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RESULTS REGARDING THE AXIOMATIZATION OF PARTIAL PROPOSITIONAL CALCULI

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In §3, §5, and §6 the problems of determining whether or not partial (partial implicational) propositional calculi may be axiomatized by one, n or fewer, or finitely many axioms are shown to be unsolvable. In §4 the split problem for partial (partial implicational) propositional calculi is shown to be unsolvable. We show further that there is a problem of each of these types of any recursively enumerable degree of unsolvability.*

§0. *Introduction*. The problem of reducing the number of axioms for a calculus seemed to us to be basic. Its study led to the results of §3, §5 and §6. The split problem for partial propositional calculi dealt with in §4 was brought to our attention by Henry Hiz after the other work was completed. He believes the formulation of the split problem to have originated with Łukasiewicz.

In §2 we prove that there is a partial (partial implicational) calculus with unsolvable decision problem. This result is known but the proof given here differs rather radically from the existing proofs. The result is included here because it was necessary to give a complete proof of it in developing the machinery for the proofs of later results. The style of the proof in §2 and in part of §3 is a parallel of Yntema's [11] proof for a less general system. The known unsolvability results for partial (partial implicational) calculi are included in the references listed at the end of this paper.

§1. *Preliminary Definitions and Remarks*. A *partial implicational propositional* calculus is a system having $\supset$, [,] and an infinite list of propositional variables $p_1, q_1, r_1, p_2, q_2, r_2, \ldots$ as primitive symbols. Its well-formed formulas are (1) a propositional variable standing alone, and (2) $[A \supset B]$, where A and B are well-formed formulas. Its axioms are a finite set of tautologies and its two rules of inference are modus ponens and substitution.

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A *partial propositional calculus* is a system having as primitive symbols all of the primitive symbols of a partial implicational propositional calculus and, in addition, the primitive symbol $\sim$. Its well-formed formulas are (1) a propositional variable standing alone, (2) $\sim A$, where A is a well-formed formula, and (3) $[A \supset B]$, where A and B are well-formed formulas. Its axioms are a finite set of tautologies and its two rules of inference are modus ponens and substitution.

A set of tautologies involving only the symbols [,], $\supset$ and the propositional variables can be taken to completely specify either a partial implicational propositional calculus or a partial propositional calculus. In the following discussions all of our calculi are specified by such sets of tautologies and our proofs are intended to be equally valid under either interpretation. For this reason we shall refer simply to the calculus specified by a certain set of tautologies. We shall say that a *calculus P*, specified by a set of tautologies *S*, *is axiomatizable by a set of tautologies* S' if and only if the calculus specified by the set S' has the same set of theorems as does P. We shall henceforth write wff as an abbreviation for well-formed formula.

In the sequel wffs are often abbreviated by the use of the heavy dot, ■, or by the omission of brackets or both. Wherever such abbreviations occur the replacement of brackets is to be done in accordance with the conventions of Church [2]. We shall also use the symbol $\mathbf{S}^{a}_{A} B|$ to denote the result of replacing the propositional variable a by the wff A at each of its occurrences in the wff B.

Wherever we give the argument which involves a given proof from a calculus we shall assume that the given proof is so arranged that all uses of substitution precede all uses of modus ponens and, furthermore, that the substitutions have been made directly in the axioms. There is no loss of generality in making this assumption.

A *semi-Thue system T* shall consist of a finite alphabet Z_T and a finite set of word pairs U_T. The members of U_T are called defining relations.

$$\begin{aligned} Z_T &: z_1, z_2, \ldots, z_n \\ U_T &: U_1 \rightarrow \overline{U}_1, U_2 \rightarrow \overline{U}_2, \ldots, U_m \rightarrow \overline{U}_m \,. \end{aligned}$$

A word is a finite (possible empty) string of symbols of Z_T, with possible repetitions. We shall define $W \vdash_T X$, where W and X are words on Z_T, to be the assertion that there exists a finite sequence of statements $W_1 \vdash_T X_1$, $W_2 \vdash_T X_2, \ldots, W_\ell \vdash_T X_\ell$ such that W_ℓ is W and X_ℓ is X, and such that each statement $W_i \vdash_T X_i$ is justified by one of the following rules.

1. *W_i is W_jY, X_i is X_jY for some j, $1 \leq j < i$ and for some word Y on Z_T.*
2. *W_i is YW_j, X_i is YX_j, for some j, $1 \leq j < i$, and for some word Y on Z_T.*
3. *W_i is X_i.*
4. *W_i is U_j and X_i is $\overline{U}_j$ for some j, $1 \leq j \leq m$.*
5. *W_i is W_j, X_i is X_k, X_j is W_k for some j and k, $1 \leq j < i$; $1 \leq k < i$.*

A less explicit summary of these rules is given as follows:

1. *If* $W \vdash X$, *then* $WY \vdash XY$.
2. *If* $W \vdash X$, *then* $YW \vdash YX$.
3. $W \vdash W$.
4. *If* $W \rightarrow X$, *then* $W \vdash X$.
5. *If* $W \vdash Y$ *and* $Y \vdash X$, *then* $W \vdash X$.

A semi-Thue system T will be called a *standard semi-Thue system* if (1) Z_T is $\{1,b\}$, and (2) no word in a defining relation of T is the empty word.

§2. *The existence of Calculi with Recursively Unsolvable Decision Problems.* We shall establish the following results.

Result 1A. *There exists a partial implicational propositional calculus with a recursively unsolvable decision problem.*

Result 1B. *There exists a partial propositional calculus with a recursively unsolvable decision problem.*

The proof of these results and subsequent results will be dependent upon the following Lemma which is due to Boone [1].

Lemma 1. (Boone). *There is a recursive construction* M^0 *such that the result of applying* M^0 *to any given recursively enumerable set of natural numbers* S *is a standard semi-Thue system* T_S *having the property that the decision problem for* S *is equivalent to the word problem for* T_S.

With Lemma 1 assumed, the proof of Results 1A and 1B are immediate from the following theorem.

Theorem 1. *There is a recursive construction* M^1 *such that the result of applying* M^1 *to any standard semi-Thue system* T *is a calculus* P_T *and a mapping* f_1 *of the non-empty words on* $\{1,b\}$ *onto a recursive subset of the wffs of* P_T. *Furthermore,* f_1 *is one-to-one, and if* W_1 *and* W_2 *are non-empty words on* $\{1,b\}$, *then* $W_1 \vdash_T W_2$ *if and only if* $\vdash_{P_T} f_1(W_1) \supset f_1(W_2)$.

We shall turn now to the task of establishing Theorem 1. Let T be a standard semi-Thue system defined by

$$U_T : U_i \rightarrow \overline{U}_i, \quad i = 1,2, \ldots, m \,.$$

If W is a non-empty word on $\{1,b\}$ then W^* is the wff defined by

$$1^* \text{ is } p_2 \supset [p_2 \supset p_2]$$
$$b^* \text{ is } p_2 \supset [p_2 \supset [p_2 \supset p_2]]$$
$$(X1)^* \text{ is } [X^* \vee 1^*],$$

and

$$(Xb)^* \text{ is } [X^* \vee b^*],$$

where X is an arbitrary non-empty word on $\{1,b\}$ and $[A \vee B]$ is an abbreviation for $[[A \supset B] \supset B]$. If W is a non-empty word on $\{1,b\}$, then $f_1(W)$ is defined to be $W^* \vee h$, where h is an abbreviation for the wff $p_2 \supset [p_2 \supset [p_2 \supset [p_2 \supset p_2]]]$. Now we specify the calculus P_T by the following set of tautologies.

1. $[[[p_1 \vee q_1] \vee r_1] \vee h] \supset [[p_1 \vee [q_1 \vee r_1]] \vee h]$
2. $[[p_1 \vee [q_1 \vee r_1]] \vee h] \supset [[[p_1 \vee q_1] \vee r_1] \vee h]$
3. $[p_1 \vee h] \supset [q_1 \vee h] \supset_{\blacksquare} [[p_1 \vee r_1] \vee h] \supset [[q_1 \vee r_1] \vee h]$
4. $[p_1 \vee h] \supset [q_1 \vee h] \supset_{\blacksquare} [[r_1 \vee p_1] \vee h] \supset [[r_1 \vee q_1] \vee h]$
5. $[p_1 \vee h] \supset [p_1 \vee h]$
6. $f_1(U_i) \supset f_1(\overline{U}_i)$, $i = 1, 2, \ldots, m$
7. $[p_1 \vee h] \supset [q_1 \vee h] \supset_{\blacksquare} [[q_1 \vee h] \supset [r_1 \vee h]] \supset [[p_1 \vee h] \supset [r_1 \vee h]]$.

