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ON KLEENE'S RECURSIVE REALIZABILITY AS AN INTERPRETATION FOR INTUITIONISTIC ELEMENTARY NUMBER THEORY

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Kleene (*Introduction to Metamathematics*, p. 501 ff.) has shown that when intuitionistic elementary number theory is interpreted in terms of recursive realizability certain elementary number theoretic statements are classically true but intuitionistically unacceptable; and that their negations are classically false but intuitionistically acceptable. Examples of such statements are (for a suitably chosen predicate A(x)): 1) excluded middle; 2) the least number principle; 3) the double negation and universal closure of (1) and (2). I shall show that a statement classically equivalent to the induction axiom has this same property, and why this is so. I shall then argue that this interpretation of intuitionistic number theory is fundamentally incorrect. And finally I shall suggest another interpretation that renders (1), (2) and (3) intuitionistically acceptable for that predicate A(x).

PART I

The formal system (**Z**) for intuitionistic elementary number theory (**I.M.**, p. 82) differs from the classical (**T**) in just one axiom:

$$\neg\neg A \supset A \quad \text{(classical)}$$
$$\neg A \supset (A \supset B) \quad \text{(intuitionistic)}$$

The induction axiom in both (**Z**) and (**T**) is:

$$(1)\quad (A(0)\,\&\,(x)(A(x) \supset A(x'))) \supset A(x)$$

The interpretation as recursive realizability proceeds as follows: (x is a variable; x is a natural number; **x** is the formal numeral corresponding to x.)

(A) 1. The number e realizes a closed prime formula P (one without free variables and logical symbols) if $e = 0$ and P is recursively true.
If A and B are any closed formulas (without free variables):

2) e realizes A & B if $e = 2^a \cdot 3^b$ where a realizes A and b realizes B.

3) e realizes A ∨ B if $e = 2^0 \cdot 3^a$ where a realizes A, or $e = 2^1 \cdot 3^b$ where b realizes B.

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4) e realizes A $\supset$ B if e is the Gödel number of a partial recursive function φ(x) such that, whenever a realizes A, $\varphi(a)$ realizes B.

5) e realizes $\neg$A if e realizes A $\supset$ 1 = 0.

If x is a variable and A(x) a formula containing only x free:

6) e realizes (Ex)A(x) if $e = 2^t \cdot 3^a$ where a realizes A(**t**).

7) e realizes (x)A(x) if e is the Gödel number of a general recursive function φ(x) such that, for every x, $\varphi(x)$ realizes A(**x**).

(B) A formula containing no free variables is recursively realizable if there exists a number e (as defined in (A)) which realizes it. A formula A($y_1, \ldots, y_m$), with only the distinct variables $y_1, \ldots, y_m$ free, is realizable if there exists a general recursive function φ of m variables such that, for every $y_1, \ldots, y_m$, $\varphi(y_1, \ldots, y_m)$ realizes A($\mathbf{y}_1, \ldots, \mathbf{y}_m$).

Kleene shows (**I.M.**, pp. 504-8) that all formulas provable in (**Z**) are recursively realizable. This is a consistency proof for (**Z**) since "1 = 0" is unrealizable. Since (**T**) can be interpreted in (**Z**) by elementary (general recursive) means, and since consistency of (**T**) cannot be shown by elementary means, this is a non-elementary result. It is non-elementary in its treatment of the induction axiom (1).

Since (1) contains x free, it is realizable (by (B)) if there exists a general recursive function φ(x) such that, for any x, $\varphi(x)$ realizes (A(0) & (x) (A(x) $\supset$ A(x'))) $\supset$ A(**x**). Since the latter is an implication, φ(x) will be the Gödel number of a partial recursive function ψ(x) (by (A), 4) such that, whenever a realizes the antecedent conjunction, $\psi(a)$ realizes A(**x**). Appropriate φ(x) and ψ(x) are defined in the proof.

(1) is also realizable when A(x) symbolizes a predicate recursively *false* for *some* natural number x, i.e. when A(**x**) is unrealizable for some **x**. Since $\varphi(x)$, for that x, realizes (A(0) & (x) (A(x) $\supset$ A(x'))) $\supset$ A(**x**), A(0) & (x) (A(x) $\supset$ A(x')) is unrealizable. (If A $\supset$ B is realizable, and B is unrealizable, A is unrealizable (**I.M.**, p. 511).) Thus ψ(x) is undefined for every a, when φ and ψ are defined as in the proof.

