\pi(p_0, q\cdot \sim q)$ for some $q \varepsilon$ **B**): $(\exists 1)$ $\mathbf{E}_\pi(p_0, \exists p_0)$. By hypothesis, $\pi(\exists(p_0\cdot \sim(q\cdot \sim q)),c) = 0$. Then by VI, I, IV, V, $\pi(\exists p_0,c) = \pi(\exists(p_0\cdot(q\cdot \sim q)),c) = 0 = \pi(\exists(p_0\cdot \sim\exists p_0),c)$. We thus have also by V, VII, $\pi(\exists p_0\cdot \exists\sim p_0,c) = 0 = \pi(\exists(\exists p_0\cdot \sim p_0),c)$. Thus $(\exists 1)$ is shown.

We next show ($\exists$2) $\mathbf{E}_\pi(p, p\cdot\exists p)$. By VIII, III, $\pi(\exists((p\cdot p)\cdot\sim(\exists p\cdot p)),c) = 0$. Then by IV, $\pi(\exists(p\cdot\sim(p\cdot\exists p),c) = 0$. This with (4.6) gives ($\exists$2).

We next show ($\exists$3) $\mathbf{E}_\pi(\exists(p\cdot\exists q), \exists p\cdot\exists q)$. This follows directly from I, V. It remains to show that $\mathbf{E}_\pi$ is a congruence relation with respect to the operation of prefixing $(\exists x)$; i.e., ($\exists$4) if $\mathbf{E}_\pi(p,q)$, then $\mathbf{E}_\pi(\exists p, \exists q)$. This follows directly from IX and X. The proof of Theorem 2 is complete.

The set $\mathbf{S}$ of syntactically closed formulas of $\mathbf{B}$ is closed under the operations of infixing $\cdot$ and prefixing $\sim$. We therefore have:

Corollary 1 (Theorem 2). $<\mathbf{S}, \cdot, \sim>$ *is a Boolean subalgebra of* $<\mathbf{B}, \cdot, \sim>$, *with respect to* $\mathbf{E}_\pi$.

Let $\mathbf{A}$ be the set of monadic formulas without constants of $\mathbf{B}$, as in Theorem 1. Then $\mathbf{A}$ is closed under the operations of infixing $\cdot$ and prefixing $\sim$ and $(\exists x)$. Therefore we have

Corollary 2 (Theorem 2.) $<\mathbf{A}, \exists, \cdot, \sim>$ *is a monadic subalgebra of* $<\mathbf{B}, \exists, \cdot, \sim>$, *with respect to* $\mathbf{E}_\pi$.

A monadic interpretation of $\mathbf{L}$ induces on the operations of forming instances of formulas, the structure of individual constants.

Theorem 3. *The mappings* $a\colon p \to p(a)$, *for* $a \varepsilon \mathbf{K}$, $p \varepsilon \mathbf{B}$, *are constants of the monadic algebra* $<\mathbf{B}, \exists, \cdot, \sim>$, *with respect to* $\mathbf{E}_\pi$.

Proof. (i) The mappings $a\colon p \to p(a)$ are endomorphisms on $\mathbf{B}$, with respect to $\mathbf{E}_\pi$, by (4.4), (4.5), I.
(ii) $\mathbf{E}_\pi(\exists p(a), \exists p)$, by (4.3), I.
(iii) $\mathbf{E}_\pi(\exists(p(a)), p(a))$, by X, I.
The proof of Theorem 3 is complete.

In the remainder of this section, π is a monadic interpretation of $\mathbf{L}$. Define

(5.1) $\mathfrak{I} = \hat{p}\,\{\pi(\exists\sim p,c) = 0\}$.

Then $\mathfrak{I}$ is a sum ideal of the Boolean algebra $<\mathbf{B}, \cdot, \sim>$, with respect to $\mathbf{E}_\pi$. For if $p \varepsilon \mathbf{B}$, and $p_1 = \sim(q\cdot\sim q)$, then

(5.2) $p \varepsilon \mathfrak{I}$ *iff* $\mathbf{E}_\pi(p, p_1)$,

which is an abbreviation of

(5.3) $\pi(\exists\sim p, c) = 0$ *iff* $\pi(\exists(p\cdot\sim p_1),c) = 0 = \pi(\exists(p_1\cdot\sim p),c)$.