These wffs are intended to be, in some sense, the logical equivalents of the rules of the semi-Thue system T and it may not be readily apparent that this is the case. Actually, Axioms 1 and 2 have no counterparts in the rules of T, but we account for them by the fact that the letters in a word on $\{1,b\}$ are not grouped as are the variables in a calculus. If one then considers h as no more than some sort of spacer, he readily sees that Axioms 3-7 are rather faithful translations of the rules 1-5 of T.

A wff A of P_T is *semi-regular* if (1) A is 1^* or A is b^*, or (2) A is of the form $A_1 \vee A_2$, where A_1 and A_2 are semi-regular wffs.

A wff A of P_T is *regular* if A is of the form $B \vee h$, where B is semi-regular. One should note that p_2 is the only propositional variable occurring in a regular wff.

If $A \vee h$ is a regular wff of P_T, then $\langle A \vee h \rangle$ is the unique word on $\{1,b\}$ obtained from $A \vee h$ by (1) abbreviating A so that it contains only [,], $\vee$, 1^* and b^*, (2) replacing all occurrences of 1^* by 1 and b^* by b, and (3) removing all occurrences of [,] and $\vee$.

Two regular wffs of P_T, A and B, are *associates* if and only if $\langle A \rangle$ is $\langle B \rangle$.

Lemma 2. *If A and B are associates, then* $\vdash_{P_T} A \supset B$ *and* $\vdash_{P_T} B \supset A$.

The proof is by mathematical induction on the number n of occurrences of 1^* and b^* in A.

If $n = 1$, A is $f_1(1)$ or A is $f_1(B)$. Hence $\langle A \rangle$ is 1 and $\langle B \rangle$ is 1 or $\langle A \rangle$ is b and $\langle B \rangle$ is b. In either event A is B. Since $[p_1 \vee h] \supset [p_1 \vee h]$ is an axiom of P_T it follows by substitution that we have $\vdash_{P_T} A \supset B$ and $\vdash_{P_T} B \supset A$.

If $n > 1$, call the number of occurrences of 1^* and b^* in A the length of A and let $\ell_{(A)}$ be an abbreviation for length of A. Since A and B are associates of $\ell_{(A)} = \ell_{(B)}$. The induction hypothesis is that if C_1 and C_2 are associates such that $\ell_{(C_1)} = \ell_{(C_2)}$ then $\vdash_{P_T} C_1 \supset C_2$.

Since $\ell_{(A)} > 1$ it follows that A is $[A_1 \vee A_2] \vee h$ and B is $[B_1 \vee B_2] \vee h$ for some semi-regular wffs A_1, A_2, B_1 and B_2. There are two cases to consider, either $\ell_{(A_1 \vee h)} = \ell_{(B_1 \vee h)}$ or $\ell_{(A_1 \vee h)} \neq \ell_{(B_1 \vee h)}$.

Assume first that $\ell_{(A_2 \vee h)} = \ell_{(B_2 \vee h)}$. Then it follows that $\ell_{(A_2 \vee h)} = \ell_{(B_2 \vee h)}$, $\langle A_1 \vee h \rangle$ must be the first $\ell_{(A_1 \vee h)}$ letters of $\langle A \rangle$, $\langle B_1 \vee h \rangle$ must be the first $\ell_{(B_1 \vee h)}$ letters of $\langle B\rangle$, and $\langle A_2 \vee h\rangle$ must be the last $\ell_{(A_2 \vee h)}$ letters of $\langle A\rangle$, $\langle B_2 \vee h\rangle$ must be the last $\ell_{(B_2 \vee h)}$ letters of $\langle B\rangle$. From this and the fact that $\langle A\rangle$ and $\langle B\rangle$ are identical we see that $A_1 \vee h$ and $B_1 \vee h$ are associates and $A_2 \vee h$ and $B_2 \vee h$ are associates. We complete the proof for this case as follows.

$\vdash_{P_T}[A_1 \vee h] \supset [B_1 \vee h]$ by hyp. ind.
$\vdash_{P_T}[[A_1 \vee A_2] \vee h] \supset [[B_1 \vee A_2] \vee h]$ by Axiom 3.
$\vdash_{P_T}[A_2 \vee h] \supset [B_2 \vee h]$ by hyp. ind.
$\vdash_{P_T}[[B_1 \vee A_2] \vee h] \supset [[B_1 \vee B_2] \vee h]$ by Axiom 4.
$\vdash_{P_T}[[A_1 \vee A_2] \vee h] \supset [[B_1 \vee B_2] \vee h]$ by Axiom 7.

i.e., $\vdash_{P_T} A \supset B$. Then, by symmetry, we also have $\vdash_{P_T} B \supset A$.

Now assume that $\ell_{(A_1 \vee h)} \neq \ell_{(B_1 \vee h)}$. Since we must prove the implication in both directions there is no loss of generality in assuming that $\ell_{(A_1 \vee h)} = \ell_{(B_1 \vee h)} + k$. Let $[A_{11} \vee A_{12}] \vee h$ be an associate of $A_1 \vee h$ such that $\ell_{(A_{11} \vee h)} = \ell_{(B_1 \vee h)}$ and $\ell_{(A_{12} \vee h)} = k$. Let $[B_{21} \vee B_{22}] \vee h$ be an associate of $B_2 \vee h$ such that $\ell_{(B_{21} \vee h)} = k$ and $\ell_{(B_{22} \vee h)} = \ell_{(A_2 \vee h)}$. Then $\langle A_{11} \vee h\rangle$ is $\langle B_1 \vee h\rangle$, $\langle A_{12} \vee h\rangle$ is $\langle B_{21} \vee h\rangle$ and $\langle A_2 \vee h\rangle$ is $\langle B_{22} \vee h\rangle$. We complete the proof for this case as follows:

$\vdash_{P_T}[[A_1 \vee A_2] \vee h] \supset [[[A_{11} \vee A_{22}] \vee A_2] \vee h]$ by previous case
$\vdash_{P_T}[[A_{11} \vee A_{12}] \vee h] \supset [[B_1 \vee B_{21}] \vee h]$ by previous case
$\vdash_{P_T}[[[A_{11} \vee A_{12}] \vee A_2] \vee h] \supset [[[B_1 \vee B_{21}] \vee A_2] \vee h]$ by Axiom 3
$\vdash_{P_T}[A_2 \vee h] \supset [B_{22} \vee h]$ by hyp. ind.
$\vdash_{P_T}[[[B_1 \vee B_{21}] \vee A_2] \vee h] \supset [[[B_1 \vee B_{21}] \vee B_{22}] \vee h]$ by Axiom 4
$\vdash_{P_T}[[[B_1 \vee B_{21}] \vee B_{22}] \vee h] \supset [[B_1 \vee [B_{21} \vee B_{22}]] \vee h]$ by Axiom 1
$\vdash_{P_T}[[B_1 \vee [B_{21} \vee B_{22}]] \vee h] \supset [[B_1 \vee B_2] \vee h]$ by previous case
$\vdash_{P_T}[[A_1 \vee A_2] \vee h] \supset [[B_1 \vee B_2] \vee h]$ by Axiom 7

i.e., $\vdash_{P_T} A \supset B$.