Consider the predicate (Ey)T(x,x,y) which, for any x, is true if and only if x is the enumeration number, i.e. the x^{th} Gödel number in the ordering of Gödel numbers, of a formula provable in (**Z**). (T(x,x,y) is a primitive recursive predicate.) The axiom:

((Ey)T(0,0,y) & (x) ((Ey)T(x,x,y) $\supset$ (Ey)T(x',x',y))) $\supset$ (Ey)T(x,x,y)

is realizable.

If (**Z**) is consistent, there is at least one x such that (Ey)T(x,x,y) is false; i.e. such that T(x,x,t) is recursively false for every t. This implies that the formula (Ey)T(**x,x**,y) is unrealizable, so that (Ey)T(x,x,y) is unrealizable. Conversely, unrealizability of (Ey)T(x,x,y), i.e. the existence of an x such that (Ey)T(**x,x**,y) is unrealizable, implies consistency. Assuming consistency, the antecedent conjunction is unrealizable in the axiom:

((Ey)T(0,0,y) & (x) ((Ey)T(x,x,y) $\supset$ (Ey)T(x',x',y))) $\supset$ (Ey)T(**x,x**,y) for that x, so that the axiom is realized by $\psi(x)$, undefined for every a. Furthermore, since realizability of the antecedent conjunction implies realizability of (Ey)T($\mathbf{x},\mathbf{x}$,y) for every x, i.e. realizability of (Ey)T(x,x,y), the conjunction is unrealizable for all x (assuming consistency). $\psi(x)$, undefined for every a, realizes the axiom for every x.

The antecedent conjunction cannot be *shown* to be unrealizable by general recursive means. For this would require either: a) proof that there exists no t such that T(0,0,t) is realizable, i.e. no t such that T(0,0,t) is recursively true, which implies consistency of (**Z**). Or b) proof that there is some x such that (Ey)T(x,x,y) is realizable while (Ey)T(x',x',y) is not. Again, unrealizability of (Ey)T(x',x',y) implies consistency.

Thus, when A(x) ≡ (Ey)T(x,x,y), (1) is realized, for every x, by $\psi(x)$, completely undefined. But there is no general recursive method of showing *that* the function *is* undefined for every a. To do so is equivalent to demonstrating consistency of (**Z**). Thus the interpretation of (**Z**) in terms of recursive realizability is not general recursive; the reasoning involved in showing that provability (**Z**) implies realizability is non-recursive. For it involves showing not only that the consequent of an axiom (1) is realizable whenever the antecedent is, a recursive result; but also, for the above axiom (1), that the antecedent is unrealizable, a non-recursive result. Information additional to that provided by $\psi(x)$ is needed to prove that the antecedent conjunction is *not* realizable; that $\psi(x)$ is totally undefined. Such information is necessary (and sufficient) to show (Ey)T(x,x,y) unrealizable, and (**Z**) consistent. This is the non-elementary portion of the proof.

PART II

As a corollary to the theorem that provability implies realizability: if $\vdash$(Ey)A($x_1, \ldots, x_n$,y) in (**Z**), then there exists a general recursive function $\varphi(x_1, \ldots, x_n) = y$ such that, for every n-tuple $x_1, \ldots, x_n$, A($\mathbf{x}_1, \ldots, \mathbf{x}_n$,y) is realizable (where $\varphi(x_1, \ldots x_n) = y$). (**I.M.**, pp. 508, 9)

Classically, (1) ≡ (2):

$$(Ez)\,(A(0)\&A(z) \supset A(z')) \supset A(x)).$$

The function that, for every x, provides a z such that A(0)&(A($\mathbf{z}$) ⊃ A($\mathbf{z}'$)) ⊃ A($\mathbf{x}$) is realizable is defined as follows:

$$\varphi(x) = \begin{cases} 0 \text{ if } A(\mathbf{x}) \text{ or if } \neg A(0) \\ \mu z_{z \le x}(A(z)\& \neg A(z')) \text{ otherwise} \end{cases}$$

When A(x) ≡ (Ey)T(x,x,y), $\varphi(x)$ is not general recursive, so that (2), with A(x) ≡ (Ey)T(x,x,y), is unrealizable. Consequently, that formula is not provable in (**Z**). (The classical proof that (1) ⊃ (2) depends upon the non-intuitionistic principle: ((x)A(x) ⊃ B) ⊃ (Ex)(A(x) ⊃ B).)

