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A SUBSTITUTION FREE AXIOM SET FOR SECOND ORDER LOGIC

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In what follows we present an adequate formulation of second order logic by means of an axiom set whose characterization does not require the notion of proper substitution either of a term for an individual variable or of a formula for a predicate variable. The axiom set is adequate in the sense of being equivalent to standard formulations of second order logic, e.g., that of Church [1]. It is clear and need not be shown here that every theorem of the present formulation is a theorem of the formulation given by Church. It of course will be shown here, however, that each of Church's axioms are theorems of the present system and that each of his primitive inference rules is either a primitive (and only *modus ponens* is taken as a primitive rule here) or a derived rule of the present system.

The importance of obtaining an axiomatic formulation such as herein described lies partly in the significance of reducing any axiom set to an equivalent one which involves fewer metalogical notions, especially such a one as proper substitution. However, of somewhat greater importance, it is highly desirable that we possess a formulation of both first and second order logic which can be extended without qualification to such areas as tense, epistemic, deontic, modal and logics of the like. Now proper substitution especially has been the main obstacle to such unqualified extensions of standard logic, and we take it to be of no little significance that at least for first order logic (with identity) a substitution free axiomatic formulation has been provided.[1] The present system extends this earlier result to the level of second order logic.[2]

A second difficulty in unqualified extensions of standard logic concerns the form which Leibniz' law, i.e., the law regarding interchangeability *salva veritate*, is to take. Generally, in the extensions of standard logic to modal logic, this law has been formulated in an unqualified form applicable to all contexts, thereby lending credence to the questionable view that only "intensions" or the like can serve adequately as values of the variables for such systems. In the substitution free formulations of first order logic

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cited, however, Leibniz' law is axiomatically formulated only for atomic contexts, and the qualified form or forms the law takes for contexts involving non-standard formula operators is given in the statement of metatheorems.[3] But again, it is a far different matter having such qualifications stipulated in the form of metatheorems as opposed to having them built directly into the characterization of the logical axioms. As we have said, it is desirable that the standard logical axioms for either first or second order logic be so that axiomatic extensions of standard logic can be made without qualification.[4] This desirable feature of the substitution free formulations of first order logic mentioned is retained in our present second order system.

§*1. Terminology* In what follows we shall take a language to be a set of predicate and operation constants of arbitrary number of places. We assume, for each natural number n, the existence of a denumerable sequence of n-place predicate variables as well as a denumerable sequence of individual variables. We shall speak of both predicate constants and predicate variables of a language $\mathfrak{L}$ as being *predicate expressions* of $\mathfrak{L}$. We note that a propositional constant or a propositional variable is a 0-place predicate constant or variable, respectively. Where $\mathfrak{L}$ is a language, we take the set of *terms* of $\mathfrak{L}$ to be the intersection of all sets Γ such that (1) every individual variable is in Γ, and (2) if n is a natural number, $\zeta_0, \ldots, \zeta_{n-1}$ are in Γ and δ is an n-place operation constant in $\mathfrak{L}$, then $\delta(\zeta_0, \ldots, \zeta_{n-1}) \epsilon \Gamma$. We say that φ is an *atomic formula* of a language $\mathfrak{L}$ if there are a natural number n, an n-place predicate expression π of $\mathfrak{L}$, and terms $\zeta_0, \ldots, \zeta_{n-1}$ of $\mathfrak{L}$ such that $\varphi = \pi(\zeta_0, \ldots, \zeta_{n-1})$. The set of *formulas* of a language $\mathfrak{L}$ is understood to be the intersection of all sets Γ such that (1) all atomic formulas of $\mathfrak{L}$ are in Γ, and (2) $\neg\varphi$, $(\varphi \rightarrow \psi)$, $\bigwedge\mu\varphi \epsilon \Gamma$ whenever φ, $\psi \epsilon \Gamma$ and μ is an individual or predicate variable. By a *term* or *formula* we understand a term or formula of some language. For convenience we shall use 'φ', 'ψ', 'χ' to refer to formulas, 'α', 'β', 'γ' to refer to individual variables, 'π', 'ρ', 'σ' to refer to predicate variables, 'ζ', 'η' to refer to terms, and 'μ', 'ν' to refer to both individual and predicate variables. We say that ψ is a *generalization* of a formula φ if there are a natural number n and predicate or individual variables $\mu_0, \ldots, \mu_{n-1}$ such that $\psi = \bigwedge\mu_0 \ldots \bigwedge\mu_{n-1}\varphi$.

