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ON THE LEIBNIZIAN MODAL SYSTEM

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The aim of the paper is to present a modal system which we will call the Leibnizian modal system and to show the completeness of the system with a restriction.

1. In my previous paper [1], in order to show an example of defence of circular definition (as analysis, not stipulative definition), I gave the following definition:

A statement is analytic if and only if it is consistent with every statement that expresses what is possible.

This definition, roughly speaking, is materially equivalent to Carnap's definition of L-truth:

A sentence $\mathfrak{S}_i$ *is* L-*true* (*in* S_1) = Df $\mathfrak{S}_i$ *holds in every state-description* (*in* S_1).

which is suggested by Leibniz' conception that a necessary truth must hold in all possible worlds (cf. Carnap [2], p. 10). If "analytic" is replaced by "necessary" in the above definition of "analytic", the definition will be as follows:

A statement is necessary if and only if it is consistent with every statement that expresses what is possible.

This is symbolized by modal signs as follows:

$$\Box p \equiv (q)[\Diamond q \supset \Diamond(p \cdot q)]$$

where p and q are propositional variables.

2. We shall construct a modal system, which will hereafter be called L (the Leibnizian modal system), consisting of the following one axiom and five rules.

A. $\vdash \Box p \supset [\Diamond q \supset \Diamond(p \cdot q)]$

R1. *If* $\vdash \Diamond p \supset \Diamond(\alpha \cdot p)$ *then* $\vdash \Box \alpha$ (p is not contained in α)

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R2. *If* α *is a tautology of the classical propositional calculus, then* $\vdash \alpha$.
R3. *Substitution for propositional variables.*
R4. *Material detachment.*
R5. *Replacement of material equivalents.*

Here α is an arbitrary formula and p, q are propositional variables. '$\Diamond$' is regarded as the abbreviation of '$\sim\Box\sim$'.

The following theorems hold in L.

Theorem. $\vdash \Box(p \supset q) \supset (\Box p \supset \Box q)$

Proof. (1) $\vdash \Box p \supset [\Diamond q \supset \Diamond(p \cdot q)]$ [A]
(2) $\vdash \Box p \supset [\sim\Diamond(p \cdot q) \supset \sim\Diamond q]$ [From (1) by R2,R3,R5]
(3) $\vdash \Box p \supset [\sim\Diamond\sim\sim(p \cdot \sim q) \supset \sim\Diamond\sim q]$ [From (2) by R2,R3,R5]
(4) $\vdash \Box p \supset [\Box(p \supset q) \supset \Box q]$ [From (3) by R2,R5]
(5) $\vdash \Box(p \supset q) \supset (\Box p \supset \Box q)$ [From (4) by R2,R3,R4]

Theorem. *If* α *is a tautology of the classical propositional calculus,* then $\vdash \Box \alpha$

Proof. (1) α is a tautology of the classical propositional calculus. [Hypothesis]
(2) $\vdash (\alpha \cdot p) \equiv p$ (p is not contained in α) [From (1) by R2]
(3) $\vdash \Diamond p \supset \Diamond p$ [By R2,R3]
(4) $\vdash \Diamond p \supset \Diamond(\alpha \cdot p)$ (p is not contained in α) [From (2), (3) by R5]
(5) $\vdash \Box \alpha$ [From (4) by R1]

The above theorems show that L contains the following system.

A. $\vdash \Box(p \supset q) \supset (\Box p \supset \Box q)$.
R1. *If* α *is a tautology of the classical propositional calculus, then* $\vdash \Box \alpha$.
R2,R3,R4,R5. (the same as those of L)

If we call this system L', we can prove that L' contains L. The following theorems hold in L'.

Theorem. $\vdash \Box p \supset [\Diamond q \supset \Diamond(p \cdot q)]$

Proof. (1) $\vdash \Box(p \supset q) \supset (\Box p \supset \Box q)$ [A]
(2) $\vdash \Box p \supset [\Box(p \supset q) \supset \Box q]$ [From (1) by R2,R3,R4]
(3) $\vdash \Box p \supset [\sim\Box q \supset \sim\Box(p \supset q)]$ [From (2) by R2,R3,R5]
(4) $\vdash \Box p \supset [\sim\Box\sim q \supset \sim\Box\sim\sim(p \supset \sim q)]$ [From (3) by R2,R3,R5]
(5) $\vdash \Box p \supset [\Diamond q \supset \Diamond(p \cdot q)]$ [From (4) by R2,R5]

Theorem. *If* $\vdash \Diamond p \supset \Diamond(\alpha \cdot p)$ *where* p *is not contained in* α, *then* $\vdash \Box \alpha$.