(2) (A(x) ≡ (Ey)T(x,x,y)) is unrealizable because of the non-existence of a recursive method of *providing* a z, for every x, such that the matrix is realizable. The difficulty occurs when (Ey)T(x,x,y) is false. As in the case

of (1), it is then impossible to produce a z for which the antecedent conjunction is unrealizable by recursive means. Because of the interpretation given to the symbol "$\supset$" in (1), it is not required that such a z be so produced, so that (1) is realizable. But the interpretation given to "(Ey)" in (2) does contain that requirement, so that (2) is unrealizable.

Finally, Kleene shows (**I.M.**, pp. 511,512) that if A is closed and unrealizable, $\neg$A is realizable. By clause 7 the closure of (2):

$$(x)(Ey)(A(0)\&(A(y) \supset A(y')) \supset A(x))\ (A(x) \equiv (Ey)T(x,x,y))$$

is unrealizable, so that its negation is realizable. But the closure is classically true, while its negation is classically false. If the negation is added to (**Z**) as an axiom, as it could be since it is supposedly intuitionistically acceptable, the resulting system (**Z**') has the very curious property of having two axioms that are classically contradictory.

PART III

(**Z**') is meant to formalize the intuitionistic interpretations of the two logical operators: "$\supset$" and "(Ex)". In general "(x)A(x) $\supset$ B" is intuitionistically true if one can construct an effective general method for deducing B from (x)A(x). Regarding (1), the partial recursive function $\psi(x)$ provides such a method for every predicate A. In general "(Ex) (A(x) $\supset$ B)" is true if and only if one can construct a proof of the statement "A(t) $\supset$ B" for some t. By restricting "construct" to general recursive constructions, (2) becomes untrue (unrealizable) when A(x) $\equiv$ (Ey)T(x,x,y).

The difficulty with accepting (**Z**') as a formal system for intuitionistic elementary number theory (or (**Z**") if $\neg$(x) (A(x) $\vee$ $\neg$A(x)), A(x) $\equiv$ (Ey)T(x,x,y), etc. be added to (**Z**')) lies in the fact that the consistency proof for (**Z**) (provability implies realizability) utilizes a non-elementary technique. Since consistency of (**Z**) implies consistency of (**T**), that technique must be as non-elementary as transfinite induction up to ε_0. Such a technique *will* provide an x such that (Ey)T(**x,x**,y) is unrealizable, i.e. an x which "unrealizes" (Ey)T(0,0,y) &(x)(T(x,x,y) $\supset$ T(x',x',y)). For example, the enumeration number of the formula "1 = 0", assuming this to be the first unprovable formula in the ordering by Gödel numbers. That same x will serve as the z' for which the matrix of (2) is realizable for every y if $\varphi(x)$ is altered to read "the enumeration number of 1 = 0 otherwise". Or, $\varphi(x)$ can simply be taken as the constant function, mapping all y onto the enumeration number of "1 = 0".

Since a non-elementary technique is used in the demonstration that provability implies realizability, and since any result regarding intuitionistic acceptability must also be intuitionistically acceptable, (Intuitionists will not accept results which utilize non-effective techniques), the interpretation must be unsatisfactory. For either: 1) transfinite induction to ε_0 is not considered effective, in which case the result is unacceptable; or 2) it is effective, in which case, since it is a technique of elementary number theory, it must be a part of the interpretation of (**Z**). But this interpretation would differ from realizability, since (2), A(x) $\equiv$ (Ey)T(x,x,y), is true in the one but false in the other.

The question whether such non-recursive techniques as transfinite induction to ε_0 are intuitionistically acceptable is indeed interesting. I believe that one is, but I won't argue the point here. If (**Z**) were to be reformulated as some (**Z**$^+$) including that technique or an equivalent (*cf*. the last chapter of Mostowski's *Sentences Undecidable* . . .), then (2), as well as excluded middle and the least number principle (A(x) $\equiv$ (Ey)T(x,x,y)), become derivable. Of course, the problem then reappears on a higher level, for another predicate A(x).

Since (**T**) and (**Z**) are each interpretable in the other (**I.M.**, pp. 492-498) and in a sense equivalent, recursive realizability is intended to pinpoint the difference believed to exist between classical and intuitionistic number theoretic inference. (**T**) and (**Z**'') clearly differ. But, if I am correct, either (**Z**) or some (**Z**$^+$) is the correct formalization of the intuitionistic number theory. And (**Z**$^+$), on the level of the predicate (Ey)T(x,x,y), does not differ from (**T**).

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