We shall understand *bondage* and *freedom* of occurrences of terms, individual variables and predicate variables to be defined in the usual manner. Similarly, we presuppose the notion of a formula ψ *being obtained from a formula* φ *by replacing a free occurrence of the term* ζ *by a free occurrence of the term* η as having already been defined in the usual manner. We say that ψ is obtained from φ by *proper substitution* of a term ζ for the individual variable α if ψ is like φ except for having free occurrences of ζ wherever φ has a free occurrence of α. Where φ is a formula, α is an individual variable, and ζ is a term, we identify $\varphi\begin{bmatrix}\alpha\\ \zeta\end{bmatrix}$, the result of (properly) substituting ζ for α in φ, with that formula ψ which is obtained from φ by proper substitution of ζ for α if there exists such a formula ψ; and otherwise $\varphi\begin{bmatrix}\alpha\\ \zeta\end{bmatrix}$ is to be φ. If there is a formula ψ which is obtained from φ by proper substitution of ζ for α, we say that ζ *can be properly substituted for* α *in* φ. Where n is a natural number, $\alpha_0, \ldots, \alpha_{n-1}, \beta_0, \ldots, \beta_{n-1}$ are pairwise distinct individual variables, φ is a formula, $\zeta_0, \ldots, \zeta_{n-1}$ are terms, $\beta_0, \ldots, \beta_{n-1}$ are the first n individual variables which do not occur in $\varphi, \zeta_0, \ldots, \zeta_{n-1}$, we define *the result of the simultaneous proper substitution* of $\zeta_0, \ldots, \zeta_{n-1}$ for $\alpha_0, \ldots, \alpha_{n-1}$, respectively, in φ, in symbols $\varphi\begin{bmatrix}\alpha_0 \ldots \alpha_{n-1}\\ \zeta_0 \ldots \zeta_{n-1}\end{bmatrix}$, to be $\varphi\begin{bmatrix}\alpha_0\\ \beta_0\end{bmatrix} \ldots \begin{bmatrix}\alpha_{n-1}\\ \beta_{n-1}\end{bmatrix}\begin{bmatrix}\beta_0\\ \zeta_0\end{bmatrix} \ldots \begin{bmatrix}\beta_{n-1}\\ \zeta_{n-1}\end{bmatrix}$ (where the iterated applications of proper substitution are associated to the left) if $\zeta_0, \ldots, \zeta_{n-1}$ can be properly substituted for $\alpha_0, \ldots, \alpha_{n-1}$, respectively, in φ; otherwise $\varphi\begin{bmatrix}\alpha_0 \ldots \alpha_{n-1}\\ \zeta_0 \ldots \zeta_{n-1}\end{bmatrix} = \varphi$. Regarding proper substitution of formulas for predicate variables, we adopt the definition given in Church [1], p. 192f. However, we shall utilize the notation

$$\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}$$

in place of Church's notation

$$\check{\mathsf{S}}\ {}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\psi}\ \varphi|$$

For convenience we state here the following useful lemmas regarding this notion of substitution. It is assumed in the statement of these lemmas that n is a natural number, π is an n-place predicate variable, and $\alpha_0, \ldots, \alpha_{n-1}$ are pairwise distinct individual variables.

Lemma 1: *If* $\pi(\zeta_0, \ldots, \zeta_{n-1})\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \varphi\end{bmatrix} \neq \pi(\zeta_0, \ldots, \zeta_{n-1})$, *then* $\pi(\zeta_0, \ldots, \zeta_{n-1})\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \varphi\end{bmatrix} = \varphi\begin{bmatrix}\alpha_0 \ldots \alpha_{n-1}\\ \zeta_0 \ldots \zeta_{n-1}\end{bmatrix}$ *and* $\zeta_0, \ldots, \zeta_{n-1}$ *can be properly substituted for* $\alpha_0, \ldots, \alpha_{n-1}$, *respectively, in* φ.

Lemma 2: *If either* β *is an individual variable which does not occur free in* ψ *or, for some natural number* $k < n$, $\beta = \alpha_k$, *then* $(\wedge\beta\varphi)\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} = \wedge\beta\left(\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}\right)$.

Lemma 3: *If* σ *is a predicate variable distinct from* π, *and* σ *does not occur free in* ψ, *then* $(\wedge\sigma\varphi)\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} = \wedge\sigma\left(\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}\right)$.

Lemma 4: *If* σ *is a predicate variable which has a free occurrence in* ψ, *then* $(\wedge\sigma\varphi)\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} = \wedge\sigma\varphi$.

§2. *Substitution Free Axioms for Second Order Logic* In our statement of the second order logical axioms and theorems to follow we shall utilize some syntactically defined or abbreviatory notation for purposes of perspicuity. Accordingly, where φ, ψ are formulas and μ is either a predicate or individual variable, we set the following as (syntactical) definitions:

$$(\varphi \longleftrightarrow \psi) = \neg((\varphi \rightarrow \psi) \rightarrow \neg(\psi \rightarrow \varphi))$$
$$\bigvee \mu \varphi = \neg \bigwedge \mu \neg \varphi$$

Definition: A formula θ is a (*second order*) *logical axiom* if and only if there are a natural number n, individual variables $\alpha_0, \ldots, \alpha_{n-1}, \beta$, an n-place predicate variable π, a 1-place predicate variable σ, a predicate or individual variable μ, terms ζ, η, and formulas φ, ψ, χ such that θ is a generalization of one of the following formulas:

(A1) $\varphi \rightarrow (\psi \rightarrow \varphi)$,
(A2) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$,
(A3) $(\neg \varphi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \varphi)$,
(A4) $\bigwedge \mu(\varphi \rightarrow \psi) \rightarrow (\bigwedge \mu \varphi \rightarrow \bigwedge \mu \psi)$,
(A5) $\varphi \rightarrow \bigwedge \mu \varphi$, where μ is a predicate or individual variable which does not occur free in φ,
(A6) $\bigvee \pi \bigwedge \alpha_0 \ldots \bigwedge \alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \longleftrightarrow \varphi)$, where $\alpha_0, \ldots, \alpha_{n-1}$ are all the distinct individual variables that occur free in φ and π is an n-place predicate variable which does not occur free in φ,
(A7) $\bigvee \beta \bigwedge \sigma(\sigma(\beta) \rightarrow \sigma(\zeta))$, where β is an individual variable which does not occur in ζ,
(A8) $\bigwedge \sigma(\sigma(\zeta) \rightarrow \sigma(\eta)) \rightarrow (\varphi \rightarrow \psi)$, where φ, ψ are atomic formulas and ψ is obtained from φ by replacing an occurrence of ζ by an occurrence of η.

We shall have only one inference rule in the present system, *viz.*, *modus ponens*. *Proofs* are understood in the usual sense of being finite sequences every constituent of which is either a logical axiom or is obtained from preceding constituents by an application of *modus ponens*. *Theorems* are, of course, formulas for which there are proofs. We express the fact that φ is a theorem by writing '$\vdash \varphi$'.

