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A FIRST ORDER TYPE THEORY FOR THE THEORY OF SETS

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In *Set Theory and its Logic* Quine presents a system of axioms for a first order simple theory of types. In that system, which we shall call "$\mathbf{T}_n$" Aussonderung and Sum are axioms. We shall present a first order simple type theory, which we shall call "**JP**", in which Aussonderung and Sum are theorems. Once this is done we introduce a simple notion for a standard model for type theory and show that the class of standard models for **JP** is the same as the class of standard models for $\mathbf{T}_n$*.

Some definitions are needed before we can continue. The notation is that of reference [4].

Definition: $(Ez)(w \epsilon^2 z \wedge x \epsilon z)$ means $(Ey)(w \epsilon y \wedge (Ez)(x \epsilon z \wedge y \epsilon z))$
Definition: $w\mathbf{PT}x$ means $(Ez)(w \epsilon^2 z \wedge x \epsilon z)$

$w\mathbf{PT}x$ is read "w precedes x in type".

Definition: $\mathbf{T}_0(x)$ means $(w) \sim (w\mathbf{PT}x)$

$\mathbf{T}_{n+1}(x)$ means $(w)(\mathbf{T}_n(w) \supset w\mathbf{PT}x)$.

$\mathbf{T}_n(x)$ is read "x is on level n".

The axioms of $\mathbf{T}_n$ are:

Extensionality: $(x)(y)(\mathbf{T}_{n+1}(x) \wedge \mathbf{T}_{n+1}(y) \wedge (z)(z \epsilon x \equiv z \epsilon y) . \supset x = y)$
Comprehension: $[Ey)(\mathbf{T}_{n+1}(y) \wedge (x)(x \epsilon y \equiv \mathbf{T}_n(x) \wedge \mathcal{B}(x)))$
All-Some: $(Ex)\,\mathbf{T}_0(x)$
Stratification: $x \epsilon y \supset (\mathbf{T}_n(x) \equiv \mathbf{T}_{n+1}(y))$
Aussonderung: $(Ey)(x)(x \epsilon y \equiv x \epsilon z \wedge \mathcal{B}(x))$
Sum: $(Ey)(x)(x \epsilon y \equiv x \epsilon^2 z)$

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In the system **JP** equality is not assumed to be part of the underlying logic as it is in $\mathbf{T}_n$. Rather, "=" is an undefined two-place predicate. The following definitions will be used:

Definition: $w\mathbf{JP}x$ means $(Ez)(w \in^2 z \wedge x \in z)$.

$w\mathbf{JP}x$ is read "w just precedes x". Note that **JP** has the same definition as **PT**. It is useful to use **JP**, however, in order to make it clear in which system we are working.

Definition: $S^2(x, y)$ means $(z)(x\mathbf{JP}z \equiv y\mathbf{JP}z)$.

$S^2(x, y)$ asserts that x and y are on the same type level.

Definition: $S^{n+1}(x_1, \ldots, x_n, x_{n+1})$ means $S^n(x_1, \ldots, x_n) \wedge S^2(x_1, x_{n+1})$, $(n \geq 2)$

$S^n(x_1, \ldots, x_n)$ asserts that $x_1, \ldots, x_n$ are on the same type level, $(n \geq 2)$.

The axioms of **JP** follow:

Axioms of Extensionality:
E-1: $(x)(y)(S^2(x, y) \wedge (z)(x\mathbf{JP}z \supset . x \in z \equiv y \in z) . \equiv . x = y)$
E-2: $(x)(y)(S^2(x, y) \wedge (Ew)(w\mathbf{JP}x) \wedge (z)(z\mathbf{JP}x \supset . z \in x \equiv z \in y) . \supset x = y)$
Comprehension: $(x)(Ey)(z)(x\mathbf{JP}y \wedge (z \in y \equiv z\mathbf{JP}y \wedge \mathcal{B}(z)))$
All-Some: $(Ex)(y) \sim (y\mathbf{JP}x)$
Stratification: $x \in y \supset x\mathbf{JP}y$
Level: $S^2(x, x)$
Level 0: $(x)(y)((z) \sim (z\mathbf{JP}x) \wedge (w) \sim (w\mathbf{JP}y) . \supset S^2(x, y))$

The following are theorems of the system **JP**:

Theorem 1 on Identity: $(x)(x = x)$

Theorem 2 on Idenity: $x = y \equiv \mathcal{A}(x) \supset \mathcal{A}(y)$ *for any wff* $\mathcal{A}$ *where* $\mathcal{A}(y)$ *arises from* $\mathcal{A}(x)$ *by replacing some free occurrences of* x *by* y, *where* y *is free for* x.