(5.3) is established as follows. In general, $\pi(\exists(p\cdot\sim p_1),c) = 0$, by IV, I, V. In the same way, if $\pi(\exists\sim p, c) = 0$, then $\pi(\exists(p_1\cdot\sim p),c) = 0$. Conversely, if $\pi(\exists(p_1\cdot\sim p), c) = 0$, then $\pi(\exists(\sim p\cdot\sim(q\,.\,\sim q), c) = 0$, so that $\pi(\exists\sim p, c) = \pi(\exists(\sim p\cdot(q\cdot\sim q)),c) = 0$.

$\mathfrak{I}$ is the set of formulas which are $\mathbf{E}_\pi$-equivalent to the unit formula p_1 of the Boolean algebra $<\mathbf{B}, \cdot, \sim>$. If $\mathfrak{I}$ is a sum ideal of $\mathbf{B}$ such that

(5.4) *If* $p \varepsilon \mathfrak{I}$, *then* $\forall p = \sim\exists\sim p \varepsilon \mathfrak{I}$, then we shall say that $\mathfrak{T}$ is a *monadic sum ideal* of **B**. If $< \mathbf{B}, \exists, \cdot, \sim >$ is a monadic Boolean algebra and $\mathfrak{T}$ is a monadic sum ideal of **B**, we shall employ the solecism of saying that $< \mathbf{B}, \mathfrak{T} >$ is a *monadic logic.* (Halmos defines monadic logics with respect to ideals, instead of sum ideals.) If $\mathfrak{T}$ is defined by (5.1), we have:

Theorem 4. $< \mathbf{B}, \mathfrak{T} >$ *is a monadic Boolean logic, with respect to* $\mathbf{E}_\pi$.

Proof. $\mathfrak{T}$ is a sum ideal of **B**, by (2.3). $\mathfrak{T}$ is monadic, since if $p \varepsilon \mathfrak{T}$, then $\pi(\exists \sim p,c) = 0 = \pi(\sim \forall p,c) = \pi(\exists \sim \forall p,c)$, by X, so that $\forall p \varepsilon \mathfrak{T}$.

Corollary 1 (Theorem 4). *If* π *is a monadic interpretation of* **L**, *satisfying, for all* $c \varepsilon \mathfrak{C}$:

(5.5) *If* $\pi(\exists p,c) = 0$, *then* $\pi(p(a),c) = 0$ *for all* $a \varepsilon \mathbf{K}$;
(5.6) *If* $\pi(p(a),c) = 1$ *for some* $a \varepsilon \mathbf{K}$, *then* $\pi(\exists p,c) = 1$,

then $< \mathbf{B}, \mathfrak{T} >$ *is consistent.*

Proof. It follows from (5.5) and (5.6) that, for all $c \varepsilon \mathfrak{C}$;

(5.7) *If* $\pi(\exists p,c) = 0$, *then* $\pi(\exists \sim p,c) = 1$.

Now suppose $p, \sim p \varepsilon \mathfrak{T}$. Then $\mathbf{E}_\pi(p,\sim p)$. Then $\pi(\exists (p \cdot \sim \sim p),c) = 0 = \pi(\exists p,c)$. But by hypothesis that $p \varepsilon \mathfrak{T}$, $\pi(\exists \sim p,c) = 0$. Then by (5.7), $\pi(\exists p,c) = 1$, against the supposition that $p, \sim p \varepsilon \mathfrak{T}$. Thus $< \mathbf{B}, \mathfrak{T} >$ is consistent.

In terms of the intended meanings of **C** and **V**, we may now assign a meaning to the set $\mathfrak{T}$ determined by a monadic interpretation π of **L**. $p \varepsilon \mathfrak{T}$ if and only if $\pi(\forall p,c) = 1$. If p is closed (and in $\mathfrak{T}$), then by X, $\pi(p,c) = 1$. $\mathfrak{T}$ is the set of formulas whose universal closures are accepted under the total evidence **C**, in the interpretation π. Since $< \mathbf{B}, \mathfrak{T} >$ is a monadic logic, it is natural to regard $\mathfrak{T}$ as the set of formulas *asserted* by π.

We may now assign a meaning to the relation $\mathbf{E}_\pi$ on **B**. By (4.8), if $\mathbf{E}_\pi(p,q)$, then $p \varepsilon \mathfrak{T}$ if and only if $q \varepsilon \mathfrak{T}$. Thus $\mathbf{E}_\pi$ may be regarded as the relation of *co-assertibility* on the formulas of **B**, relative to the monadic interpretation π.