We presuppose the notion of a tautology or *tautologous formula* without going into the definition here.[5] Because of (A1)-(A3) and the completeness of sentential logic, we have the following theorem:

Theorem 1: *If φ is a tautologous formula, then* $\vdash \varphi$.

Where proofs are not given for the remaining theorems, it is understood that they proceed in the standard fashion. We utilize our convention of having specific groups of Greek letters for the different kinds of expressions in what follows by not bothering to specify in each case the kind of expression involved.

Theorem 2: *If* $\vdash\varphi$ *and* $\vdash(\varphi \to \psi)$, *then* $\vdash\psi$.

Theorem 3: *If* $\vdash\varphi$, *then* $\vdash \wedge\mu\,\varphi$.

Theorem 4: *If* ψ *is obtained from* φ *by replacing a free occurrence of* ζ *by a free occurrence of* η, *then* $\vdash \wedge\pi(\pi(\zeta) \to \pi(\eta)) \to (\varphi \to \psi)$ *and* $\vdash \wedge\pi(\pi(\zeta) \to \pi(\eta)) \to (\psi \to \varphi)$.

Proof: Assume the hypothesis and let $\mathfrak{L}$ be the language whose predicate and operation constants are just those of φ, ζ, or η. Let Γ be the set of formulas φ of $\mathfrak{L}$ such that for all formulas ψ of $\mathfrak{L}$ and all terms ζ, η of $\mathfrak{L}$, if ψ is obtained from φ by replacing a free occurrence of ζ by a free occurrence of η, then $\vdash \wedge\pi(\pi(\zeta) \to \pi(\eta)) \to (\varphi \to \psi)$ and $\vdash \wedge\pi(\pi(\zeta) \to \pi(\eta)) \to (\psi \to \varphi)$. It suffices to show that every formula of $\mathfrak{L}$ is in Γ. That all atomic formulas of $\mathfrak{L}$ are in Γ is an immediate consequence of (A8). Moreover, by Theorems 1 and 2, $\neg\varphi$ and $(\varphi \to \chi)$ belong to Γ whenever $\varphi, \chi \in \Gamma$. Assume then that $\varphi \in \Gamma$ and that α, σ are an individual and a predicate variable, respectively. The proof of Theorem 4 is completed if we show that $\wedge\alpha\varphi, \wedge\sigma\varphi \in \Gamma$. Assume therefore that χ is obtained from $\wedge\alpha\varphi$ by replacing a free occurrence of ζ by a free occurrence of η and that θ is obtained from $\wedge\sigma\varphi$ by replacing a free occurrence of ζ by a free occurrence of η. Then, by definition, there is a formula φ' of $\mathfrak{L}$ such that $\chi = \wedge\alpha\varphi'$, $\theta = \wedge\sigma\varphi'$, φ' is obtained from φ by replacing a free occurrence of ζ by a free occurrence of η, and α does not occur in either ζ or η. Accordingly,

$\vdash \wedge\pi(\pi(\zeta) \to \pi(\eta)) \to (\varphi \to \varphi')$ since $\varphi \in \Gamma$,
$\vdash \wedge\sigma\,[\wedge\pi(\pi(\zeta) \to \pi(\eta)) \to (\varphi \to \varphi')]$ and
$\vdash \wedge\alpha\,[\wedge\pi(\pi(\zeta) \to \pi(\eta)) \to (\varphi \to \varphi')]$ by Theorem 3,
$\vdash \wedge\sigma\, \wedge\pi(\pi(\zeta) \to \pi(\eta)) \to (\wedge\sigma\varphi \to \wedge\sigma\varphi')$ and
$\vdash \wedge\alpha\, \wedge\pi(\pi(\zeta) \to \pi(\eta)) \to (\wedge\alpha\varphi \to \wedge\alpha\varphi')$ by (A4) and Theorems 1 and 2,
$\vdash \wedge\pi(\pi(\zeta) \to \pi(\eta)) \to \wedge\sigma\wedge\pi(\pi(\zeta) \to \pi(\eta))$ and
$\vdash \wedge\pi(\pi(\zeta) \to \pi(\eta)) \to \wedge\alpha\,\wedge\pi(\pi(\zeta) \to \pi(\eta))$ by (A5), and therefore
$\vdash \wedge\pi(\pi(\zeta) \to \pi(\eta)) \to (\wedge\sigma\varphi \to \wedge\sigma\varphi')$ and
$\vdash \wedge\pi(\pi(\zeta) \to \pi(\eta)) \to (\wedge\alpha\varphi \to \wedge\alpha\varphi')$ by Theorems 1 and 2.

By an entirely similar argument, $\vdash \wedge\pi(\pi(\zeta) \to \pi(\eta)) \to (\wedge\sigma\varphi' \to \wedge\sigma\varphi)$ and $\vdash \wedge\pi(\pi(\zeta) \to \pi(\eta)) \to (\wedge\alpha\varphi' \to \wedge\alpha\varphi)$, and therefore $\wedge\alpha\varphi, \wedge\sigma\varphi \in \Gamma$. (Q.E.D.)

Theorem 5: *If* α *does not occur in* ζ *and* ψ *is obtained from* φ *by proper substitution of* ζ *for* α, *then* $\vdash \wedge\alpha\varphi \to \psi$.