Proof: Suppose $\mathcal{A}(x) \supset \mathcal{A}(y)$; then $x = x \supset x = y$; but $x = x$, and hence $x = y$. To prove the converse, suppose $x = y$, $\mathcal{A}(x)$ and $\sim\mathcal{A}(y)$. Since $(u)(Ev)(w)(S^2(u, v) \wedge (w \in v \equiv w\mathbf{JP}v \wedge \mathcal{B}(w)))$ and $(Ez)(x\mathbf{JP}z)$, we write $x\mathbf{JP}t$ yielding $(Ev)(w)(S^2(t, v) \wedge (w \in v \equiv . w\mathbf{JP}v \wedge \mathcal{B}(w)))$. This gives us $(w)(S^2(t, r) \wedge (w \in r \equiv . w\mathbf{JP}r \wedge \mathcal{B}(w)))$. Now $x\mathbf{JP}r \wedge y\mathbf{JP}r$, so recalling $\mathcal{A}(x)$ and $\sim\mathcal{A}(y)$, we let $\mathcal{B}$ be $\mathcal{A}$ yielding $x \in r$, $\sim(y \in r)$, which yields $x \neq y$. This contradiction completes the proof.

Comprehension: $(Ey)(z)(z \in y \equiv . z\mathbf{JP}y \wedge \mathcal{B}(z))$

Proof: $(x)(Ey)(z)(x\mathbf{JP}y \wedge (z \in y \equiv . z\mathbf{JP}y \wedge \mathcal{B}(z)))$. Therefore $(Ey)(z)(x\mathbf{JP}y \wedge (z \in y \equiv . z\mathbf{JP}y \wedge \mathcal{B}(z)))$, so that $(z)(x\mathbf{JP}t \wedge (z \in t \equiv . z\mathbf{JP}t \wedge \mathcal{B}(z)))$. From this we get $(x\mathbf{JP}t \wedge (z \in t \equiv . z\mathbf{JP}t \wedge \mathcal{B}(z)))$. Hence $z \in t \equiv . z\mathbf{JP}t \wedge \mathcal{B}(z)$, yielding $(Ey)(z)(z \in y \equiv . z\mathbf{JP}y \wedge \mathcal{B}(z))$.

By choosing $\mathscr{B}$ appropriately we can prove:

Pairing: $(x)\,(y)\,(\mathrm{E}z)\,(w)\,(w \in z \equiv .\, w\mathbf{JP}z \wedge S^3(x,y,w) \wedge (w = x \vee w = y))$
Union: $(x)\,(y)\,(\mathrm{E}z)\,(w)\,(w \in z \equiv .\, w\mathbf{JP}z \wedge S^2(x,y) \wedge (w \in x \vee w \in y))$
Power Set: $(x)\,(\mathrm{E}y)\,(z)\,(z \in y \equiv .\, z\mathbf{JP}y \wedge S^2(x,z) \wedge (w)\,(w \in z \supset w \in x))$
Replacement: $(x)\,(y)\,(z)\,(S^3(x,y,z) \wedge x \in w \wedge \mathscr{F}(x,y) \wedge \mathscr{F}(x,z) \supset y = z) \supset (\mathrm{E}v)$ $(y)\,(y \in v \equiv y\mathbf{JP}v \wedge (\mathrm{E}x)\,(x \in w \wedge \mathscr{F}(x,y))$.
Intersection: $(x)\,(y)\,(\mathrm{E}z)\,(w)\,(w \in z \equiv w\mathbf{JP}z \wedge (w \in x \wedge w \in y))$
Empty Sets: $(\mathrm{E}y)\,(x)\,(x \in y \equiv x\mathbf{JP}y \wedge x \neq x)$
Universal Sets: $(\mathrm{E}y)\,(x)\,(x \in y \equiv x\mathbf{JP}y \wedge x = x)$
Complements: $(z)\,(\mathrm{E}y)\,(x)\,(x \in y \equiv x\mathbf{JP}y \wedge x\mathbf{JP}z \wedge \sim(x \in z))$
Aussonderung: $(z)\,(\mathrm{E}y)\,(x)\,(x \in y \equiv .\, x\mathbf{JP}y \wedge x \in z \wedge \mathscr{B}(x))$
Sum: $(z)\,(\mathrm{E}y)\,(x)\,(x \in y \equiv .\, x\mathbf{JP}y \wedge (\mathrm{E}w)\,(x \in w \wedge w \in z))$

