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ON PROVABLE RECURSIVE FUNCTIONS

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A provable recursive function is, roughly, a recursive function which is not only a total function (on the natural numbers), but can even be *proven* to be total in some formal system, e.g. first-order Peano arithmetic. In this note we discuss the problem of making this definition precise. See the references for discussion of the properties of this class of functions.

In [2] Fischer proposes that a recursive function f be called *provable* in the formal system $\mathbf{S}$ if there is an index e of f such that

$$\vdash_{\mathbf{S}} \forall x \exists y \exists z \, \mathbf{M}(e, x, y, z)$$

where $\mathbf{M}$ is a formula which binumerates (i.e. numeralwise expresses) in $\mathbf{S}$ the primitive recursive relation holding of a, b, c, d iff the a'*th* partial recursive function applied to input b gives output c in not more than d steps. The problem here is: Which such $\mathbf{M}$? If the notion does not depend on the choice of $\mathbf{M}$, the problem vanishes. But that is far from true. For simplicity assume that the formal system in question is in fact first-order Peano arithmetic $\mathbf{P}$.

Theorem: (a) *We can choose* $\mathbf{M}$ *as above such that no function is provable.*
(b) *For any total recursive* f *we can choose* $\mathbf{M}$ *as above such that* f *is provable.*

Proof: (a) Let $\mathbf{M}_0(w, x, y, z)$ be your first choice for $\mathbf{M}$. Let $\mathbf{A}(x)$ be a formula such that

$$\vdash_{\mathbf{P}} \mathbf{A}(n) \text{ for each } n, \text{ but } \nvdash_{\mathbf{P}} \forall x \, \mathbf{A}(x) \quad .$$

Let $\mathbf{M}(w, x, y, z)$ be

$$\mathbf{M}_0(w, x, y, z) \wedge \mathbf{A}(x) \quad .$$

This binumerates the same relation, but for any e

$$\nvdash_{\mathbf{P}} \forall x \exists y \exists z \, \mathbf{M}(e, x, y, z)$$

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(b) Let e be an index of f. This time take $\mathbf{M}(w, x, y, z)$ to be

$$\mathbf{M}_0(w, x, y, z) \vee [w \approx e \wedge \neg \exists y \exists z \, \mathbf{M}_0(e, x, y, z)] \; .$$

Since for every n

$$\vdash_{\mathbf{P}} \exists y \exists z \, \mathbf{M}_0(e, n, y, z) \; ,$$

$\mathbf{M}$ binumerates the same relation $\mathbf{M}_0$ does. But it is easy to see that for this e

$$\vdash_{\mathbf{P}} \forall x \exists y \exists z \, \mathbf{M}(e, x, y, z) \; .$$

So some further restriction on $\mathbf{M}$ is needed. For example one could restrict $\mathbf{M}$ to be a PR formula and then ask if the class of provable functions is well-defined. (For definition of PR and RE formulas, see [1, p. 53].) Of course one could even construct one specific $\mathbf{M}$ and make the definition of "provable" use that one formula. (In effect this is what has been done in the literature.) But surely such an extreme step is not really necessary; surely the class of provable recursive functions has some stability.

Another approach to defining the class of provable recursive functions is suggested by a reading of Kreisel's [5, p. 157]. A function f might have the property that for some formula $\mathbf{A}(x, y)$

$$\text{i)} \; f(a) = \mu b \models \mathbf{A}(a, b)$$
$$\text{ii)} \; \vdash_{\mathbf{P}} \forall x \exists y \, \mathbf{A}(x, y) \; .$$

(Here $\models \mathbf{A}(a, b)$ means of course that the sentence is true in the intended interpretation of the (applied) language.) But what further requirement should be placed on $\mathbf{A}$? An example of too weak a requirement is

iii) $\mathbf{A}$ defines a recursive set.

For if we let $\mathbf{B}(x, y)$ be

$$\mathbf{A}(x, y) \vee [y \approx O \wedge \neg (\exists z) \, \mathbf{A}(x, z)]$$

then automatically

$$\vdash_{\mathbf{P}} \forall x \exists y \, \mathbf{B}(x, y) \; .$$

But if $\models \forall x \exists y \, \mathbf{A}(x, y)$ then $\mathbf{A}$ and $\mathbf{B}$ define the same relation.

On the other hand it would be undesirable to impose restrictions which would force $\mathbf{A}$ to define a primitive recursive set. For the following is easily verified:

Proposition: (a) *A partial function f is of the form $f(x) \simeq \mu y \, \mathbf{P}(x, y)$, $\mathbf{P}$ primitive recursive iff the graph of f is a primitive recursive relation.*

(b) *Any primitive recursive function has a primitive recursive graph, but so do some total (recursive) functions which are not primitive recursive.*

(c) *Not every total recursive function has primitive recursive graph.*

So if we expected **A** to define a primitive recursive set then some recursive functions would be prevented from being provable, no matter how strong the axiomatic system.

A reasonable restriction would seem to be that **A** must be an RE formula [1]. This is at least as general as the first definition discussed, with **M** required to be PR. We suspect that it is in fact equivalent.

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