We also have:

$\vdash_{P_T}[[B_1 \vee B_2] \vee h] \supset [[B_1 \vee [B_{21} \vee B_{22}]] \vee h]$ by previous case
$\vdash_{P_T}[[B_{21} \vee B_{22}] \vee h] \supset [[A_{12} \vee A_2] \vee h]$ by previous case
$\vdash_{P_T}[[B_1 \vee [B_{21} \vee B_{22}]] \vee h] \supset [[B_1 \vee [A_{12} \vee A_2]] \vee h]$ by Axiom 4
$\vdash_{P_T}[B_1 \vee h] \supset [A_{11} \vee h]$ by hyp. ind.
$\vdash_{P_T}[[B_1 \vee [A_{12} \vee A_2]] \vee h] \supset [[A_{11} \vee [A_{12} \vee A_2]] \vee h]$ by Axiom 3
$\vdash_{P_T}[[A_{11} \vee [A_{12} \vee A_2]] \vee h] \supset [[[A_{11} \vee A_{12}] \vee A_2] \vee h]$ by Axiom 2
$\vdash_{P_T}[[[A_{11} \vee A_{12}] \vee A_2] \vee h] \supset [[A_1 \vee A_2] \vee h]$ by previous case
$\vdash_{P_T}[[B_1 \vee B_2] \vee h] \supset [[A_1 \vee A_2] \vee h]$ by Axiom 7

i.e., $\vdash_{P_T} B \supset A$.

Lemma 3. *For non-empty words W and X on $\{1,b\}$ if $W \vdash_T X$, then $\vdash_{P_T} f_1(W) \supset f_1(X)$.*

If $n = 1$, wither W is X or $W \rightarrow X$ is a defining relation of T. In either case $\vdash_{P_T} f_1(W) \supset f_1(X)$ by Axiom 5 or Axiom 6.

Suppose $n > 1$. Let $W_1 \vdash_T X_1, \ldots, W_{n-1} \vdash_T X_{n-1}, W \vdash_T X$ be a proof in T, then by the induction hypothesis we have $\vdash_{P_T} f_1(W\ ) \supset f_1(X\ )$ for $i = 1$,

$2, \ldots, n-1$. If $W \vdash X$ is justified by rule 3 or rule 4, then $\vdash_{P_T} f_1(W) \supset f_1(X)$ by Axiom 5 or Axiom 6 respectively. If $W \vdash_T X$ is justified by rule 1, then $\vdash_{P_T} f_1(W) \supset f_1(X)$ by Axiom 3 and Lemma 2. If $W \vdash_T X$ is justified by rule 2, then $\vdash_{P_T} f_1(W) \supset f_1(X)$ by Axiom 4 and Lemma 2. If $W \vdash_T X$ is justified by rule 5, then $\vdash\ f_1(W) \supset f_1(X)$ by Axiom 7 and modus ponens.

The following definition is crucial in the proof of Theorem 1.

If A is a wff of P_T, then A is *valid* if and only if A is of the form $A_1 \supset A_2$, A is not of the form $B_1 \vee B_2$ and (1) A_1 is regular, A_2 is regular and $\langle A_1 \rangle \vdash_T \langle A_2 \rangle$ or (2) A_1 is not regular, A_2 is not regular and, if A_1 is valid, then A_2 is valid.

In the following proposition we single out certain simple properties of wffs which will be used rather extensively in the remainder of the paper.

Proposition 1. *No wff of any one of the following forms may be abbreviated in the form $B_1 \vee B_2$, where B_1 and B_2 are wffs.*

Form **a**. $[A_1 \vee H] \supset [A_2 \vee H]$
Form **b**. $[A_1 \vee H] \supset [A_2 \vee H] \supset_{\blacksquare} [A_3 \vee H] \supset [A_4 \vee H]$
Form **c**. $[A_1 \vee H] \supset [A_2 \vee H] \supset_{\blacksquare} [[A_2 \vee H] \supset [A_3 \vee H]] \supset [[A_1 \vee H] \supset [A_3 \vee H]]$.

For the proof for Form **a** we simply recall that $[B_1 \vee B_2]$ is an abbreviation for $[B_1 \supset B_2] \supset B_2$ and hence, if $[A_1 \vee H] \supset [A_2 \vee H]$ were of this form, then B_2 would necessarily be identified with H and also with $A_2 \vee H$, which is impossible. The proof for Form **b** follows from the result for Form **a**, since in this case B_2 would necessarily be identified with $A_2 \vee H$ and also with $[A_3 \vee H] \supset [A_4 \vee H]$. The proof for Form **c** follows from the result for Form **b**, since in this case B_2 would necessarily be identified with $A_2 \vee H$ and also with $[[A_2 \vee H] \supset [A_3 \vee H]] \supset [[A_1 \vee H] \supset [A_3 \vee H]]$.

Lemma 4. *Every theorem of P_T is of Form* **a**, **b** *or* **c** *of* Proposition 1, *where H is a substitution instance of h.*

First we note that substitution instances of Axioms 1, 2, 5 and 6 are of Form **a**, substitution instances of Axioms 3 and 4 are of Form **b**, and substitution instances of Axiom 7 are of Form **c**. Then from Proposition 1 it follows that Forms **b** and **c** can never serve as the minor premise in a use of modus ponens where a formula of Form **a**, **b** or **c** is the major premise. Likewise we see that a formula of Form **a** can never serve as minor premise in a use of modus ponens where another formula of Form **a** is the major premise. If a formula of Form **a** is the minor premise and a formula of Form **b** or **c** is the major premise in a use of modus ponens, then the resulting theorem is in Form **a** or **b** respectively. The proof of Lemma 4 is now complete if we take into account our assumption on the arrangement of proofs in the calculus.

Lemma 5. *If A is a regular wff of P_T and if B is a wff distinct from p_2, then $\mathbf{S}^{p_2}_{B} A|$ is not regular and is not valid.*

$\mathbf{S}^{p_2}_{B}\, h|$ is distinct from h and hence $\mathbf{S}^{p_2}_{B} A|$ is not of the form $B_1 \vee h$

and, therefore, cannot be regular. On the other hand, $\mathbf{S}^{p_2}_{B} A|$ is of the form $B_1 \vee B_2$ and hence is not valid.

Lemma 6. *All substitution instances of the axioms are valid.*

For the proof we shall consider the axioms individually. From Proposition 1 and Lemma 4 it follows that no substitution instance of an axiom is of the form $A \vee B$. Let P, Q and R be the wffs substituted for p_1, q_1, and r_1 respectively, and let H be the substitution instance of h in each case.

Axiom 1. $[[[P \vee Q] \vee R] \vee H] \supset [[P \vee [Q \vee R]] \vee H]$.

If $[[P \vee Q] \vee R] \vee H$ is regular, then P, Q and R are all semi-regular and H is h. Hence $[P \vee [Q \vee R]] \vee H$ is also regular. And since $<[[P \vee Q] \vee R] \vee H>$ is $<[P \vee [Q \vee R]] \vee H>$ we also have $<[[P \vee Q] \vee R] \vee H> \vdash_T <[P \vee [Q \vee R]] \vee H>$ by rule 3 of T. If $[[P \vee Q] \vee R] \vee H$ is not regular, either P, Q or R is not semi-regular or H is not h. In any event $[P \vee [Q \vee R]] \vee H$ is not regular, and $[P \vee [Q \vee R]] \vee H$ is not valid since it is of the form $A \vee B$. In either case $[[[P \vee Q] \vee R] \vee H] \supset [[P \vee [Q \vee R]] \vee H]$ is valid.

Axiom 2. $[[P \vee [Q \vee R]] \vee H] \supset [[[P \vee Q] \vee R] \vee H]$.

The proof here is similar to the proof for Axiom 1.

Axiom 3. $[P \vee H] \supset [Q \vee H] \supset_{\blacksquare} [[P \vee R] \vee H] \supset [[Q \vee R] \vee H]$.

By Proposition 1 $[P \vee H] \supset [Q \vee H]$ cannot be regular and $[[P \vee R] \vee H] \supset [[Q \vee R] \vee H]$ cannot be regular. We shall assume, therefore, that $[P \vee H] \supset [Q \vee H]$ is valid and show that $[[P \vee R] \vee H] \supset [[Q \vee R] \vee H]$ must also be valid.