Proof: Assume the hypothesis. Then,

$\vdash \wedge\pi(\pi(\alpha) \to \pi(\zeta)) \to (\varphi \to \psi)$ by repeated use of Theorems 4, 1, and 2,
$\vdash \wedge\alpha\big(\neg\psi \to [\varphi \to \neg\wedge\pi(\pi(\alpha) \to \pi(\zeta))]\big)$ by Theorems 1, 2, and 3,
$\vdash \wedge\alpha\neg\psi \to [\wedge\alpha\varphi \to \wedge\alpha\neg\wedge\pi(\pi(\alpha) \to \pi(\zeta))]$ by (A4) and Theorems 1 and 2,
$\vdash \neg\psi \to \wedge\alpha\neg\psi$ by (A5),
$\vdash \neg\wedge\alpha\neg\wedge\pi(\pi(\alpha) \to \pi(\zeta)) \to \big(\wedge\alpha\varphi \to \psi\big)$ by Theorems 1 and 2,
$\vdash \neg\wedge\alpha\neg\wedge\pi(\pi(\alpha) \to \pi(\zeta))$ by (A7), and therefore
$\vdash \wedge\alpha\varphi \to \psi$ by Theorem 2, (Q.E.D.)

Theorem 6: *If β is an individual variable which does not occur free in φ and ψ is obtained from φ by proper substitution of β for α, then* $\vdash \wedge\alpha\varphi \rightarrow \wedge\beta\psi$ *and* $\vdash \wedge\beta\psi \rightarrow \wedge\alpha\varphi$.

Proof: Assume the hypothesis. If α has no free occurrences in φ, then, by definition, $\psi = \varphi$, and Theorem 6 holds trivially by (A5) and Theorems 1, 2, and 5. Assume therefore that α has free occurrences in φ. Then $\alpha \neq \beta$; and, accordingly,

$\vdash \wedge\alpha\varphi \rightarrow \psi$	by Theorem 5,
$\vdash \wedge\beta(\wedge\alpha\varphi \rightarrow \psi)$	by Theorem 3,
$\vdash \wedge\beta\ \wedge\alpha\varphi \rightarrow \wedge\beta\psi$	by (A4) and Theorem 2,
$\vdash \wedge\alpha\varphi \rightarrow \wedge\beta\wedge\alpha\varphi$	by (A5), and therefore
$\vdash \wedge\alpha\varphi \rightarrow \wedge\beta\psi$	by Theorems 1 and 2.

By a similar argument it is shown that $\vdash \wedge\beta\psi \rightarrow \wedge\alpha\varphi$. (Q.E.D.)

Theorem 7: *If α is an individual variable which has no free occurrences in φ, β is an individual variable which does not occur in φ, and ψ is obtained from φ by replacing each occurrence of α by an occurrence of β, then* $\vdash \varphi \rightarrow \psi$ *and* $\vdash \psi \rightarrow \varphi$.

Proof: Theorem 7 is easily seen to hold by a simple inductive argument using Theorems 1, 2, and 6. (Q.E.D.)

Theorem 8: *If ψ' is obtained from ψ by replacing an occurrence of φ by an occurrence of φ',* $\vdash \varphi \rightarrow \varphi'$ *and* $\vdash \varphi' \rightarrow \varphi$, *then* $\vdash \psi \rightarrow \psi'$ *and* $\vdash \psi' \rightarrow \psi$.

Proof: Assume the hypothesis and let $\mathfrak{L}$ be the language consisting of all the predicate and operation constants occurring in ψ, φ, or φ'. Let Γ be the set formulas ψ of $\mathfrak{L}$ such that for all formulas ψ', φ, φ' of $\mathfrak{L}$, if ψ' is obtained from ψ by replacing an occurrence of φ by an occurrence of φ', $\vdash \varphi \rightarrow \varphi'$ and $\vdash \varphi' \rightarrow \varphi$, then $\vdash \psi \rightarrow \psi'$ and $\vdash \psi' \rightarrow \psi$. By a simple inductive argument using Theorems 1, 2, 3, and (A4)-(A5), it is easily shown that every formula of $\mathfrak{L}$ is in Γ. (Q.E.D.)

Theorem 9: (Rule of alphabetic change of bound individual variables): *If α is an individual variable which has no free occurrences in φ, β is an individual variable which does not occur in φ, φ' is obtained from φ by replacing each occurrence of α by an occurrence of β, and ψ' is obtained from ψ by replacing an occurrence of φ by an occurrence of φ', then* $\vdash \psi \rightarrow \psi'$ *and* $\vdash \psi' \rightarrow \psi$.

Proof: Theorem 9 follows trivially from Theorems 7 and 8. (Q.E.D.)

Theorem 10: *If ψ is obtained from φ by proper substitution of ζ for α, then* $\vdash \wedge\alpha\varphi \rightarrow \psi$.

Proof: Assume the hypothesis. If α does not occur in ζ, then the desired result follows from Theorem 5. Assume therefore that α occurs in ζ. Let β be an individual variable which does not occur in either ζ or in φ, and let

ζ' be the result of replacing in ζ each occurrence of α by an occurrence of β. Finally, let ψ' be the result of properly substituting ζ' for α in φ. Then,

$\vdash \wedge\alpha\varphi \rightarrow \psi'$ by Theorem 5,
$\vdash \wedge\beta(\wedge\alpha\varphi \rightarrow \psi')$ by Theorem 3,
$\vdash \wedge\beta(\wedge\alpha\varphi \rightarrow \psi') \rightarrow (\wedge\alpha\varphi \rightarrow \psi)$ by Theorem 5, and therefore
$\vdash \wedge\alpha\varphi \rightarrow \psi$ by Theorem 2.
(Q.E.D.)

Corollary: *If* $\vdash\varphi$, *then* $\vdash\varphi\begin{bmatrix}\alpha\\ \zeta\end{bmatrix}$.

Proof: By Theorems 3, 10, and 2. (Q.E.D.)