Proof. (1) $\vdash \Diamond p \supset \Diamond(\alpha \cdot p)$ (p is not contained in α) [Hypothesis]
(2) $\vdash \sim\Diamond(\alpha \cdot p) \supset \sim\Diamond p$ (p is not contained in α) [From (1) by R2,R3,R5]
(3) $\vdash \sim\Diamond\sim\sim(\alpha \cdot \sim p) \supset \sim\Diamond\sim p$ (p is not contained in α) [From (2) by R2,R3,R5]

(4) $\vdash\Box(\alpha \supset p) \supset \Box p$ (p is not contained in α)
[From (3) by R2,R3,R5]

(5) $\vdash\Box(\alpha \supset \alpha) \supset \Box\alpha$ [From (4) by R3]

(6) $\vdash\Box(\alpha \supset \alpha)$ [By R1,R3]

(7) $\vdash\Box\alpha$ [From (5), (6) by R4]

3. We call L_0 the system obtained from L' with the restriction that if $\Box\alpha$ is a formula of L_0 then α does not contain $\Box$. We shall discuss the completeness of L_0 in the following.

We write $\alpha, \beta, \gamma, \ldots$ for the formulas of L_0 which do not contain $\Box$. We write $\alpha', \beta', \gamma', \ldots$ for formulas of L_0. ($\alpha', \beta', \ldots$ are composed of propositional variables and $\Box\alpha, \Box\beta, \ldots$ with $\sim, \cdot, \vee, \supset$.) Let γ be an arbitary formula not containing $\Box$. We call a "γ-valuation" a manner of truth value assignments to all the respective formulas which satisfies the following condition:

$$\text{truth value of } \Box\alpha = \begin{cases} \text{true, if } \gamma \supset \alpha \text{ is a tautology;} \\ \text{false, otherwise,} \end{cases}$$

where α is an arbitrary formula not containing $\Box$, and γ is called "axiom". (Truth value of $\Box\alpha$ depends only on γ, α and is independent of γ-valuation.) For a formula α', the following definition is given.

α' is a γ-tautology, if and only if, for a fixed γ, α' is true for all γ-valuation. (α, which does not contain $\Box$, is a γ-tautology if and only if α is a tautology.)

We now state the following theorems.

Theorem 1. *If α' is provable in L_0, then α' is a γ-tautology for all γ.*

Theorem 2. *If α' is a γ-tautology for all γ, then α' is provable in* L_0.

4. Let us mention the following lemmas for the sake of the proof of the above theorems.

Lemma 1. *$\alpha' \cdot \beta'$ is γ-tautology, if and only if α', β' are γ-tautologies.*

The proof is evident.

Lemma 2. *If δ does not contain $\Box$ and*

$$\text{(I)} \quad \sim\Box\alpha_1 \vee \sim\Box\alpha_2 \vee \ldots \vee \sim\Box\alpha_m \vee \Box\beta_1 \vee \Box\beta_2 \vee \ldots \vee \Box\beta_n \vee \delta$$

is a γ-tautology, then

$$\text{(II)} \quad \sim\Box\alpha_1 \vee \sim\Box\alpha_2 \vee \ldots \vee \sim\Box\alpha_m \vee \Box\beta_1 \vee \Box\beta_2 \vee \ldots \vee \Box\beta_n$$

is a γ-tautology or δ is a tautology.

Proof. If δ is not a tautology, then there exist a γ-valuation by which δ is false. For such a γ-valuation, (II) is true. Therefore, (II) is a

γ-tautology, because truth value of (II) depends only on $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_n, \gamma$ and is independent of γ-valuation, Q.E.D.

Lemma 3.

(II) $\sim\Box\alpha_1 \vee \sim\Box\alpha_2 \vee \ldots \vee \sim\Box\alpha_m \vee \Box\beta_1 \vee \Box\beta_2 \vee \ldots \vee \Box\beta_n$ *is a* γ*-tautology for all* γ,

if and only if,

(III) $(\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_m) \supset \beta_i$ *is a tautology for some* i $(1 \leqq i \leqq n)$.

Proof. When (II) is a γ-tautology for all γ, let us take $\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_m$ as axiom γ. (if $m = 0$ then it means a tautology). Then since

$$\Box\alpha_1, \Box\alpha_2, \ldots, \Box\alpha_m$$

are true,

$$\Box\beta_1 \vee \Box\beta_2 \vee \ldots \vee \Box\beta_n$$

is true. Therefore for some $i(1 \leqq i \leqq n)$ $\Box\beta_i$ is true, that is, (III) is a tautology.