Case 1. Assume that $[P \vee H]$ is regular, $[Q \vee H]$ is regular and $<R \vee H> \vdash_T <Q \vee H>$. If $R \vee H$ is also regular, then P, Q and R are all semi-regular and H is h. It follows that $[[P \vee R] \vee H]$ and $[[Q \vee R] \vee H]$ are also regular. With $<P \vee H> \vdash_T <Q \vee H>$ we also have $<[P \vee R] \vee H> \vdash_T [Q \vee R] \vee H>$ by rule 1 for T. If $R \vee H$ is not regular it follows that R is not semi-regular and hence neither $[P \vee R] \vee H$ nor $[Q \vee R] \vee H$ is regular. Then, since $[P \vee R] \vee H$ cannot be valid, we see that in either event $[[P \vee R] \vee H] \supset [[Q \vee R] \vee H]$ is valid.

Case 2. Assume that $P \vee H$ is not regular and $Q \vee H$ is not regular. Then either H is not h or neither P nor Q is semi-regular. In either event neither $[P \vee Q] \vee H$ nor $[Q \vee R] \vee H$ is regular, and, since $[P \vee R] \vee H$ cannot be valid, we see that $[[P \vee R] \vee H] \supset [[Q \vee R] \vee H]$ is valid.

Axiom 4. $[P \vee H] \supset [Q \vee H] \supset_{\blacksquare} [[R \vee P] \vee H] \supset [[R \vee Q] \vee H]$.

The proof here is similar to that for Axiom 3.

Axiom 5. $[P \vee H] \supset [P \vee H]$.

If $P \vee H$ is regular, then $P \vee H$ is regular and $<P \vee H> \vdash_T <P \vee H>$ by rule 3 of T. If $[P \vee H]$ is not regular, then $[P \vee H]$ is not regular and $P \vee H$ cannot be valid. In either event $[P \vee H] \supset [P \vee H]$ is valid.

Axiom 6. $\mathbf{S}^{p_2}_{A}\, f_1(U_i)| \supset \mathbf{S}^{p_2}_{A}\, f_1(\overline{U}_i)|$.

If A is p_2, then $\mathbf{S}^{p_2}_{A}\, f_1(U_i)|$ is $f_1(U_i)$ and $\mathbf{S}^{p_2}_{A}\, f_1(\overline{U}_i)|$ is $f_1(\overline{U}_i)$. Now $f_1(U_i)$ and $f_1(\overline{U}_i)$ are both regular and $\langle f_1(U_i)\rangle \vdash_T \langle f_1(\overline{U}_i)\rangle$ by rule 4 for T. If A is not p_2 it follows from Lemma 5 that neither $\mathbf{S}^{p_2}_{A}\, f_1(U_i)|$ nor $\mathbf{S}^{p_2}_{A}\, f_1(\overline{U}_i)|$ is regular and $\mathbf{S}^{p_2}_{A}\, f_1(U_i)|$ is not valid. In either event $\mathbf{S}^{p_2}_{A}\, f_1(U_i)| \supset \mathbf{S}^{p_2}_{A}\, f_1(\overline{U}_i)|$ is valid.

Axiom 7. $[P \vee H] \supset [Q \vee H] \supset_{\bullet} [[Q \vee H] \supset [R \vee H]] \supset [[P \vee H] \supset [R \vee H]]$.

From proposition 1 it follows that neither $[P \vee H] \supset [Q \vee H]$ nor $[[Q \vee H] \supset [R \vee H]] \supset [[P \vee H] \supset [R \vee H]]$ can be regular. Hence it suffices to show that if the former is valid then the latter is also valid. Again from Proposition 1 we see that neither $[Q \vee H] \supset [R \vee H]$ nor $[P \vee H] \supset [R \vee H]$ can be regular. Therefore, in order to show that $[[Q \vee H] \supset [R \vee H]] \supset [[P \vee H] \supset [R \vee H]]$ is valid it is only necessary to show that if $[Q \vee H] \supset [R \vee H]$ is valid, then $[P \vee H] \supset [R \vee H]$ is valid. Hence we shall assume that $[P \vee H] \supset [Q \vee H]$ and $[Q \vee H] \supset [R \vee H]$ are both valid and show that $[P \vee H] \supset [R \vee H]$ must then be valid also.

Case 1. Assume $P \vee H$ is regular. Then since $[P \vee H] \supset [Q \vee H]$ is valid, $Q \vee H$ is regular and $\langle P \vee H\rangle \vdash_T \langle Q \vee H\rangle$. But then, since $[Q \vee H] \supset [R \vee H]$ is also valid, $R \vee H$ is regular and $\langle Q \vee H\rangle \vdash_T \langle R \vee H\rangle$. Now $P \vee H$ and $R \vee H$ are both regular so we need only show that $\langle P \vee H\rangle \vdash_T \langle R \vee H\rangle$, but this follows from $\langle P \vee H\rangle \vdash_T \langle Q \vee H\rangle$ and $\langle Q \vee H\rangle \vdash_T \langle R \vee H\rangle$ by rule 5 for T.

Case 2. Assume $P \vee H$ is not regular. Then, since $[P \vee H] \supset [Q \vee H]$ is assumed to be valid, $Q \vee H$ is not regular. But then, since $[Q \vee H] \supset [R \vee H]$ is assumed to be valid, $R \vee H$ is not regular. Then, since $P \vee H$ cannot be valid, we see that $[P \vee H] \supset [R \vee H]$ is valid.

Lemma 7. *If A_1 and A_2 are wffs of P_T such that A_1 is valid and $A_1 \supset A_2$ is valid, then A_2 is valid.*

A_1 is not regular for if it were it would be of the form $B \vee H$ and hence not valid. The result then follows from the fact that $A_1 \supset A_2$ is valid.

Lemma 8. *If A and B are regular wffs of P_T and $\vdash_{P_T} A \supset B$, then $\langle A\rangle \vdash_T \langle B\rangle$.*

By Lemma 6 all substitution instances of the Axioms are valid. By Lemma 7 modus ponens preserves validity. Hence $A \supset B$ is valid. Then since A is regular we have $\langle A\rangle \vdash_T \langle B\rangle$ from the definition of validity.

Lemma 9. $W_1 \vdash_T W_2$ *if and only if* $\vdash_{P_T} f_1(W_1) \supset f_1(W_2)$.

This is merely a restatement of Lemmas 3 and 8.

§3. *Recursive Unsolvability of the Problem of Determining Whether or not an Arbitrary Calculus is Axiomatizable by a Single Axiom*. We shall establish the following results.

Result 2A. *For each recursively enumerable degree of unsolvability D there exists a class of partial implicational propositional calculi C_D such that the problem to determine of an arbitrary member P of C_D whether or not P is axiomatizable by a single axiom is of degree D.*

Result 2B. *For each recursively enumerable degree of unsolvability D there exists a class of partial propositional calculi C_D such that the problem to determine of an arbitrary member P of C_D whether or not P is axiomatizable by a single axiom is of degree D.*

These results are immediate from Lemma 1 and the following theorem.

Theorem 2. *There is a recursive construction M^2 such that the result of applying M^2 to any standard semi-Thue system T is a recursive class of calculi C_T and a mapping f_T of the pairs of non-empty words on $\{1,b\}$ onto C_T. Furthermore, f_T is one-to-one, and if W_1 and W_2 are non-empty words on $\{1,b\}$, then $W_1 \vdash_T W_2$ if and only if $f_T(W_1, W_2)$ is axiomatizable by a single axiom.*

We turn now to the task of establishing Theorem 2. In order to fascilitate this task we find it convenient to introduce several new notions here.

A recursive (possibly empty) set of tautologies S is said to be *sterile* if (1) no substitution instance of a wff of S is a substitution instance of any other wff of S, and (2) no substitution instance of a wff of S is a substitution instance of the antecedent of any wff of S.

Lemma 10. *The minimum number of axioms necessary to axiomatize a calculus $P(S)$ specified by a sterile set of tautologies S is the cardinality of S.*

From condition (2) of the definition of a sterile set it follows that modus ponens is vacuous in $P(S)$. Then from condition (1) we see that any set of axioms for $P(S)$ must contain at least one substitution instance of each wff of S.

A wff A is said to be *completely untrue* with respect to a calculus P if no substitution instance of A is a theorem of P.