§3. *Proof of Adequacy* We note at this point that Church's first inference rule *500, *viz*., *modus ponens*, is a primitive rule of the present system. His inference rules *501-*503 are all derived rules here according to Theorems 3, 9, and the Corollary to Theorem 10, respectively. Church's axioms †505-†507 are, of course, all theorems here by (A1)-(A3). Every instance of Church's schematic axioms $*508_n$, for each natural number n, as well as his axiom †508, are easily seen to be theorems of the present system by (A4), (A5), and Theorems 1 and 2. Church's axiom †509 is, of course, an instance of our Theorem 10. It remains therefore to show that each instance of Church's schematic axioms $*509_n$, for each natural number n, is a theorem of the present system. The remainder of the (meta) theorems shown here all lead to establishing this result in the form of Theorem 16.

We remark that Theorems 11, 12, and 16, and their development from one to the other, are the obvious analogues of Theorems 4, 5, and 10, respectively, and their development from one to the other. This is noteworthy since the manner in which we justify our specification principle for predicate variables (Theorem 16) parallels that in which we justify our specification principle for individual variables (Theorem 10); and, accordingly, the novelty of our major result, *viz*., a substitution free axiom set for second order logic, is derived from the novelty of this parallel reasoning.

We remind the reader of our convention of having specific groups of Greek letters for the different kinds of expressions, a convention we continue to use in what follows. In addition, for the remainder of this paper it will be assumed that n is a natural number, π is an n-place predicate variable, and $\alpha_0, \ldots, \alpha_{n-1}$ are pairwise distinct individual variables.

Theorem 11: *If* $\alpha_0, \ldots, \alpha_{n-1}$ *are all the individual variables that have free occurrences in* ψ, *then* $\vdash \wedge\alpha_0 \ldots \wedge\alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \psi) \rightarrow \left(\varphi \rightarrow \left(\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}\right)\right)$ *and* $\vdash \wedge\alpha_0 \ldots \wedge\alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \psi) \rightarrow \left(\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} \rightarrow \varphi\right)$.

Proof: Assume the hypothesis and let $\mathfrak{L}$ be the language consisting of the predicate and operation constants occurring in φ or ψ. Let Γ be the set of formulas φ of $\mathfrak{L}$ such that for all formulas ψ of $\mathfrak{L}$, if $\alpha_0, \ldots, \alpha_{n-1}$ are all the individual variables occurring free in ψ, then

$$\vdash \Lambda\alpha_0 \ldots \Lambda\alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \psi) \rightarrow \left(\varphi \rightarrow \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}\right)$$

and

$$\vdash \Lambda\alpha_0 \ldots \Lambda\alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \psi) \rightarrow \left(\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} \rightarrow \varphi\right).$$

It suffices to show that every formula of $\mathfrak{L}$ is in Γ. Suppose φ is an arbitrary atomic formula of $\mathfrak{L}$. Then $\varphi = \rho(\zeta_0, \ldots, \zeta_{k-1})$ for some natural number k, some k-place predicate expression ρ of $\mathfrak{L}$, and some terms $\zeta_0, \ldots, \zeta_{k-1}$ of $\mathfrak{L}$. If $\rho \neq \pi$, then for all formulas ψ, $\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} = \varphi$; and therefore $\varphi \epsilon \Gamma$ by Theorem 1. Assume then that $\rho = \pi$ and that ψ is an arbitrary formula of $\mathfrak{L}$ whose only free individual variables are $\alpha_0, \ldots, \alpha_{n-1}$. Accordingly, $k = n$ and $\varphi = \pi(\zeta_0, \ldots, \zeta_{n-1})$. If $\pi(\zeta_0, \ldots, \zeta_{n-1})$ $\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} = \pi(\zeta_0, \ldots, \zeta_{n-1})$, then $\varphi \epsilon \Gamma$ by Theorem 1. Assume therefore that $\pi(\zeta_0, \ldots, \zeta_{n-1})\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} \neq \pi(\zeta_0, \ldots, \zeta_{n-1})$. Then, by Lemma 1, $\pi(\zeta_0, \ldots, \zeta_{n-1})\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} = \psi\begin{bmatrix}\alpha_0 \ldots \alpha_{n-1}\\ \zeta_0 \ldots \zeta_{n-1}\end{bmatrix}$ and $\zeta_0, \ldots, \zeta_{n-1}$ can be properly substituted for $\alpha_0, \ldots, \alpha_{n-1}$, respectively, in ψ. Let $\beta_0, \ldots, \beta_{n-1}$ be the first n individual variables which do not occur in $\psi, \zeta_0, \ldots, \zeta_{n-1}$ and which do not belong to $\{\alpha_0, \ldots, \alpha_{n-1}\}$. Then by repeated application of Theorems 10 and 3, (A4), Theorems 1 and 2, and (A5),

$$\vdash \Lambda\alpha_0 \ldots \Lambda\alpha_{n-1}(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \psi) \rightarrow \Lambda\beta_0 \ldots \Lambda\beta_{n-1}\left(\pi(\beta_0, \ldots, \beta_{n-1}) \leftrightarrow \psi\begin{bmatrix}\alpha_0\\ \beta_0\end{bmatrix} \ldots \begin{bmatrix}\alpha_{n-1}\\ \beta_{n-1}\end{bmatrix}\right).$$

Moreover, by repeated application of Theorems 10, 1, and 2,

$$\vdash \Lambda\beta_0 \ldots \Lambda\beta_{n-1}\left(\pi(\beta_0, \ldots, \beta_{n-1}) \leftrightarrow \psi\begin{bmatrix}\alpha_0\\ \beta_0\end{bmatrix} \ldots \begin{bmatrix}\alpha_{n-1}\\ \beta_{n-1}\end{bmatrix}\right) \rightarrow \left(\pi(\zeta_0, \ldots, \zeta_{n-1}) \leftrightarrow \psi\begin{bmatrix}\alpha_0\\ \beta_0\end{bmatrix} \ldots \begin{bmatrix}\alpha_{n-1}\\ \beta_{n-1}\end{bmatrix}\begin{bmatrix}\beta_0\\ \zeta_0\end{bmatrix} \ldots \begin{bmatrix}\beta_{n-1}\\ \zeta_{n-1}\end{bmatrix}\right).$$

