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A NOTE ON THESES OF THE FIRST-ORDER FUNCTIONAL CALCULUS

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In paper [4] I have presented some generalization of the usual definition of satisfiability and as a conclusion a possibility of approximation of the first-order functional calculus by many valued propositional calculi, see [3].

In the following I am describing another way of obtaining such conclusions which I proved in 1957/8.

We are using the notation given in [2] and in particular:

(*01*) variables: (1') free: $x_1, \ldots$ (simply x); (2') apparent: $a_1, \ldots$ (simply a)

(*02*) relation signs: $f_1^c, \ldots, f_q^m$,

(*03*) logical constants: ', +, Π,

(*04*) $w(E)$—the number of free ($p(E)$—apparent) variables occurring in the expression E,

(*05*) $\{i_m\}$—the sequence $i_1, \ldots, i_m$,

(*06*) $\{i_{w(E)}\}$ or $\{j_{w(E)}\}$, or $\{l_{w(E)}\}$—different indices of all free variable occurring in E,

(*07*) $n(E) = \max\{w(E) + p(E), \{i_{w(E)}\}\}$,

(*08*) S_E—the set of all symbols occurring in E,

(*09*) $E(u/z)$—the expression resulting from E by the substitution of u for each z in E with knowing conditions,

(*010*) Skt—the set of all Skolem normal forms $\Sigma a_1 \ldots \Sigma a_i \Pi a_{i+1} \ldots \Pi a_k F$, where F is quantifier, and free variable, free and $\Sigma a_j G = (\Pi a_j G')$, $j = 1, \ldots i$,

(*011*) $C(E)$—the set of significant parts of E: $H \varepsilon C(E)$.≡. $h = E$ or there exist F, G and G_1 such that $F \varepsilon C(E)$ and: $(\exists i)$ $\{H = G(x_i/a)\} \wedge \{(F = \Pi a G) \vee (F = \Sigma a G)\} \vee \{(H = G)\} \vee (H = G_1)\} \wedge (F = G + G_1)$

(*012*) $\mathbf{M}, \mathbf{M}_1, \ldots$—models; $T, T_1, \ldots$—tables of given rank; $Q, Q_1, \ldots$—non-empty sets of tables of the same rank.

It is known that if E is normal form or in an alternation of such forms,

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then E is a thesis if and only if it may be obtained from theses of the propositional calculus by means of the following proof rules:

(11) *If $(F + G) + H$ is a thesis, then $(F + H) + G$ is a thesis.*
(12) *If $(F + G) + G$ is a thesis, then $F + G$ is a thesis.*
(13) *If $(F + E) + G$ is a thesis, $x \bar{\varepsilon} S_{F+E}$ and F is a quantifierless formula, then $(F + E + \Pi a G(a/x)$ is a thesis.*
(14) *If $F + G(x_t/a)$ is a thesis and $\Sigma a G \varepsilon C(F)$ then $F + \Sigma a G$ is a thesis.*

In the above rules F may not occur, see [1], Of course:

L.1. If the length of a formal proof of the formula E is k, then the length of some formal proof of $E(x/x_i)$ also is k.

The sequence $<D, F_1^c, \ldots, F_q^m>$ denotes a model, i.e. that the domain D is an arbitrary non-empty set and $F_1^c, \ldots, F_q^m$ is an arbitrary finite sequence of relations on D such that F_i^j is a j-ary relation, $i = 1, \ldots, q$ and $j = c, \ldots, m$. A table of the rank k is a model whose domain has exactly k elements which are numbers $\leq k$.