A set S of tautologies is said to be *completely independent* of a calculus P if (1) the set S is sterile, (2) every wff of S is completely untrue with respect to P, (3) the antecedent of every wff of S is completely untrue with respect to P, and (4) the antecedent of every theorem of P is completely untrue with respect to the calculus specified by S. One should note that every subset of a completely independent set is completely independent.

Lemma 11. *If P is a calculus and S is a set of tautologies completely independent of P, then the minimum number of axioms necessary to axiomatize the system resulting from the addition of the wffs of S to the* *axioms of P is equal to the minimum number of axioms necessary to axiomatize P plus the cardinality of S.*

From properties (1) and (2) of the definition of a completely independent set it follows that no theorem of P is a theorem of the calculus specified by S and vice versa. Properties (3) and (4) guarantee that there is no modus ponens interaction between the theorems of P and the theorems of the calculus specified by S. Therefore any set of axioms sufficient to axiomatize the enriched system must contain mutually independent sets of axioms for P and for the calculus specified by S. The result then follows from Lemma 10.

We shall use the symbol Ł as an abbreviation for the wff

$$[[p_1 \supset q_1] \supset r_1] \supset_{\blacksquare} [r_1 \supset p_1] \supset [q_2 \supset p_1].$$

Łukasiewicz [8] has shown that Ł is sufficient to axiomatize the complete implicational propositional calculus.

Let T be a standard semi-Thue system and construct P_T from T as in the proof of Theorem 1. For each pair of non-empty words W_1, W_2 on $\{1,b\}$ we shall designate $f_T(W_1, W_2)$ to be the system resulting from the addition of $[f_1(W_1) \supset f_1(W_2)] \supset$ Ł to the axioms of P_T. The class C_T will then consist of all systems of this form. We complete the proof of Theorem 2 by showing that $f_T(W_1, W_2)$ is axiomatizable by a single axiom if and only if $W_1 \vdash_T W_2$.

Lemma 12. *If $W_1 \vdash_T W_2$, then $f_T(W_1, W_2)$ is axiomatizable by a single axiom.*

From Lemma 3 and $W_1 \vdash_T W_2$ we have $\vdash_{P_T} f_1(W_1) \supset f_1(W_2)$. Hence $f_1(W_1) \supset f_1(W_2)$ is a theorem of $f_T(W_1, W_2)$ and by definition $[f_1(W_1) \supset f_1(W_2)] \supset$ Ł is also a theorem of $f_T(W_1, W_2)$. Hence by modus ponens Ł is a theorem of $f_T(W_1, W_2)$. It follows that $f_T(W_1, W_2)$ is axiomatizable by any set of axioms sufficient to axiomatize the complete implicational calculus. Specifically, Ł is sufficient to axiomatize $f_T(W_1, W_2)$.

Lemmas 13 through 20 lead to a proof of the contrapositive of the converse of Lemma 12. Lemmas 13, 14, 18 and 19 are the necessary steps in establishing the complete independence of $[f_1(W_1) \supset f_1(W_2)] \supset$ Ł with respect to P if it is not the case that $\vdash_{P_T} f_1(W_1) \supset f_1(W_2)$.

Lemma 13. *For arbitrary non-empty words on $\{1,b\}$, W_1 and W_2, the wff $[f_1(W_1) \supset f_1(W_2)] \supset$ Ł is a sterile set.*

Suppose some substitution instance of $[f_1(W_1) \supset f_1(W_2)] \supset$ Ł were a substitution instance of $f_1(W_1) \supset f_1(W_2)$. Then some substitution instance of $f_1(W_1) \supset f_1(W_2)$ would necessarily be a substitution instance of $f_1(W_1)$. Recalling that $f_1(W_1)$ is of the form $A_1 \vee h$ and that $f_1(W_2)$ is of the form $A_2 \vee h$ we readily see from Proposition 1 that this is impossible. Hence $[f_1(W_1) \supset f_1(W_2)] \supset$ Ł is a sterile set.

Lemma 14. *For arbitrary non-empty words W_1 and W_2 on $\{1,b\}$ the wff $[f_1(W_1) \supset f_1(W_2)] \supset$ Ł is completely untrue with respect to P_T.*

For the proof we shall show that no substitution instance of $[f_1(W_1) \supset f_1(W_2)] \supset$ Ł is of the form **a**, **b** or **c** of Proposition 1, and hence by Lemma 4 it follows that no substitution instance of $[f_1(W_1) \supset f_1(W_2)] \supset \text{Ł}$ is a theorem of P_T. We shall consider the forms separately.

Form **a**. If some substitution instance of $[f_1(W_1) \supset f_1(W_2)] \supset \text{Ł}$ were of the form $[A_1 \vee H] \supset [A_2 \vee H]$, then some substitution instance of $[f_1(W_1) \supset f_1(W_2)]$ would necessarily be of the form $A_1 \vee H$. This is impossible by Proposition 1.

Form **b**. Suppose some substitution instance of $[f_1(W_1) \supset f_1(W_2)] \supset \text{Ł}$ were of the form $[A_1 \vee H] \supset [A_2 \vee H] \supset_{\blacksquare} [A_3 \vee H] \supset [A_4 \vee H]$. Recalling that Ł is an abbreviation for $[[p_1 \supset q_1] \supset r_1] \supset_{\blacksquare} [r_1 \supset p_1] \supset [q_2 \supset p_1]$ we see that some substitution instance of $[r_1 \supset p_1] \supset [q_2 \supset p_1]$ must then be identified with $A_4 \vee H$. But then, since $A_4 \vee H$ is an abbreviation for $[A_4 \supset H] \supset H$, H must be identified with the substitution instance of p_1 and also with the substitution instance of $q_2 \supset p_1$. This is clearly impossible.

Form **c**. Suppose some substitution instance of $[f_1(W_1) \supset f_1(W_2)] \supset \text{Ł}$ were of the form $[A_1 \vee H] \supset [A_2 \vee H] \supset_{\blacksquare} [[A_2 \vee H] \supset [A_3 \vee H]] \supset [[A_1 \vee H] \supset [A_3 \vee H]]$. Then some substitution instance of Ł is of the form $[[A_2 \vee H] \supset [A_3 \vee H]] \supset [[A_1 \vee H] \supset [A_3 \vee H]]$. Hence, from the first occurrence of p_1 in Ł, the substitution instance of p_1 must be identified with $A_2 \supset H$. While, from the second occurrence of p_1 in Ł, the substitution instance of p_1 must be identified with H. Clearly these conditions are incompatible.

Lemmas 15 through 18 below constitute a proof that $f_1(W_1) \supset f_1(W_2)$ is completely untrue with respect to P_T if it is not the case that $\vdash_{P_T} f_1(W_1) \supset f_1(W_2)$. In order to establish this result we first introduce two new definitions.

A wff B of P is *S-regular* if and only if there is a regular wff B_1, and a wff A such that B is $\mathbf{S}^{p_2}_{A} B_1|$.

Recalling that p_2 is the only variable occurring in a regular wff and the fact that every regular wff is of the form $C \vee h$, one easily sees that if B is S-regular, then there is a unique B_1 and a unique A such that B is $\mathbf{S}^{p_2}_{A} B_1|$.

A wff B of P_T is *S-valid* if and only if B is of the form $B_1 \supset B_2$, B is not of the form $A_1 \vee A_2$ and (1) there are regular wffs C_1 and C_2 and a wff A such that B_1 is $\mathbf{S}^{p_2}_{A} C_1|$, B_2 is $\mathbf{S}^{p_2}_{A} C_2|$, and $\vdash_{P_T} C_1 \supset C_2$, or (2) B_1 is not S-regular, B_2 is not S-regular, and if B_1 is S-valid, then B_2 is S-valid.

Lemma 15. *All substitution instances of the axioms of* P_T *are S-valid.*

For the proof we shall again consider the axioms individually. Let P, Q and R be the substitution instances of p_1, q_1, and r_1 respectively, and let H be the substitution instance of h.

Axiom 1. $[[[P \vee Q] \vee R] \vee H] \supset [[P \vee [Q \vee R]] \vee H]$.