Let $\mathfrak{T}_0 = \mathfrak{T} \cap \mathbf{S}$. Then $\mathfrak{T}_0$ is the set of closed formulas which are $\mathbf{E}_\pi$-equivalent to the unit closed formula of the Boolean subalgebra $< \mathbf{S}, \cdot, \sim >$ of $< \mathbf{B}, \cdot, \sim >$. Thus $\mathfrak{T}$ is a sum ideal. This proves

Corollary 2 (Theorem 4). $< \mathbf{S}, \mathfrak{T}_0 >$ *is a Boolean logic with respect to* $\mathbf{E}_\pi$.

$< \mathbf{S}, \mathfrak{T}_0 >$ is consistent if and only if $< \mathbf{B}, \mathfrak{T} >$ is.

We shall define the *completeness* of $< \mathbf{B}, \mathfrak{T} >$ as the maximality of $\mathfrak{T}_0$ in $< \mathbf{S}, \mathfrak{T}_0 >$. For to require that $\mathfrak{T}$ be maximal in $< \mathbf{B}, \mathfrak{T} >$ is to require that, for all $p \varepsilon \mathbf{B}$, either $\pi(\exists p,c) = 0$ or $\pi(\exists \sim p,c) = 0$. If $< \mathbf{B}, \mathfrak{T} >$ is complete in the sense that $\mathfrak{T}_0$ is maximal in $< \mathbf{S}, \mathfrak{T}_0 >$, then for each $s \varepsilon \mathbf{S}$, either $s \varepsilon \mathfrak{T}_0$ or $\sim s \varepsilon \mathfrak{T}_0$. In terms of the monadic interpretation π, this is to say that, for each $s \varepsilon \mathbf{S}$, $\pi(s,c) \neq 2$: the set **C** of conditions of π is germane for all $s \varepsilon \mathbf{S}$.

For $\pi(s,c) = 1$ iff $\pi(\exists \sim s,c) = 0$ iff $s \varepsilon \mathfrak{T}_0$; and $\pi(s,c) = 0$ iff $\pi(\exists \sim \sim s,c) = 0$ iff $\sim s \varepsilon \mathfrak{T}_0$. We have therefore shown:

(5.8) $< \mathbf{B}, \mathfrak{T} >$ *is complete iff* $\mathbf{C}$ *is germane for* $\mathbf{S}$.

An element p of a monadic algebra is defined by Halmos to be *closed* if $\exists p = p$. If the monadic interpretation π determines the monadic algebra $< \mathbf{B}, \mathfrak{T} >$ with respect to $\mathbf{E}_\pi$, and if $p \varepsilon \mathbf{S}$, then by X of D2, $\mathbf{E}_\pi(p, \exists p)$. But the converse is not true. For example, if $p \varepsilon \mathfrak{T}$, then $\mathbf{E}_\pi(p, \exists p)$, even if $p \notin \mathbf{S}$. Thus the property of being *syntactically* closed is included in the property of being *algebraically* closed, but not conversely. Moreover, if $\exists(\mathbf{B})$ is the set of algebraically closed formulas of $\mathbf{B}$ (i.e. $p \varepsilon \exists(\mathbf{B})$ iff $\mathbf{E}_\pi(p, \exists p)$), then $\mathfrak{T} \subset \exists(\mathbf{B})$.

The completeness of any monadic logic $< \mathbf{A}, \mathbf{M} >$ is defined by Halmos to be the maximality of $\mathbf{M} \cap \exists(\mathbf{A})$ in $(\exists(\mathbf{A}), \mathbf{M} \cap \exists(\mathbf{A}))$. Let $< \mathbf{B}, \mathfrak{T} >$ be a monadic algebra, as in Theorem 4. Then $\mathfrak{T} \cap \exists(\mathbf{B}) = \mathfrak{T}$, since $\mathfrak{T} \subset \exists(\mathbf{B})$, as we have just seen. Then we may show that the completeness of $< \mathbf{B}, \mathfrak{T} >$ via $< \mathbf{S}, \mathfrak{T}_0 >$ is equivalent to the completeness of $< \mathbf{B}, \mathfrak{T} >$ via $< \exists(\mathbf{B}), \mathfrak{T} >$.