By a *standard model* for a type theory we mean a model[1] in which every element is on some level, and furthermore, that every level is either level zero (a level containing elements, z, such that $(x) \sim x\mathbf{JP}z$) or else is a finite successor of level zero. (That is, a level such that given x_0 on that level there exists a finite collection of elements, $x_1, \ldots, x_n$, such that $x_1\mathbf{JP}x_0, x_2\mathbf{JP}x_1, \ldots, x_n\mathbf{JP}x_{n-1}, (z) \sim z\mathbf{JP}x_n$). Let us express the statement "x is on the n^{th} level" by $S_n(x)$.

Theorem: *Every consistent type theory*, **T**, *admits non-standard models*.

Proof: Using a procedure due to Skolem[2] we add to **T** a new individual constant, θ, and the list of axioms: $\mathscr{A}_0$: $\sim S_0(\theta)$, $\mathscr{A}_1$: $\sim S_1(\theta), \ldots, \mathscr{A}_n$: $\sim S_n(\theta), \ldots$. When we add θ together with any finite subset of $\{\mathscr{A}_i\}_{i=1}^{\infty}$ to **T** we obtain a consistent system. Therefore if we add θ and $\{\mathscr{A}_i\}_{i=1}^{\infty}$ to **T** we have a consistent system[3], **T'**. Any model for **T'** is a model for **T**, and is clearly non-standard.

Theorem: *Any model for* **JP** *is a model for* $\mathbf{T}_n$.

The proof is a straightforward derivation of the axioms of $\mathbf{T}_n$ from the axioms of **JP** with appropriate changes in notation.

Lemma: $\mathbf{T}_n x$ *and* $\mathbf{T}_m x$ *implies* $n = m$.
Corollary 1: $x\mathbf{JP}y \wedge x\mathbf{JP}z \supset S^2(x,y)$.
Corollary 2: $x\mathbf{JP}y \wedge z\mathbf{JP}y \supset S^2(x,z)$.

Theorem: *Any standard model for* **JP** *is a standard model for* $\mathbf{T}_n$.

Proof: $\mathbf{T}_n \Rightarrow$ *All-Some* (**JP**), $\mathbf{T}_n \Rightarrow$ *level 0*, $\mathbf{T}_n \Rightarrow$ *Level*, $\mathbf{T}_n \Rightarrow$ *Comprehension*, $\mathbf{T}_n \Rightarrow$ *E-1*, and $\mathbf{T}_n \Rightarrow$ *E-2* require only straight forward syntactical proofs. We shall now show that *Stratification* (**JP**) holds in any standard model for $\mathbf{T}_n$. Suppose $x \in y$. Since x is on some level say $\mathbf{T}_n$, we have $\mathbf{T}_{n+1}y$. To show $x\mathbf{JP}y$ we show $(\mathrm{E}w)\,(x \in w \wedge (\mathrm{E}z)\,(y \in z \wedge w \in z))$. Letting $\mathscr{B}(v)$ from *Comprehension* ($\mathbf{T}_n$) say $v = y$, $(\mathrm{E}u)\,(\mathbf{T}_{n+2}(u) \wedge (v)\,(v \in u \equiv \mathbf{T}_{n+1}(v) \wedge v = y))$. Letting u be z, $\mathbf{T}_{n+2}(z) \wedge (v)\,(v \in z \equiv \mathbf{T}_{n+1}(v) \wedge v = y)$, and we have $y \in z$. But $x \in y$, so that $(\mathrm{E}w)\,(x \in w \wedge (\mathrm{E}z)\,(y\,\mathrm{E}\,z \wedge w \in z))$, and hence $x\mathbf{JP}y$.

NOTES

1. For the definition of model as well as a deeper meaning of standard model see [2].
2. See reference [6].
3. Reference [3] pp. 424-425.

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