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THE CHURCH ROSSER THEOREM FOR STRONG REDUCTION IN COMBINATORY LOGIC

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The Church Rosser theorem concerns a property relating to certain preordering relations [2a]. Originally it was stated for lambda conversions in a paper by Church and Rosser [1].

To define the property let Γ be a preordering and = be its symmetric closure. The property in question states

(C R). *If $M = N$, then there is an L such that $M \; \Gamma \; L$ and $N \; \Gamma \; L$.*

In this paper we give a proof that strong reduction (as modified by the author in a previous paper [3]) has the property (**C R**). For strong reduction, the symmetric closure is simply combinatory logic with equality [2b]. The following results were proved in [3] and [5] and will be used here.

Lemma 1. If $X = [x]\mathfrak{X}$, then $\lambda x.Xx \succ\!\!- \lambda x.\mathfrak{X}$ by Type I steps only. In other words, the contractum of a Type III step may be reversed to the original redex by a single Type II step followed by Type I steps.

Lemma 2. The contraction of a Type II redex P may be reversed provided there are no intervening steps interior to the contractum of P.

Lemma 3. (Theorem 2.II of [5]) *If there is a standard reduction from M_0 to M_n and if there is a single step of Type I or III from M_0 to N_0, then there is an N_n such that there is a standard reduction $N_0 \succ\!\!- N_n$ and M_n.*

Lemma 4. (Lemma 5 of [5]) *If there is a reduction from M_0N_0 beginning with a Type II step yielding $(\lambda x.M_1)N_0$ and continuing to $(\lambda x.M_m)N_n$, then there is a reduction from M_0N_0 to $[N_n/x]M_m$ (where $[N_n/x]M_m$ means the substitution of N_n for x in M_m) in exactly the same number of steps.*

Lemma 5. (Theorem 3 of [5]) *If there is a strong reduction from X to Y where neither X nor Y contain lambda expressions, then there is a Z such that there is a standard reduction from X to Z and $Y \succ\!\!- Z$.*

Lemma 6. (Corollary A of [5]) *If there is a strong reduction from M to N, then there is a Z such that there is a reduction consisting of zero or*

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more Type III steps from M to X and a standard reduction from X to Z and $N \succ Z$.

THE CHURCH ROSSER THEOREM

Theorem 1. *If* $M = N$, *then there is an* L *such that* $N \succ L$ *and* $M \succ L$.

The proof will be in the form of an induction with Theorem 2 providing the induction step.

Theorem 2. *If* $M_0 \succ N_0$ *in a single step and* $M_0 \succ M_m$, *then there is an* N_n *such that* $M_m \succ N_n$ *and* $N_0 \succ N_n$.

Proof of Theorem 1: If $M = N$, then there is a finite sequence of reductions and expansions (converses of reductions) beginning with M and ending with N. These reductions and expansions may be in any order. The proof is by induction on the number of contractions following the first expansion. If there are no contractions following the first expansion, then the original proof of equality is already in the desired form.

If there are contractions following the first expansion, let the stage at which the first expansion sequence begins be M_m and let the end of this sequence be M_0. Following this expansion M_0 will be contracted to N_0. Now apply Theorem 2 to get N_n such that the first expansion sequence begins with N_n and ends with N_0. We have reduced the induction index by one and the induction can be completed.

Proof of Theorem 2: First replace the reduction from M_0 to M_n by a reduction of the type derived in Lemma 6 from M_0 to X by a sequence of Type III steps followed by a standard reduction from X to Y where $M_n \succ Z$.

If the reduction from M_0 to N_0 is by a Type III step, this redex is contracted in the reduction from M_0 to X. Hence by at most a reordering of these steps we may reduce this one first and have $N_0 \succ X \succ M_m$.

If the reduction from M_0 to N_0 is by a Type II step, apply Lemma 2 to reverse this step getting $N_0 \succ M_0 \succ M_n$.

Suppose that M_0 reduces to N_0 by the contraction of a single Type I redex P with initial combinator p. Let the contractum of P be Q. If P is not contained in any Type III redexes, then it has a single residual in X which is of the same type as P. Let Y be the result of reducing all Type III redexes in N_0. In this case any Type III redexes overlapping P will be contained in arguments of p and will occur in Q with at most a change of multiplicity. If Y is the result of reducing all Type III redexes in N_0, the residual of Q in Y will be Q with Type III redexes in the arguments contracted. This is precisely the same as if the residual of P were contracted in X, a single step reduction. Now apply Lemma 3 to get Z.