(5.9) $< \mathbf{S}, \mathfrak{T}_0 >$ *is complete iff* $< \exists(\mathbf{B}), \mathfrak{T} >$ *is complete*.

The proof from right to left is obvious. Let $< \mathbf{S}, \mathfrak{T}_0 >$ be complete. Let $p \varepsilon \exists(\mathbf{B})$. Then $\mathbf{E}_\pi(p, \exists p)$. Then $\pi(\exists(\exists p \cdot \sim p),c) = 0 = \pi(\exists p \cdot \exists \sim p,c)$. Now $\pi(\exists p,c) \neq 2 \neq \pi(\exists \sim p,c)$, by hypothesis and (5.8), recalling that the completeness of $< \mathbf{B}, \mathfrak{T} >$ in (5.8) is defined as the completeness of $< \mathbf{S}, \mathfrak{T}_0 >$. Therefore either $\pi(\exists p,c) = 0$ or $\pi(\exists \sim p,c) = 0$. Therefore either $\sim p \varepsilon \mathfrak{T}$ or $p \varepsilon \mathfrak{T}$. The proof of (5.9) is complete.

6. *Monadic Interpretants* In this section a pragmatic foundation is established for the theory of monadic interpretations of $\mathbf{L}$, by relating such interpretations to the set $\mathbf{U}$ of users of $\mathbf{L}$ and set $\mathbf{T}$ of the times of their valuations of the expressions of $\mathbf{L}$. If $\mathbf{D}$ is defined as in (3.1), it is natural to ask whether a function may be defined from $\mathbf{L}$ to $\mathbf{D}$, which induces a monadic Boolean structure on a subset $\mathbf{B}$ of $\mathbf{L}$. By Theorem 2, it is sufficient to find a function from $\mathbf{L}$ to $\mathbf{D}$ which determines a monadic interpretation of $\mathbf{L}$. We shall call the range of such a function, a *system of monadic interpretants* of $\mathbf{L}$, following the terminology of Peirce. It is convenient to begin by defining the concept of a *system of quasi-monadic interpretants*.

D4. *$g(\mathbf{L})$ is a system of quasi-monadic interpretants iff $g(\mathbf{L})$ is a system of quasi-$\mathbf{S}$-interpretants, for some $\mathbf{S} \subset \mathbf{L}$, and there are disjoint subsets $\mathbf{M}$ and $\mathbf{K}$ of $\mathbf{L}$, and unique expressions $\exists$ and x in $\mathbf{L}$ such that, for all $Q, a, s \varepsilon \mathbf{L}$:*

I. *If $Q \varepsilon \mathbf{M}$, then $g((\exists x)(Q(x)))(u,t,c) \neq 2$ for some $u \varepsilon \mathbf{U}$, $t \varepsilon \mathbf{T}$, $c \varepsilon \mathfrak{C}$.*
II. *If $a \varepsilon \mathbf{K}$, then $g(Q(a))(u,t,c) \neq 2$ for some $Q \varepsilon \mathbf{M}$, $u \varepsilon \mathbf{U}$, $t \varepsilon \mathbf{T}$, $c \varepsilon \mathfrak{C}$.*
III. *$s \varepsilon \mathbf{S}$ iff s is constructed from $Q_1(t_1), \ldots, Q_k(t_k)$, where $Q_1, \ldots,$*

*$Q_k \varepsilon$ **M**, and $t_1, \ldots, t_k \varepsilon$ **K** $\cup \{x\}$, by infixing $\cdot$ or prefixing $\sim$ or $(\exists x)$, and if $t_i \varepsilon \{x\} (1 \leq i \leq k)$, then Q_i is in a part of s of the form $(\exists t_i)(A)$.*

IV. $\sim, \cdot, \exists, x \notin$ **M**, *K*.

Theorem 5. *If g(**L**) is a system of quasi-monadic interpretants, then there is a quasi-monadic interpretation of **L**.*

Proof. By hypothesis, g(**L**) is a system of quasi-**S**-interpretants for some **S** $\subset$ **L**. Then by Theorem 4 of Part I, there is a quasi-**S**-interpretation π of **L** which is defined by

$$(6.1)\quad \pi(e,c) \equiv \begin{cases} v, \text{ if } g(e)\,(u,t,c) = v \neq 2 \text{ for some } u \varepsilon \mathbf{U},\ t \varepsilon \mathbf{T}. \\ 2, \text{ otherwise.} \end{cases}$$