For each model $\mathsf{M} = <D, F_1^c, \ldots, F_q^m>$ by $\mathsf{M}/s_1, \ldots, s_k/$—or briefly $\mathsf{M}/\{s_k\}$—we shall denote a table $<D_k, \phi_1^c, \ldots, \phi_q^m>$ of the rank k such that for each $r_1, \ldots, r_j \leq k$:

$$\phi_i^j(r_1, \ldots, r_j) \,.\!\equiv\!.\, F_i^j(r_1, \ldots, r_j),\ i = 1, \ldots, q \text{ and } j = 1, \ldots, m$$

Therefore $\mathsf{M}/\{s_k\} = <D_k, \phi_1^c, \ldots, \phi_q^m>$; if s_k is empty then it holds for all models. $\mathsf{M}/\{s_k\}$ is a submodel of M in the sense of homomorphism.

Of course:

L.2. $\mathsf{M}/\{s_k\}/\{j_m\} = \mathsf{M}/\{s_{j_m}\}$.

D.0. $T \varepsilon \mathsf{M}[k] \,.\!\equiv\!.\, (\exists \{s_k\}) \{T = \mathsf{M}/\{s_k\}\}$.

$\mathsf{M}[k]$ is the set of all $\mathsf{M}/\{s_k\}$. We assume:

1. $\mathsf{M}\{E\} = 0$ i.e. E' is true in the model M.

2. $\mathsf{M}(E\{s_k\}) = 0$ i.e. $\{s_k\}$ are elements of the domain of M, x_i are names s_i, $i = 1, \ldots, k$ and $s_1, \ldots, s_k$ do not satisfy E in the model M.

It is known:

T.1. A formula is a thesis if and only if it is true.

D.1. $R(k, Q, T_1, T_2, \{i_t\}, i) \,.\!\equiv\!.$ *(Q is non-empty set of talbes of the rank k) $\wedge$ $(T_1, T_2 \varepsilon Q) \wedge (T_1/\{i_t\} = T_2/\{i_t\}) \wedge$ (if $\{i_t\}$, i are different natural numbers $\leq k$, then for each j, if $\{i_t\}$, j are different natural numbers $\leq k$, then there exists $T_3 \varepsilon Q$ such that $T_3/1, \ldots, j-1, j+1, \ldots, k/ = T_1/1, \ldots, j-1, j+1, \ldots, k/$ and $T_3/\{i_t\}, j/ = T_2/\{i_t\}, i/$).*

We note that T_3 in *D.1.* is a common extension of T_1 and T_2.

For an arbitrary non-empty set Q of tables of the rank k, for an arbitrary table $T = <D_k, F_1^c, \ldots, F_q^m> \varepsilon D$ and for an arbitrary formula E

whose indices of free variables occurring in it are $\leq k$ we introduce the following inductive definition of the functional V:

(*1d*) $V\{k, Q, T, f_r^j(x_{r_1}, \ldots, x_{r_m})\} = 1 .\equiv. F_r^j(r_1, \ldots, r_m)$,
(*2d*) $V\{k, Q, T, F'\} = 1 .\equiv. \sim V\{k, Q, T, F\} = 1 .\equiv. V\{k, Q, T, F\} = 0$,
(*3d*) $V\{k, Q, T, F + G\} = 1 .\equiv. V\{k, Q, T, F\} = 1 \vee V\{k, Q, T, G\} = 1$,
(*4d*) $V\{k, Q, T, \Pi aF\} = 1 .\equiv. (i)\ (T_1)\ \{(i \leq k) \wedge R(k, Q, T, T_1\ \{i_{w(F)}\}, i) \rightarrow V\{k, Q, T_1, F(x_i/a)\} = 1\}$.

D.2. $F \varepsilon P(k, Q) .\equiv. (T)\ \{(T \varepsilon Q) \rightarrow V\{k, Q, T, F\} = 1\}$.
D.3. $N(k, Q, E) .\equiv. (T_1)\ (T_2)\ \{(T_1, T_2 \varepsilon Q) \wedge (T_1/\{i_{w(E)}\} = T_2/\{i_{w(E)}\}) \wedge V\{k, Q, T_1, E\} = 1 \rightarrow V\{k, Q, T_2, E\} = 1\}$.
D.4. $F \varepsilon P[F, k] .\equiv. (Q)\ (N(k, Q, E) \rightarrow \{F \varepsilon P(k, Q)\})$.
D.5. $F \varepsilon P|E| .\equiv. (\exists k)\ (\{k \geq n(F)\} \wedge \{F \varepsilon P[E, k]\})$.
D.7. $E \varepsilon P .\equiv. E \varepsilon P|E|$.