Finally we consider the case in which a Type I redex is contained in one or more Type III redexes. First we define the residual of P for this situation. If P does not contain the indeterminate x removed by the Type III step, then it will be a subcomponent of a component of the form $[x]R$ or else of the form $[x]Rx$. In either case the contractum contains a subcomponent congruent to P and this will be the residual. If P contains x, the residual of P is $[x]P$. This has an initial combinator p_1. If a residual is contained in other Type III redexes we simply define the residual of a residual to be a residual. The initial combinator p of the redex P may be in the interior of its residual, but it can be identified by going through the steps of the reduction in any case. If in succeeding steps of the reduction some of the combinators introduced by the Type III steps are reduced, but p is not reduced, the contractum will still be a residual of P.

To complete the proof we let Y be the result of contracting all Type III redexes in N_0. Then apply Lemma 1 to reduce the residual of Q to Q. Call the resulting stage N_1. This is exactly like X with Q replacing the residual of P since Y was exactly like X with the residual of Q replacing the residual of P. Follow N_1 with the reduction from X to M_n except that each residual of P will be replaced by Q. Let the end of this reduction be N_k'. We modify this reduction so that we still have $N_0 \succ N_n$ and in addition we have $M_m \succ N_n$.

As an induction hypothesis we assume that M_{i-1} reduces to N_{i-1} by reducing the residuals of P to P and then reducing P to Q. From this we show that M_i reduces to N_i of the modified reduction. We already observed that the induction hypothesis holds for $X(=M_1)$ and N_1. We break up the consideration into five cases.

Case 1. If the step from M_{i-1} to M_i is disjoint from any residual of P, then the reduction from N_{i-1} to N_i is unchanged.

Case 2. If the step from M_{i-1} to M_i contains one or more residuals of P in an argument place, then there may be a change of multiplicity of residuals of P. In the reduction from N_{i-1} to N_i Q appears in the place of residuals of P. Hence the same change of multiplicity will be made for Q. Therefore reducing the residuals of P in M_i will give N_i.

Case 3. The step from M_{i-1} to M_i is a (partial) reduction of a residual of P. N_{i-1} and N_i are identical in this case. Here we compare the reduction stage from M_{i-1} to M_i with the corresponding step of the reduction of the residual of P in M_1 to P. Call the redex reduced at this stage P_i and let its initial combinator be p_i.

Case 3a. The combinator p_i is not a descendent of p. Then p_i is a combinator introduced into the residual of P by a Type III step. If there are more than one residual of P at this point we look at the redexes in each of them headed by p_i. If the redex headed by p_i in the reduction from the residual of P in M_1 to P has at least as many arguments as in any of the reductions following M_{i-1}, then we make no changes in the reduction N reductions. If the step from M_{i-1} to M_i has the same number of arguments as the corresponding step from the residual of P to P, then this step is one of the steps in the reduction of M_{i-1} to $N_{i-1} = N_i$. If the step from M_{i-1} to M_i uses fewer arguments than in P_i, it is Type II and we can reverse the step by a Type II step to M_{i-1} and this reduces as before to $N_{i-1} = N_i$.

If the step from M_{i-1} to M_i or one of the other instances of p_i in another residual of P uses more arguments than P we make the transformation of Lemma 4 to all residuals of P from the point where the indeterminate is introduced on. Since P and Q contain the same indeterminates, the indeterminate is also introduced between Y and N_1. Make the same transformation on the N reductions from the point where this indeterminate is introduced. The residuals of P in the modified reduction reduce to the residuals of Q in the modified N reduction, since the same changes have been made in both reductions up to M_{i-1} and N_{i-1}. Now the reduction of P_1 of the index is just a step of the reduction from P_i to Q as modified. At subsequent stages of the reduction the modified reduction will serve as before without further changes.

Case 3b. P_i has the same head as P. If P_i is the same as P, then this is simply one step of the reduction of the residuals of P to Q. If P_i is not P it is a Type II step (P is Type I) and a Type III step will reverse this step and the reduction from M_{i-1} gives the remaining reduction.

Case 4. The step from M_{i-1} to M_i takes place within an argument of P. Arguments of P occur in Q unchanged for multiplicity. Hence if we make the same reductions in the arguments of Q corresponding to the particular residual of P in which the reduction is taking place, we still have M_i reducing to N_i.

Case 5. If the step from M_{i-1} to M_i is a Type III step, it necessarily occurs after all Type I and II steps, and applying Lemma 1 gives $M_i \succ M_{i-1} \succ N_{i-1} = N_i$.

This completes the induction. We can now drop duplications in the N reductions and we have the theorem proved.

BIBLIOGRAPHY AND NOTES

[1] Church, A., and J. B. Rosser, "Some Properties of Conversion." *Transactions of the American Mathematical Society*, 1936, 472-482.

[2] Curry, H. B., and Robert Feys, *Combinatory Logic Vol. 1.* (Studies in Logic and Foundations of Mathematics Series) North Holland Publishing Co., Amsterdam, 1958.

[2a] Curry and Feys use the term quasi-ordering instead of pre-ordering.

[2b] See section 6f, pp. 218 ff.

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