If (II) is not a γ-tautology for some γ, (II) is false for some γ, say for γ_1. Then for γ_1 $\Box\alpha_i (i = 1, \ldots, m)$ is true for all i and for γ_1 $\Box\beta_i (i = 1, \ldots, n)$ is false for all i. Therefore for all i $\gamma_1 \supset \alpha_i$ is a tautology. Accordingly $\gamma_1 \supset (\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_m)$ is a tautology. On the other hand, for all i $\gamma_1 \supset \beta_i$ is not a tautology. Therefore, for all i (III) is not a tautology. Thus, if (II) is not a γ-tautology for some γ, then (III) is not a tautology for all i. Consequently if (III) is a tautology for some i, then (II) is a γ-tautology for all γ. Q.E.D.

5. Proof of Theorem 1. As to the axiom

$$\text{A. } \Box(p \supset q) \supset (\Box p \supset \Box q)$$

the following holds. If $\gamma \supset (p \supset q)$ and $\gamma \supset p$ are both tautologies, then $\gamma \supset q$ is a tautology. Therefore A is a γ-tautology for all γ. As to R1, the following holds. If α is a tautology, then $\gamma \supset \alpha$ is a tautology. Therefore, if α is a tautology, then $\Box\alpha$ is a γ-tautology. As to R3, the following holds. An arbitrary formula α' is reduced to the conjunction of the formulas of the following form.

$$\text{(I)} \quad \sim\Box\alpha_1 \vee \sim\Box\alpha_2 \vee \ldots \vee \sim\Box\alpha_m \vee \Box\beta_1 \vee \Box\beta_2 \vee \ldots \vee \Box\beta_n \vee \delta$$

where δ does not contain $\Box$.

In case δ in (I) is a tautology, even if any formula is substituted for a propositional variable in (I), then (I) remains a γ-tautology for all γ. From Lemma 1 and Lemma 2, therefore, for the proof as to R3, it is sufficient to consider

$$\text{(II)} \quad \sim\Box\alpha_1 \vee \sim\Box\alpha_2 \vee \ldots \vee \sim\Box\alpha_m \vee \Box\beta_1 \vee \Box\beta_2 \vee \ldots \vee \Box\beta_n$$

as α'. If (II) is a γ-tautology for all γ, then, from Lemma 3, for some i

$$\text{(III)} \quad (\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_m) \supset \beta_i \ (1 \leqq i \leqq n)$$

is a tautology. Even if any formula, which does not contain $\Box$, is substituted for a propositional variable contained in (III), then (III) remains a tautology. From Lemma 3, therefore, (II) remains a γ-tautology for all γ, even if any formula, which does not contain $\Box$, is substituted for a propositional variable contained in (II). Further as to the other rules similar results hold evidently. Q.E.D.

6. Proof of Theorem 2. An arbitrary formula α' is reduced to the conjunction of the formulas of the form.

$$\text{(I)} \quad \sim\Box\alpha_1 \vee \sim\Box\alpha_2 \vee \ldots \vee \sim\Box\alpha_m \quad \Box\beta_1 \vee \Box\beta_2 \vee \ldots \vee \Box\beta_n \vee \delta$$

where δ does not contain $\Box$. If δ in (I) is a tautology, then (I) is provable in L_0. From Lemma 1 and Lemma 2, therefore, for the proof of Theorem 2, it is sufficient to consider

$$\text{(II)} \quad \sim\Box\alpha_1 \vee \sim\Box\alpha_2 \vee \ldots \vee \sim\Box\alpha_m \vee \Box\beta_1 \vee \Box\beta_2 \vee \ldots \vee \Box\beta_n$$

as α'. If (II) is a γ-tautology, from Lemma 3, for some i

$$(\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_m) \supset \beta_i (1 \leqq i \leqq n)$$

is a tautology. Accordingly

$$\alpha_1 \supset (\alpha_2 \supset (\ldots (\alpha_m \supset \beta_i) \ldots))$$

is a tautology too. Therefore

$$\Box\alpha_1 \supset (\Box\alpha_2 \supset (\ldots (\Box\alpha_m \supset \Box\beta_i) \ldots))$$

that is

$$\sim\Box\alpha_1 \vee \sim\Box\alpha_2 \vee \ldots \vee \sim\Box\alpha_m \vee \Box\beta_i$$

is provable in L_0, and so (II) is provable in L_0.

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