We may read:

1. $V\{k, Q, T, E\} = 1$ as: T satisfies E relative to Q[1].
2. $E \varepsilon P(k, Q)$ as: E is true relative to Q.
3. $N(k, Q, E)$ is an invariant relation.
4. $E \varepsilon P$ as: E is P-true.

Of course:

(*3d'*) $V\{k, Q, T, F + G\} = 0 .\equiv. V\{k, Q, T, F\} = 0 \wedge V\{k, Q, T, G\} = 0$,
(*4d'*) $V\{k, Q, T, \Pi aF\} = 0 .\equiv. (\exists i)\ (\exists T_1)\ \{(i \leq k) \wedge R(k, Q, T, T_1\ \{i_{w(F)}\}, i) \wedge V\{k, Q, T_1, F(x_i/a)\} = 0\}$,
(*5d'*) $V\{k, Q, T, \Sigma aF\} = 0 .\equiv. (i)\ (T_1)\ \{(i \leq k) \wedge R(k, Q, T, T_1, \{i_{w(F)}\}, i) \rightarrow V\{k, Q, T_1, F(x_i/a)\} = 0\}$.

T.2. If $E \varepsilon Skt$, $F \varepsilon C(E)$, $n(E) \leq k$, $\mathsf{M}\{E\} = 0$, $Q = \mathsf{M}[k]$, *then*:

(*1*) *If* $\mathsf{M}/\{s_{i_{w(F)}}\} = T/\{i_{w(F)}\}$, $T \varepsilon Q$ *and* $\mathsf{M}(F\{s_{i_{w(F)}}\}) = 0$, *then* $V\{k, Q, T, F\} = 0$.
(*2*) $E' \varepsilon P[k, Q]$ *and* $E \bar{\varepsilon} P$.

Proof: First of all we note that (*2*) follows from (*1*). We shall prove (*1*) by induction on the number of quantifiers occurring in F. If $F \varepsilon C(E)$ and F is a quantifierless formula, then (*1*) holds. It remains to verify that if (*1*) holds for $F(x_i/a)\ \varepsilon G(E)$, then it also holds for formulas belonging to $C(E)$ of the form:

(*1'*) ΠaF

and

(*2'*) ΣaF.

In the case (*1'*) by virtue of the definition of satisfiability, of the assumption *L.2.* and (*4d'*) we have:

If $\mathsf{M}(\Pi aF\{s_{i_{w(F)}}\}) = 0$, then $(\exists i)\ (\exists s_i)\ \{(x_i \bar{\varepsilon}\ S_F) \wedge (i \leq k) \wedge \mathsf{M}(F(x_i/a)\ \{s_{i_{w(F)}}\}, s_i) = 0\}$; hence $(\exists i)\ (\exists s_i)\ (\exists T_1)\ \{(x_i \bar{\varepsilon} S_F) \wedge (i \leq k) \wedge (\mathsf{M}/\{s_{i_{w(F)}}\}, s_i/ = T_1/\{i_{w(F)}\}, i/) \wedge R(k, Q, T, T_1, i_{w(F)}, i) \wedge \mathsf{M}(F(x_i/a)\ \{s_{w(F)}\}, s_i) = 0\}$[2]; hence

$(\exists i)(\exists T_1)\{(x_i \bar{\varepsilon} S_F) \wedge (i \leq k) \wedge R(k, Q, T, T_1, \{i_{w(F)}\}, i) \wedge V\{k, Q, T_1, F(x_i/a)\} = 0\}$ and therefore $V\{k, Q, T, \Pi aF\} = 0$.