We shall consider two cases. For the first case assume that $[[P \vee Q] \vee R] \vee H$ is S-regular. Then there is a wff A and a regular wff C_1 such that

$[[P \vee Q] \vee R] \vee H$ is $\mathsf{S}^{p_2}_{|A} C_1|$. Thus C_1 is $[[C_{11} \vee C_{12}] \vee C_{13}] \vee h$ for some semi-regular wffs C_{11}, C_{12} and C_{13}. Hence $[P \vee [Q \vee R]] \vee H$ is $\mathsf{S}^{p_2}_{A} [C_{11} \vee [C_{12} \vee C_{13}]] \vee h]|$, and $[C_{11} \vee [C_{12} \vee C_{13}]] \vee h$ is regular. Then since $<[[C_{11} \vee C_{12}] \vee C_{13}] \vee h>$ is $<[C_{11} \vee [C_{12} \vee C_{13}]] \vee h>$, we have $\vdash_{P_T} [[[C_{11} \vee C_{12}] \vee C_{13}] \vee h] \supset [[C_{11} \vee [C_{12} \vee C_{13}]] \vee h]$ by Lemma 2. For the second case assume that $[[P \vee Q] \vee R] \vee H$ is not S-regular. Then $[P \vee [Q \vee R]] \vee H$ is not S-regular, for if it were $[[P \vee Q] \vee R] \vee H$ would be also by an argument similar to the one given above. Since $[[P \vee Q] \vee R] \vee H$ is of the form $A_1 \vee A_2$ it is not S-valid. Hence in either case $[[[P \vee Q] \vee R] \vee H] \supset [[P \vee [Q \vee R]] \vee H]$ is S-valid.

Axiom 2. $[[P \vee [Q \vee R]] \vee H] \supset [[[P \vee Q] \vee R] \vee H]$.

The proof here is similar to the proof for Axiom 1.

Axiom 3. $[P \vee H] \supset [Q \vee H] \supset_{\blacksquare} [[P \vee R] \vee H] \supset [[Q \vee R] \vee H]$.

From Proposition 1 we see that neither $[P \vee H] \supset [Q \vee H]$ nor $[[P \vee R] \vee H] \supset [[Q \vee R] \vee H]$ can be S-regular. Therefore it is sufficient to assume that the former is S-valid and to prove under this assumption that the latter must be also. We consider two cases.

Case 1. $P \vee H$ is $\mathsf{S}^{p_2}_{|A} C_1|$ and $Q \vee H$ is $\mathsf{S}^{p_2}_{|A} C_2|$ where C_1 and C_2 are both regular and $\vdash_{P_T} C_1 \supset C_2$. Then C_1 is of the form $C_{11} \vee h$ and C_2 is of the form $C_{21} \vee h$ where C_{11} and C_{21} are both semi-regular. We shall consider two subcases.

Case 1a. R is $\mathsf{S}^{p_2}_{|A} R_1|$ where R_1 is semi-regular. Then $[P \vee R] \vee H$ is $\mathsf{S}^{p_2}_{|A} [C_{11} \vee R_1] \vee h|$ and $[Q \vee R] \vee H$ is $\mathsf{S}^{p_2}_{|A} [C_{21} \vee R_1] \vee h|$ where $[C_{11} \vee R_1] \vee h$ and $[C_{21} \vee R_1] \vee h$ are regular. By assumption we have $\vdash_{P_T} C_1 \supset C_2$, i.e., $\vdash_{P_T} [C_{11} \vee h] \supset [C_{21} \vee h]$. Hence from Axiom 3, substitution and modus ponens we obtain $\vdash_{P_T} [[C_{11} \vee R_1] \vee h] \supset [[C_{21} \vee R_1] \vee h]$. And we see that the result holds in this case.

Case 1b. There is no semi-regular wff R_1 such that R is $\mathsf{S}^{p_2}_{|A} R_1|$. Then neither $[P \vee R] \vee H$ nor $[Q \vee R] \vee H$ is S-regular and, since $[P \vee R] \vee H$ cannot be S-valid, the result holds in this case.

Case 2. Neither $P \vee H$ nor $Q \vee H$ is S-regular. Then neither $[P \vee R] \vee H$ nor $[Q \vee R] \vee H$ is S-regular, and, since $[P \vee R] \vee H$ cannot be S-valid, the result follows.

Axiom 4. $[P \vee H] \supset [Q \vee H] \supset_{\blacksquare} [[R \vee P] \vee H] \supset [[R \vee Q] \vee H]$.

The proof here is similar to that for Axiom 3.

Axiom 5. $[P \vee H] \supset [P \vee H]$.

The proof here is immediate.

Axiom 6. $\mathsf{S}^{p_2}_{|A} f_1(U_i)| \supset \mathsf{S}^{p_2}_{A} f_1(\overline{U}_i)|$.

Both $f_1(U_i)$ and $f_1(\bar{U}_i)$ are regular and $\vdash_{P_T} f_1(U_i) \supset f_1(\bar{U}_i)$ by Axiom 6. Hence $\mathbf{S}_A^{p_2} f_1(U_i)| \supset \mathbf{S}_A^{p_2} f_1(\bar{U}_i)$ is S-valid for every wff A.

Axiom 7. $[P \vee H] \supset [Q \vee H] \supset_{\blacksquare} [[Q \vee H] \supset [R \vee H]] \supset [[P \vee H] \supset [R \vee H]]$.

From Proposition 1 it follows that neither the antecedent nor the consequent is S-regular. Since we also have from Proposition 1 that neither $[Q \vee H] \supset [R \vee H]$ nor $[P \vee H] \supset [R \vee H]$ is S-regular it is sufficient to prove that if $[P \vee H] \supset [Q \vee H]$ and $[Q \vee H] \supset [R \vee H]$ are both S-valid, then $[P \vee H] \supset [R \vee H]$ is S-valid. We consider two cases.

Case 1. If $P \vee H$ is $\mathbf{S}_{|A}^{|p_2} P_1 \vee h|$ for some semi-regular wff P_1 and some wff A, then, since $[P \vee H] \supset [Q \vee H]$ is assumed to be S-valid, $Q \vee H$ is $\mathbf{S}_{|A}^{|p_2} Q_1 \vee h|$ for some semi-regular wff Q_1 and $\vdash_{P_T} [P_1 \vee h] \supset [Q_1 \vee h]$. But then, since $[Q \vee H] \supset [R \vee H]$ is S-valid, $R \vee H$ is $\mathbf{S}_{|A}^{|p_2} R_1 \vee h|$ for some semi-regular wff R_1 and $\vdash_{P_T} [Q_1 \vee h] \supset [R_1 \vee h]$. Hence by Axiom 7, substitution, and modus ponens we obtain $\vdash_{P_T} [P_1 \vee h] \supset [R_1 \vee h]$, and it follows that $[P \vee H] \supset [R \vee H]$ is S-valid.

Case 2. If $P \vee H$ is not S-regular, then, since $[P \vee H] \supset [Q \vee H]$ is assumed to be S-valid, $Q \vee H$ is not S-regular. But then, since $[Q \vee H] \supset [R \vee H]$ is assumed to be S-valid, it follows that $R \vee H$ is not S-regular. Hence neither $P \vee H$ nor $R \vee H$ is S-regular and since $P \vee H$ cannot be S-valid it follows that $[P \vee H] \supset [R \vee H]$ is S-valid.

Lemma 16. *If A is S-valid and $A \supset B$ is S-valid, then B is S-valid.*

Since A is S-valid it cannot be S-regular. Hence from the fact that $A \supset B$ is S-valid it follows that B is S-valid.

Lemma 17. *All theorems of P_T are S-valid.*

By Lemma 15 all substitution instances of the axioms are S-valid and by Lemma 16 modus ponens preserves S-validity. The conclusion follows from our assumption on the form of the proofs.

Lemma 18. *If it is not the case that $W_1 \vdash_T W_2$ then $f_1(W_1) \supset f_1(W_2)$ is completely untrue with respect to P_T.*

We shall prove the contrapositive. Suppose $\vdash_{P_T} \mathbf{S}_{|A}^{p_2} f_1(W_1) \supset f_1(W_2)|$ for some wff A. Then from Lemma 16 and the fact that $f_1(W_1)$ and $f_1(W_2)$ are regular we have $\vdash_{P_T} f_1(W_1) \supset f_1(W_2)$. Hence from Lemma 9 we also have $W_1 \vdash_1 W_2$. This establishes the result.