But, by definition, $\psi\begin{bmatrix}\alpha_0\\ \beta_0\end{bmatrix} \ldots \begin{bmatrix}\alpha_{n-1}\\ \beta_{n-1}\end{bmatrix}\begin{bmatrix}\beta_0\\ \zeta_0\end{bmatrix} \ldots \begin{bmatrix}\beta_{n-1}\\ \zeta_{n-1}\end{bmatrix} = \psi\begin{bmatrix}\alpha_0 \ldots \alpha_{n-1}\\ \zeta_0 \ldots \zeta_{n-1}\end{bmatrix}$ and therefore,

$$\vdash \Lambda\beta_0 \ldots \Lambda\beta_{n-1}\left(\pi(\beta_0, \ldots, \beta_{n-1}) \leftrightarrow \psi\begin{bmatrix}\alpha_0\\ \beta_0\end{bmatrix} \ldots \begin{bmatrix}\alpha_{n-1}\\ \beta_{n-1}\end{bmatrix}\right) \rightarrow \left(\pi(\zeta_0, \ldots, \zeta_{n-1}) \leftrightarrow \psi\begin{bmatrix}\alpha_0 \ldots \alpha_{n-1}\\ \zeta_0 \ldots \zeta_{n-1}\end{bmatrix}\right).$$

and, accordingly, by Theorems 1 and 2.

$$\vdash \wedge\alpha_0 \ldots \wedge\alpha_{n-1}\,(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \psi) \rightarrow \left(\pi(\zeta_0, \ldots, \zeta_{n-1}) \leftrightarrow \psi\begin{bmatrix}\alpha_0 \ldots \alpha_{n-1}\\ \zeta_0 \ldots \zeta_{n-1}\end{bmatrix}\right),$$

from which it follows, by Theorems 1 and 2, that $\varphi \epsilon \Gamma$. Since $(\neg\varphi)\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} = \neg\left(\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}\right)$ and since $(\varphi \rightarrow \chi)\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} = \left(\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} \rightarrow \chi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}\right)$, or $(\varphi \rightarrow \chi)\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} = (\varphi \rightarrow \chi)$, then by Theorems 1 and 2, $\neg\varphi$ and $(\varphi \rightarrow \chi)$ are in Γ whenever $\varphi, \chi \epsilon \Gamma$. Assume now that $\varphi \epsilon \Gamma$ and that γ, σ are an individual and a predicate variable, respectively. It suffices to show that $\wedge\gamma\varphi$, $\wedge\sigma\varphi \epsilon \Gamma$. Suppose ψ is a formula of $\mathfrak{L}$ whose only free individual variables are $\alpha_0, \ldots, \alpha_{n-1}$. Then,

$$\vdash \wedge\alpha_0 \ldots \wedge\alpha_{n-1}\,(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \psi) \rightarrow \left(\varphi \rightarrow \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}\right)$$

since $\varphi \epsilon \Gamma$.

$$\vdash \wedge\gamma\ \wedge\alpha_0 \ldots \wedge\alpha_{n-1}\,(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \psi) \rightarrow \left(\wedge\gamma\varphi \rightarrow \wedge\gamma\left(\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}\right)\right)$$

by Theorem 3, (A4), and Theorems 1 and 2,

$$\vdash \wedge\alpha_0 \ldots \wedge\alpha_{n-1}\,(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \psi) \rightarrow \wedge\gamma\wedge\alpha_0 \ldots \wedge\alpha_{n-1}\,(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \psi)$$

by (A5).

We note that either γ does not have a free occurrence in ψ or $\gamma = \alpha_i$, for some $i < n$. In either case, by Lemma 2, $\wedge\gamma\left(\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}\right) = (\wedge\gamma\varphi)\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}$, and therefore

$$\vdash \wedge\alpha_0 \ldots \wedge\alpha_{n-1}\,(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \psi) \rightarrow \left(\wedge\gamma\alpha \rightarrow (\wedge\gamma\varphi)\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}\right).$$

By an entirely similar argument,

$$\vdash \wedge\alpha_0 \ldots \wedge\alpha_{n-1}\,(\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \psi) \rightarrow \left((\wedge\gamma\varphi)\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} \rightarrow \wedge\gamma\varphi\right),$$

and therefore $\wedge\gamma\varphi\epsilon\Gamma$. In regard to showing that $\wedge\sigma\varphi\epsilon\Gamma$ we consider two cases, depending on whether $\sigma = \pi$ or $\sigma \neq \pi$. If $\sigma = \pi$, then, by definition, $(\wedge\sigma\varphi)\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} = \wedge\sigma\varphi$, from which it trivially follows by Theorems 1 and 2 that $\wedge\sigma\varphi\epsilon\Gamma$. Assume then that $\sigma \neq \pi$. If σ has a free occurrence in ψ, then, by Lemma 4, $(\wedge\sigma\varphi)\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} = \wedge\sigma\varphi$, from which it follows by Theorems 1 and 2 that $\wedge\sigma\varphi\epsilon\Gamma$. Suppose σ has no free occurrences in ψ. Accordingly, by Lemma 3, $(\wedge\sigma\varphi)\begin{bmatrix}\pi(\alpha_0 \; . \; . \; . \; , \alpha_{n-1}) \\ \psi\end{bmatrix} =$ $\wedge\sigma\left(\varphi\begin{bmatrix}\pi(\alpha_0, \; . \; . \; . \; , \alpha_{n-1}) \\ \psi\end{bmatrix}\right)$. Furthermore,