In the case (*2'*) by virtue of $\Sigma aF \varepsilon C(F)$, $E \varepsilon Skt$, of the satisfiability definition, $\mathbf{M}\{E\} = 0$ and of the assumption we obtain that for an arbitrary $i \leq k$ and for each $T_1 \varepsilon Q$ we have $V\{k, Q, T_1, F(x_i/a)\} = 0$ and so by (*5d'*) for each $T \varepsilon Q$: $V\{k, Q, T, \Sigma aF\} = 0$; q.e.d.

T.3. If $E \varepsilon Skt$, $E = \Sigma a_1 \ldots \Sigma a_i \Pi a_{i+1} F$ *and E is a thesis, then* $E \varepsilon P$.

Proof: Let E , . . . , E_n be a formal proof of the formula E by rules (*11*)-(*14*). We shall prove by induction on n that: (*) $E_i \varepsilon P|E|$, $i = 1, \ldots, n$. If E_i is a thesis of the propositional calculus, then it is obviously that (*) holds; therefore (*) holds for $i = 1$. Let (*) hold for $i < r$; we shall prove it for r. In view of the rules (*11*)-(*14*) it suffices to verify:

(*1'*) If $(F + E) + G \varepsilon P|E|$, $x_t \bar{\varepsilon} S_{F+E}$ and F, G are quantifierless formulas, then $F + E + \Pi aG(a/x_t) \varepsilon P|E|$.

and

(*2'*) If $F + G(x/a) \varepsilon P|E|$, $\Sigma aG \varepsilon C(F)$, then $F + \Sigma aG \varepsilon P|E|$.

(*1'*): Let $x_t \bar{\varepsilon} S_{F+E}$, $x_t \varepsilon S_G$, $k \geq n(F + E + G) \geq n(F + E + \Pi aG(a/x_t))$, F, G are quantifierless formulas, $N(k, Q, E)$ and $V\{k, Q, T, F + E + \Pi aG(a/x_t)\} = 0$. Hence by (*3d'*): $V\{k, Q, T, F\} = 0$, $V\{k, Q, T, E\} = 0$ and $V\{k, Q, T, \Pi aG(a/x_t)\} = 0$; therefore in view of (*4d'*) there exist $i \leq k$ and $T_1 \varepsilon Q$ such that $R(k, Q, T, T_1, \{i_{w(G(a/x_t))}\}, i$ and $V\{k, Q, T_1, G(x_i/a)\} = 0$.

We consider two cases:

(1^0) $i = i_j$, for some j, $1 \leq j \leq w(G(a/x_t))$,
(2^0) i is different from $i_1, \ldots, i_{w(G(a/x_t))}$.

The case (1^0): For the shortest writing we assume $i = i_1$. By virtue of *D.1.* we have $T/\{i_{w(G(a/x_t))}\} = T_1/\{i_{w(G(a/x_t))}\}$. Because $G(x_{i_1}/x_t)$ is a quantifierless formula, then in view of the above $V\{k, Q, T, G(x_{i_1}/x_t)\} = 0$ and by (*3d'*): $V\{k, Q, T, (F + E) + G(x_{i_1}/x_t)\} = 0$. Because $(F + E) + G(x_{i_1}/x_t)$ is a substitution of some formula occurring in the formal proof for which (*) holds, therefore in view of *L.1.* and the induction hypothesis we have a contradiction with the assumption of (*1'*).