Lemma 19. *For arbitrary non-empty words W_1 and W_2 on $\{1,b\}$ no substitution instance of $[f_1(W_1) \supset f_1(W_2)] \supset Ł$ is a substitution instance of the antecedent of a theorem of P_T.*

For the proof we shall consider the forms of the theorems as given in Lemma 4. We consider these separately.

Form **a**. $[A_1 \vee H] \supset [A_2 \vee H]$. If $[f_1(W_1) \supset f_1(W_2)] \supset \text{Ł}$ had a substitution instance of the form $A_1 \vee H$, then some substitution instance of Ł would necessarily be a substitution instance of h but one easily sees that this is impossible.

Form **b**. $[A_1 \vee H] \supset [A_2 \vee H] \supset_{\blacksquare} [A_3 \vee H] \supset [A_4 \vee H]$. Note that the antecedent of this form is of Form **a**. Then the result follows from Lemma 14.

Form **c**. $[A_1 \vee H] \supset [A_2 \vee H] \supset_{\blacksquare} [[A_2 \vee H] \supset [A_3 \vee H]] \supset [[A_1 \vee H] \supset [A_3 \vee H]]$.

The proof here is similar to that for Form **b**.

Lemma 20. *If it is not the case that* $W_1 \vdash_T W_2$, *then* $[f_1(W_1) \supset f_1(W_2)] \supset \text{Ł}$ *is completely independent of* P_T.

This follows from Lemmas 13, 14, 18 and 19 and the fact that modus ponens is vacuous in the calculus specified by $[f_1(W_1) \supset f_1(W_2)] \supset \text{Ł}$.

Lemma 21. *If it is not the case that* $W_1 \vdash_T W_2$, *then at least two axioms are required to axiomatize* $f_T(W_1, W_2)$.

This is immediate from Lemmas 11 and 20.

Lemma 22. $f\ (W_1, W_2)$ *is axiomatizable by a single axiom if and only if* $W_1 \vdash_T W_2$.

This is a restatement of Lemmas 12 and 21.

§4. *Recursive Unsolvability of the Split Problem for Propositional Calculi.* A calculus P is said to allow a *split* if P is axiomatizable by a set of tautologies S and the set S can be divided into two non-empty sets S_1 and S_2 such that every theorem of P is a theorem of the calculus specified by S_1 or of the calculus specified by S_2 and the calculi specified by S_1 and S_2 have no theorem in common. We shall establish the following results.

Result 3A. *For each recursively enumerable degree of unsolvability D there exists a class of partial implicational propositional calculi C_D such that the problem to determine of an arbitrary member P of C_D whether or not P allows a split is of degree D.*

Result 3B. *For each recursively enumerable degree of unsolvability D there exists a class of partial propositional calculi C_D such that the problem to determine of an arbitrary member P or C_D whether or not P allows a split is of degree D.*

These results are immediate from Lemma 1, the proof of Theorem 2 and the following theorem.

Theorem 3. *Consider a class of calculi C_T constructed from a semi-Thue system T as in the proof of Theorem* 2. *An arbitrary member $P(W_1, W_2)$ of C_T allows a split if and only if $P(W_1, W_2)$ is not axiomatizable by a single axiom.*

We turn now to the relatively easy task of establishing Theorem 3.

Lemma 23. *If a calculus P is axiomatizable by a single axiom A, then P allows no split.*

For the proof assume there is a calculus P axiomatizable by a single axiom A which allows a split. Let P_1 and P_2 be the two calculi resulting from the split of P. Then A is a theorem of either P_1 or P_2. Without loss of generality assume A is a theorem of P_1. Then all of the theorems of P are theorems of P_1 and this is clearly a contradiction.

Lemma 24. *Let $P(W_1, W_2)$ be a calculus of C_T which is not axiomatizable by a single axiom. Then $P(W_1, W_2)$ allows a split.*

From Lemma 12 we have that it is not the case that $W_1 \vdash_T W_2$. Then from Lemma 20 it follows that $[f_1(W_1) \supset f_1(W_2)] \supset Ł$ is completely independent of P_T. Clearly P_T and the calculus specified by the single axiom $[f_1(W_1) \supset f_1(W_2)] \supset Ł$ constitute a split of $P(W_1, W_2)$.

§5. *Recursive Unsolvability of the Problem of Determining Whether or not an Arbitrary Calculus is Axiomatizable by n or Fewer Axioms.* We shall establish the following results.

Result 4A. *For each recursively enumerable degree of unsolvability D and each natural number n there exists a class of partial implicational propositional calculi $C_{D,n}$ such that the problem to determine of an arbitrary member P of $C_{D,n}$ whether or not P is axiomatizably by n or fewer axioms is of degree D.*

Result 4B. *For each recursively enumerable degree of unsolvability D and each natural number n there exists a class of partial propositional calculi $C_{D,n}$ such that the problem to determine of an arbitrary member P of $C_{D,n}$ whether or not P is axiomatizable by n or fewer axioms is of degree D.*

These results are immediate from Lemma 1 and the following theorem.

Theorem 4. *There is a recursive construction M^3 such that the result of applying M^3 to any standard semi-Thue system T and any natural number n is a recursive class of calculi $C_{T,n}$ and a mapping $f_{T,n}$ of the pairs of non-empty words on $\{1,b\}$ onto $C_{T,n}$. Furthermore, $f_{T,n}$ is one-to-one, and for non-empty words W_1 and W_2 on $\{1,b\}$ $W_1 \vdash_T W_2$ if and only if $f_{T,n}(W_1, W_2)$ is axiomatizable by n or fewer axioms.*

We turn now to the task of establishing Theorem 4. With each natural number n we recursively associate a wff L_n as follows. L_1 is $p_2 \supset [p_2 \supset [p_2 \supset [p_2 \supset [p_2 \supset p_2]]]]$ and L_{n+1} is $p_2 \supset L_n$.

Note that no substitution instance of L_i is a substitution instance of L_j for $i \neq j$, and that no substitution instance of L_i, for any natural number i, can be abbreviated in the form $A \vee B$. For each natural number n let K_n be the class of wffs of the form $L_j \supset L_j$ for $1 \leq j \leq n$. Let K_∞ be the class of formulas of the form $L_j \supset L_j$ for $1 \leq j < \infty$. Now let T be an arbitrary standard semi-Thue system and construct P_T from T. We shall prove that if W_1 and W_2 are arbitrary non-empty words on $\{1,b\}$ and it is not the case that $W_1 \vdash_T W_2$, then L_∞ is completely independent of $f_T(W_1, W_2)$.

Lemma 25. *The class of wffs K_∞ is sterile.*

From the fact that no substitution instance of L_i is a substitution instance of L_j for $i \neq j$ we see that no substitution instance of a wff of K_∞ is a substitution instance of any other. Now all members of K_∞ are of the form $A \supset A$ while the antecedents of these wffs are all of the form L_j. From the fact that no substitution instance of a wff of the form $A \supset A$ can be a substitution instance of a wff of the form L_j we see that no substitution instance of a wff of K_∞ is a substitution instance of the antecedent of a wff of K_∞.

Lemma 26. *If it is not the case that $W_1 \vdash_T W_2$, then every wff of the class K_∞ is completely untrue with respect to $f_T(W_1, W_2)$.*

By Lemma 20 $[f_1(W_1) \supset f_1(W_2)] \supset Ł$ is completely independent of P_T in this case and it follows that the theorems of $f_T(W_1, W_2)$ are the theorems of P_T and substitution instances of $[f_1(W_1) \supset f_1(W_2)] \supset Ł$. It is sufficient, therefore, to prove that the class K_∞ is completely untrue with respect to P_T and that no substitution instance of a member of K_∞ is a substitution instance of $[f_1(W_1) \supset f_1(W_2)] \supset Ł$. Now every wff of K_∞ and every substitution instance of such a formula is of the form $A \supset A$. If we consider the forms the theorems of P_T may take as given in Lemma 4, we see that only theorems of Form **a** or **b** could be of the form $A \supset A$. But the antecedent of every wff of Form **a** contains more symbols than the consequent of the antecedent and this is untrue with respect to every substitution instance of a wff of K_∞. Also the antecedent of the consequent of the antecedent of every wff of Form **b** contains more symbols than the consequent of the consequent of the antecedent and this is untrue with respect to every substitution instance of a wff of K_∞. Therefore every wff of K_∞ is completely untrue with respect to P_T.