$$\vdash \wedge\alpha_0 \ldots \wedge\alpha_{n-1}(\pi(\alpha_0, \ldots , \alpha_{n-1}) \longleftrightarrow \psi) \to \left(\varphi \to \varphi\begin{bmatrix}\pi(\alpha_0, \ldots , \alpha_{n-1}) \\ \psi\end{bmatrix}\right)$$

since $\varphi \epsilon \Gamma$,

$$\vdash \wedge\sigma \wedge\alpha_0 \ldots \wedge\alpha_{n-1}(\pi(\alpha_0, \ldots , \alpha_{n-1} \longleftrightarrow \psi) \to \left(\wedge\sigma\varphi \to (\wedge\sigma\varphi)\begin{bmatrix}\pi(\alpha_0, \ldots , \alpha_{n-1}) \\ \psi\end{bmatrix}\right)$$

by Theorem 3, (A4), and Theorems 1 and 2.

$$\vdash \wedge\alpha_0 \ldots \wedge\alpha_{n-1}(\pi(\alpha_0, \ldots , \alpha_{n-1}) \longleftrightarrow \psi) \to \wedge\sigma \wedge\alpha_0 \ldots \wedge\alpha_{n-1}(\pi(\alpha_0, \ldots , \alpha_{n-1}) \longleftrightarrow \psi)$$

by (A5), and therefore

$$\vdash \wedge\alpha_0 \ldots \wedge\alpha_{n-1}(\pi(\alpha_0, \ldots , \alpha_{n-1}) \longleftrightarrow \psi) \to \left(\wedge\sigma\varphi \to (\wedge\sigma\varphi)\begin{bmatrix}\pi(\alpha_0, \ldots , \alpha_{n-1}) \\ \psi\end{bmatrix}\right)$$

by Theorems 1 and 2.

By an entirely similar argument,

$$\vdash \wedge\alpha_0 \ldots \wedge\alpha_{n-1}(\pi(\alpha_0, \ldots , \alpha_{n-1}) \longleftrightarrow \psi) \to \left((\wedge\sigma\varphi)\begin{bmatrix}\pi(\alpha_0, \ldots , \alpha_{n-1}) \\ \psi\end{bmatrix} \to \wedge\sigma\varphi\right),$$

and therefore $\wedge\sigma\varphi\epsilon\Gamma$. (Q.E.D.)

Theorem 12: *If* $\alpha_0, \ldots , \alpha_{n-1}$ *are all the individual variables that have free occurrences in* ψ, *and* π *does not have a free occurrence in either* ψ *or in* $\varphi\begin{bmatrix}\pi(\alpha_0, \ldots , \alpha_{n-1}) \\ \psi\end{bmatrix}$, *then* $\vdash \wedge\pi\varphi \to \varphi\begin{bmatrix}\pi(\alpha_0, \ldots , \alpha_{n-1}) \\ \psi\end{bmatrix}$.

Proof: Similar to the proof of Theorem 5, using Theorem 11 in place of Theorem 4 and (A6) in place of (A7). (Q.E.D).

Theorem 13: $\vdash \wedge\pi\varphi \to \varphi$.

Proof: Let ρ be the first n-place predicate variable distinct from π and which does not occur in φ. Then

$$\vdash \wedge\pi\varphi \to \varphi\begin{bmatrix}\pi(\alpha_0, \ldots , \alpha_{n-1}) \\ \rho(\alpha_0, \ldots , \alpha_{n-1})\end{bmatrix}$$

by Theorem 12,

$$\vdash \wedge\rho \wedge\pi\varphi \to \wedge\rho\left(\varphi\begin{bmatrix}\pi(\alpha_0, \ldots , \alpha_{n-1}) \\ \rho(\alpha_0, \ldots , \alpha_{n-1})\end{bmatrix}\right)$$

by Theorem 3, (A4), and Theorems 1 and 2,

$$\vdash \wedge\pi\varphi \to \wedge\rho \wedge\pi\varphi$$

by (A5),

$$\vdash \wedge\pi\varphi \to \wedge\rho\left(\varphi\begin{bmatrix}\pi(\alpha_0, \ldots , \alpha_{n-1}) \\ \rho(\alpha_0, \ldots , \alpha_{n-1})\end{bmatrix}\right)$$

by Theorems 1 and 2,

$$\vdash \wedge\rho\left(\varphi\begin{bmatrix}\pi(\alpha_0, \ldots , \alpha_{n-1}) \\ \rho(\alpha_0, \ldots , \alpha_{n-1})\end{bmatrix}\right) \to \varphi\begin{bmatrix}\pi(\alpha_0, \ldots , \alpha_{n-1}) \\ \rho(\alpha_0, \ldots , \alpha_{n-1})\end{bmatrix}\begin{bmatrix}\rho(\alpha_0, \ldots , \alpha_{n-1}) \\ \pi(\alpha_0, \ldots , \alpha_{n-1})\end{bmatrix}$$

by Theorem 12 and therefore,

$\vdash \bigwedge\pi\varphi \rightarrow \varphi$ by Theorems 1, 2, and the fact that $\varphi = \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \rho(\alpha_0, \ldots, \alpha_{n-1})\end{bmatrix}\begin{bmatrix}\rho(\alpha_0, \ldots, \alpha_{n-1})\\ \pi(\alpha_0, \ldots, \alpha_{n-1})\end{bmatrix}$. (Q.E.D.)

Theorem 14: *If* $\alpha_0, \ldots, \alpha_{n-1}$ *are all the individual variables occurring free in* ψ, *then* $\vdash \bigwedge\pi\varphi \rightarrow \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}$.