It remains to show that π is a quasi-monadic interpretation of **L**. Let $Q \varepsilon$ **M**. Then by D4 (I), $g((\exists x)\,(Q(x)))\,(u,t,c) \neq 2$, for some $u \varepsilon$ **U**, $t \varepsilon$ **T**, $c \varepsilon \mathfrak{C}$. Then $\pi((\exists x)\,(Q(x)),c) \neq 2$, for some $c \varepsilon \mathfrak{C}$, by (6.1). Thus π satisfies D1 (I). In the same way, by D4 (II), π is shown to satisfy D1 (II). D1 (III, IV) are identical to D4 (III, IV). Thus π satisfies the conditions of D1; the proof of Theorem 5 is complete.

D5. *g(**L**) is a system of monadic interpretants iff g(**L**) is a system of quasi-monadic interpretants such that, for all p, q, $r \varepsilon$ **B**; $c \varepsilon \mathfrak{C}$; $u \varepsilon$ **U**; $t \varepsilon$ **T**:*

I. $g(\exists (p \cdot \sim p)) = d_0$.
II. *If* $g(\exists (p \cdot \sim q))(u,t,c) = 0 = g(\exists (q \cdot \sim r))(u,t,c)$, *then* $g(\exists (p \cdot \sim r))(u,t,c) = 0$.
III. *If* $g(\exists (p \cdot \sim q))\,(u,t,c) = 0$, *then* $g(\exists ((p \cdot r) \cdot \sim (q \,.\, r)))\,(u,t,c) = 0$.
IV. *If p and q are tantologically equivalent, then* $g(e(p)) = g(e(q))$.
V. *If* $g(\exists p)\,(u,t,c) = 0$, *then* $g(\exists (p \cdot q))\,(u,t,c) = 0$.
VI. *If* $g(\exists (p \cdot \sim q))\,(u,t,c) = 0$, *then* $g(\exists p)\,(u,t,c) = g(\exists (p \cdot q))\,(u,t,c)$.
VII. $g(e(\exists (p \cdot \exists q))) = g(e(\exists p \cdot \exists q))$.
VIII. $g(\exists (p \cdot \sim \exists p)) = d_0$.
IX. *If* $g(\exists (p \cdot \sim q))\,(u,t,c) = 0 = g(\exists (q \cdot \sim p))\,(u,t,c)$, *then* $g(\exists p \cdot \sim \exists q)(u,t,c) = 0 = g(\exists q \cdot \sim \exists p)\,(u,t,c)$.
X. *If p is closed,* $g(e(p)) = g(e(\exists p))$.

If g(**L**) is a system of monadic interpretants, then a subset $\mathfrak{T}_g$ of **B** is determined as follows.

$$(6.2)\quad \mathfrak{T}_g = \hat{p}\{g(\exists \sim p)\,(u,t,c) = 0, \text{ for some } u \varepsilon \mathbf{U},\ t \varepsilon \mathbf{T}\}.$$

The following theorem establishes a relation between the interpreting functions g and π, in virtue of which the desired monadic structure is induced on **B** by g. If π is a monadic interpretation of **L**, let $\mathfrak{T}_\pi$ be defined by (5.1).

Theorem 6. *If g(**L**) is a system of monadic interpretants, then there is a monadic interpretation π of **L**, such that $\mathfrak{T}_g = \mathfrak{T}_\pi$.*

Proof. By hypothesis, g(**L**) is a system of quasi-monadic interpretants which, by T5, determines a quasi-monadic interpretation π of **L**, defined by (6.1). It remains to show that π is a monadic interpretation of **L**, and that $\mathfrak{T}_\pi = \mathfrak{T}_g$. The conditions of D2 correspond to those of D5 in such a way that, just as in the proofs of Theorems 4 and 5 of Part I, each condition of D5 implies the corresponding one of D2, via (6.1). Moreover, $\mathfrak{T}_\pi = \mathfrak{T}_g$. For by (6.1) $g(\exists \sim p)\ (u_0,t_0,c) = 0$ iff $\pi(\exists \sim p, c) = 0$, for fixed $u_0 \varepsilon \mathbf{U}$, $t_0 \varepsilon \mathbf{T}$. Thus by (5.1) and (6.3), $\mathfrak{T}_\pi = \mathfrak{T}_g$.

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