If some substitution instance of $[f_1(W_1) \supset f_1(W_2)] \supset Ł$ were a substitution instance of a wff of K_∞ it follows from the form of the L_n and the form of Ł that the wff identified with $[p_1 \supset q_1] \supset r_1$ in the substitution instance of Ł would also have to be identified with $r_1 \supset p_1$ in this substitution instance of Ł, but this is impossible.

Lemma 27. *If it is not the case that $W_1 \vdash_T W_2$, then the antecedent of every wff of K_∞ is completely untrue with respect to $f_T(W_1, W_2)$.*

As in the proof of Theorem 4 we use the fact that here in the theorems of $f_T(W_1, W_2)$ are the theorems of P_T and substitution instances of $[f_1(W_1) \supset f_1(W_2)] \supset Ł$. We shall first show that the antecedent of every wff of K_∞ is completely untrue with respect to P_T. For this purpose we shall consider the forms the theorems of P_T may take as given in Lemma 4. We consider the forms separately.

Form **a**. $[A_1 \vee H] \supset [A_2 \vee H]$.

Recall that every wff of K_∞ has an antecedent of the form L_j for some j. The antecedent of the consequent of every wff of Form **a** contains more symbols than the consequent of the consequent, but this is untrue with respect to every substitution instance of a wff of the form L_j.

Form **b**. $[A_1 \vee H] \supset [A_2 \vee H] \supset_{\blacksquare} [A_3 \vee H] \supset [A_4 \vee H]$. The antecedent of the consequent of the consequent of every wff of Form **b** contains more symbols than does the consequent of the consequent of the consequent, but this is untrue with respect to every substitution instance of a wff of the form L_j.

Form **c**. $[A_1 \vee H] \supset [A_2 \vee H] \supset_{\blacksquare} [[A_2 \vee H] \supset [A_3 \vee H]] \supset [[A_1 \vee H] \supset [A_3 \vee H]]$. The antecedent of the consequent of the consequent of the consequent of every wff of Form **a** contains more symbols than does the consequent of the consequent of the consequent of the consequent, but this is untrue with respect to every substitution instance of a wff of the form L_j.

Now suppose some substitution instance of $[f_1(W_1) \supset f_1(W_2)] \supset Ł$ were a substitution instance of a wff of the form L_j. Then from the form of L_j it follows that in the substitution instance of $[f_1(W_1) \supset f_1(W_2)] \supset Ł$ the substitution instance of $[p_1 \supset q_1] \supset r_1$ would necessarily be identical to the substitution instance of $[r_1 \supset p_1]$, but this is impossible.

Lemma 28. *If it is not the case that $W_1 \vdash_T W_2$, then no substitution instance of the antecedent of a theorem of $f_T(W_1, W_2)$ is a substitution instance of a wff of K_∞.*

Again recall the fact that the theorems of $f_T(W_1, W_2)$ are the theorems of P_T and substitution instances of $[f_1(W_1) \supset f_1(W_2)] \supset Ł$ in this case. That the result holds for wffs of Form **a** of Lemma 4 follows from the fact that the antecedent of the antecedent of every wff of Form **a** contains more symbols than the consequent of the antecedent while every substitution instance of a wff of K_∞ is of the form $A \supset A$. For all wffs of Form **b** or **c** of Lemma 4 and all substitution instances of $[f_1(W_1) \supset f_1(W_2)] \supset Ł$ it is the case that the antecedent of the consequent of the antecedent contains more symbols than the consequent of the consequent of the antecedent but for every substitution instance of a wff of K_∞ the antecedent of the consequent contains fewer symbols than the consequent of the consequent.

Lemma 29. *If it is not the case that $W_1 \vdash_T W_2$, then the class K_∞ is completely independent of $f_T(W_1, W_2)$.*

This is immediate from Lemmas 25, 26, 27 and 28 and the fact that modus ponens is vacuous in a calculus specified by a sterile class of wffs.

Now let T be an arbitrary standard semi-Thue system and let n be any natural number. If n is 1 and W_1 and W_2 are arbitrary non-empty words on $\{1,b\}$, then $f_{T,n}(W_1, W_2)$ is $f_T(W_1, W_2)$. If $n > 1$ and W_1 and W_2 are arbitrary non-empty words on $\{1,b\}$ then $f_{T,n}(W_1, W_2)$ is to be the calculus resulting from the addition of K_{n-1} to the axioms of $f_T(W_1, W_2)$. In any case the class $C_{T,n}$ shall consist of all calculi of the form $f_{T,n}(W_1, W_2)$.

Lemma 30. *If $W_1 \vdash_T W_2$, then $f_{T,n}(W_1, W_2)$ is axiomatizable by a single axiom.*

This follows from Lemma 12.

Lemma 31. *If it is not the case that* $W_1 \vdash_T W_2$, *then* $f_{T,n}(W_1, W_2)$ *is axiomatizable by no fewer than* $n + 1$ *axioms.*

This follows from Lemma 29, the fact that every subset of a completely independent class is completely independent, Lemma 11 and Lemma 21.

Lemma 32. $f_{T,n}(W_1, W_2)$ *is axiomatizable by n or fewer axioms if and only if* $W_1 \vdash_T W_2$.

This is immediate from Lemmas 30 and 31.

§6. *Recursive Unsolvability of the Problem to Determine Whether or not an Arbitrary Infinite Calculus is Axiomatizable by a Finite Set of Axioms.* We now relax the condition that the axioms of a partial (partial implicational) propositional calculus be a finite set of tautologies and require only that the set be recursive. If the set of axioms is infinite we then call the system an *infinite partial (partial implicational) propositional calculus.* We shall establish the following results.

Result 5A. *For each recursively enumerable degree of unsolvability D there exists a class of infinite partial implicational propositional calculi* $C_{D,\infty}$ *such that the problem to determine of an arbitrary member P of* $C_{D,\infty}$ *whether or not P is finitely axiomatizable is of degree D.*

Result 5B. *For each recursively enumerable degree of unsolvability D there exists a class of infinite partial propositional calculi* $C_{D,\infty}$ *such that the problem to determine of an arbitrary member P of* $C_{D,\infty}$ *whether or not P is finitely axiomatizable is of degree D.*

These results are immediate from Lemma 1 and the following theorem.

Theorem 5. *There is a recursive procedure* M^4 *such that the result of applying* M^4 *to any standard semi-Thue system T is a recursive class of infinite calculi* $C_{T,\infty}$ *and a mapping* $f_{T,\infty}$ *of the pairs of non-empty words on* $\{1,b\}$ *onto* $C_{T,\infty}$. *Furthermore,* $f_{T,\infty}$ *is one-to-one, and for non-empty words* W_1 *and* W_2 *on* $\{1,b\}$, $W_1 \vdash_T W_2$ *if and only if* $f_{T,\infty}(W_1, W_2)$ *is finitely axiomatizable.*

Let T be a standard semi-Thue system. If W_1 and W_2 are arbitrary non-empty words on $\{1,b\}$ let $f_{T,\infty}(W_1, W_2)$ be the infinite calculus resulting from the addition of the class K_∞ to the axioms of $f_T(W_1, W_2)$.

Lemma 33. *If* $W_1 \vdash_T W_2$, *then* $f_{T,n}(W_1, W_2)$ *is axiomatizable by a single axiom.*

This follows from Lemma 12.

Lemma 34. *If it is not the case that* $W_1 \vdash_T W_2$, *then* $f_{T,\infty}(W_1, W_2)$ *is not finitely axiomatizable.*

This follows from Lemmas 29, 11 and 21.

Lemma 35. $f_{T,\infty}(W_1, W_2)$ *is finitely axiomatizable if and only if* $W_1 \vdash_T W_2$.

This is immediate from Lemmas 33 and 34.

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