Proof: Assume the hypothesis. If π does not have a free occurrence in either ψ or in $\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}$, then the desired result follows from Theorem 12. If π does not have a free occurrence in ψ but does have a free occurrence in $\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}$ or if π has a free occurrence in ψ but does not have one in $\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}$, then, by definition, $\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} = \varphi$; and therefore the desired result follows from Theorem 13. Suppose then that π has a free occurrence in both ψ and $\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}$. If $\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} = \varphi$, then

$\vdash \bigwedge\pi\varphi \rightarrow \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}$ by Theorem 13.

Assume therefore that $\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} \neq \varphi$. Let ρ be the first n-place predicate variable distinct from π and which does not occur in either φ or ψ, and let $\psi' = \psi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \rho(\alpha_0, \ldots, \alpha_{n-1})\end{bmatrix}$. Accordingly, by definition, π does not have a free occurrence in ψ'; and, furthermore, π does not have a free occurrence in $\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi'\end{bmatrix}$, since the latter is in effect $\left(\varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}\right)\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \rho(\alpha_0, \ldots, \alpha_{n-1})\end{bmatrix}$. Then

$\vdash \bigwedge\pi\varphi \rightarrow \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi'\end{bmatrix}$ by Theorem 12,

$\vdash \bigwedge\rho\left(\bigwedge\pi\varphi \rightarrow \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi'\end{bmatrix}\right)$ by Theorem 3,

$\vdash \bigwedge\rho\left(\bigwedge\pi\varphi \rightarrow \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi'\end{bmatrix}\right) \rightarrow$
$\left(\bigwedge\pi\varphi \rightarrow \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi'\end{bmatrix}\right)\begin{bmatrix}\rho(\alpha_0, \ldots, \alpha_{n-1})\\ \pi(\alpha_0, \ldots, \alpha_{n-1})\end{bmatrix}$ by Theorem 12.

$\vdash \left(\bigwedge\pi\varphi \rightarrow \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi'\end{bmatrix}\right)\begin{bmatrix}\rho(\alpha_0, \ldots, \alpha_{n-1})\\ \pi(\alpha_0, \ldots, \alpha_{n-1})\end{bmatrix}$
by Theorem 2, and therefore

$\vdash \bigwedge\pi\varphi \rightarrow \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}$ by definition. (Q.E.D.)

Theorem 15: *If* $\chi = ((\pi(\alpha_0, \ldots, \alpha_{n-1}) \longleftrightarrow \pi(\alpha_0, \ldots, \alpha_{n-1})) \rightarrow \psi)$, *then* $\vdash \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \chi\end{bmatrix} \rightarrow \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}$ *and* $\vdash \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} \rightarrow \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \chi\end{bmatrix}$.

Proof: Theorem 15 is easily seen to hold by a simple inductive argument using Theorems 1, 2, 3, and (A4). (Q.E.D.)

Theorem 16: $\vdash \Lambda\pi\varphi \rightarrow \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}$.

Proof: Assume the hypothesis. Let $\beta_0, \ldots, \beta_{k-1}$ be (without repetition) all the distinct individual variables that are free in ψ and that are not members of $\{\alpha_0, \ldots, \alpha_{n-1}\}$. Let $\zeta_0, \ldots, \zeta_{k-1}$ be pairwise distinct individual constants (i.e., 0-place operation constants) which do not occur in φ or ψ. In addition, let $\chi = ((\pi(\alpha_0, \ldots, \alpha_{n-1}) \longleftrightarrow \pi(\alpha_0, \ldots, \alpha_{n-1})) \rightarrow \psi\begin{bmatrix}\beta_0 & \ldots & \beta_{k-1}\\ \zeta_0 & \ldots & \zeta_{k-1}\end{bmatrix})$. Accordingly, $\alpha_0, \ldots, \alpha_{n-1}$ are all the individual variables occurring free in χ. Therefore,

$$\vdash \Lambda\pi\varphi \rightarrow \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \chi\end{bmatrix} \qquad \text{by Theorem 14,}$$

$$\vdash \Lambda\pi\varphi \rightarrow \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{k-1})\\ \psi\begin{bmatrix}\beta_0 & \ldots & \beta_{k-1}\\ \zeta_0 & \ldots & \zeta_{k-1}\end{bmatrix}\end{bmatrix} \qquad \text{by Theorem 15,}$$

$$\vdash \Lambda\beta_0 \ldots \Lambda\beta_{k-1}\left(\Lambda\pi\varphi \rightarrow \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix}\right)$$

by repeated application of (A5) and Theorem 2, and therefore,

$$\vdash \Lambda\pi\varphi \rightarrow \varphi\begin{bmatrix}\pi(\alpha_0, \ldots, \alpha_{n-1})\\ \psi\end{bmatrix} \qquad \text{by Theorem 10. (Q.E.D.)}$$

NOTES

1. Such a formulation is given by A. Tarski in [2] and developed by D. Kalish and R. Montague in [3]. The present author in [4] and [5] has also formulated a substitution free axiomatization of first order logic without "existential presuppositions."

2. Of course, when extending either first or second order logic to tense, epistemic, deontic, or modal logic, qualifications in metatheorems regarding principles of proper substitution will be required. Nevertheless, it is a far different matter having such qualifications stipulated in the form of metatheorems than it is having them built directly into the characterization of the logical axioms themselves.

3. cf. [4], lemma 4.27 (p. 108) and the discussion on page 106. The objections against an unqualified, general version of Leibniz' principle (or interchangeability *salva veritate*) are applicable when certain special "opaque" contexts are involved, be they modal or otherwise. But all such contexts are—or should be when properly

formalized—other than atomic, their "opacity" being generated within the scope of special formula operators. Atomic formulas, because they are atomic, will contain no occurrences of such operators and consequently will uphold *par excellence* the Leibnizian principle unqualifiedly.

4. Examples of such unqualified extensions of standard logic can be found in [4], [6], and [7].

5. cf. [2].

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