The case (2^0): In view of the asuumption and *D.1.* there exists $T_3 \varepsilon Q$ such that $T_3/1, \ldots, t-1, t+1, \ldots, k/ = T/1, \ldots, t-1, t+1, \ldots, k/$ and $T_3/\{i_{w(G(a/x_t))}\}, t/ = T_1/\{i_{w(G(a/x_t))}\}, i/$. Because $N(k, Q, E)$, $x_t \bar{\varepsilon} S_{F+E}$ and F, G are quantifierless formulas and from the above: $T_3/\{j_{w(E)}\} = T/\{j_{w(E)}\}$, $T/\{l_{w(F)}\} = T_3/\{l_{w(F)}\}$ and $T_3/\{i_{w(G(x_{i_1}/x_t))}, t/ = T/\{i_{w(G(x_{i_1}/x_t))}, i/$, therefore $V\{k, Q, T_3, E\} = 0$, $V\{k, Q, T_3, F\} = 0$ and $V\{k, Q, T_3, G\} = 0$. Hence by (*3d'*) $V\{k, Q, T_3, (F + E) + G\} = 0$ which is inconsistent with the assumption of (*1'*).

(*2'*): Let $x_t \varepsilon S_{G(x_t/a)}$, $\Sigma aG \varepsilon C(F)$, $k \geq n(F + G(x_t/a)) \geq n(F + \Sigma aG)$, $N(k, Q, E)$,

$T \varepsilon Q$ and $V\{k, Q, T, F + \Sigma aG\} = 0$; hence and in view of (*3d'*) we have $V\{k, Q, T, F\} = 0$ and $V\{k, Q, T, \Sigma aG\} = 0$. We note that if $\{i_{w(G)}\}$, t, and $\{i_{w(G)}\}$, j are sequences of different natural numbers $\leq k$, then assuming $T_3 = T/1, \ldots, j-1, t, j+1, \ldots, k/$, in view of *L.2.* and *D.1.* we obtain $R(k, Q, T, T, \{i_{w(G)}\}, t)$. Therefore by virtue of (*5d'*) we have $V\{k, Q, T, G\,x_t/a)\} = 0$; hence by (*3d'*) $V\{k, Q, T, F + G(x_t/a)\} = 0$ which is inconsistent with the assumption of (*2'*); q.e.d.

T.4. If $E \varepsilon Skt$, $E = \Sigma a_1 \ldots \Sigma a_{i-1} \Pi a_i F$, *then E is a thesis if and only if* $E \varepsilon P$.

T.4. is a simple conclusion from *T.1-3.* Of course, *T.4.* remains true, if we replace E by an alternation of formulas considered in *T.4.* Because it is easy to show that the above class of theses is equivalent with the class of all theses, therefore *T.4.* gives a characterization of theses of the first-order functional calculus, see [1]. We note that from the proof of *T.4.* it follows that be suitable formulation of *D.3.* and *D.4.* we shall obtain *T.4.* for normal forms.

T.4. gives a certain proof that the Kleene-Mostowski class of theses is P_1^1 and simultaneously a simple proof that it is possible to approximate the first-order functional calculus by many-valued propositional calculi; the Boolean many-valued propositional calculi are determined by the product of tables belonging to Q, see (*1d*)-(*4d*) and *D.1.-6.*

NOTES

1. If Q has one element, then V is the usual satisfiability function. Therefore the above definitions are certain generalizations of the usual satisfiability definition.
2. If $T = M/\{x_k\}$, where $z_{i_1} = s_{i_1}, \ldots, z_{i_{w(F)}} = s_{i_{w(F)}}$, then $T_1 = M/\{u_k\}$, where $u_{i_1} = s_{i_1} = z_{i_1}, \ldots, u_{i_{w(F)}} = s_{i_{w(F)}} = z_{i_{w(F)}}$, $u_i = s_i$, $u_1 = s_1$ for others 1, and if $\{i_{w(F)}\}$, i are different natural numbers $\leq k$, then $T_3 = M/\{v_k\}$, where $v_1 = z_1, \ldots, v_{j-1} = z_{j-1}$, $v_j = s_i$, $v_{j+1} = z_{j+1}, \ldots, v_k = z_k$.

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