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NORMAL FORM GENERATION OF S5 FUNCTIONS VIA TRUTH FUNCTIONS

GERALD J. MASSEY

1. *Generating the Singulary* S5 *Functions.* The fact that the number of n-ary S5 functions (connectives, if you prefer) is equal to the number of m-ary truth functions, where $m = 2^n + n - 1$ (Carnap [2] p. 48), has led Canty and Scharle in [1] to pose the fascinating problem now to be described. Consider the schema

$$T(p, F(p)).$$

Does there exist an S5 function F such that, as T runs through the binary truth functions, the expression $T(p, F(p))$ generates all the singulary S5 functions? Let us call such a function F a *solution to the singulary-function-generation problem.* In [1], Canty and Scharle correctly state that the modal function $\otimes_1$ is a solution to the singulary-function-generation problem (we use the names introduced in Massey [4] for the S5 functions), where the semantics of $\otimes_1$ is given by the following complete set of truth tables (Cf. Massey [4] on using complete sets of truth tables to define the semantics of an S5 connective):

p	$\otimes_1\ p$
t	t

p	$\otimes_1\ p$
f	f

p	$\otimes_1\ p$
t	f
f	t

It will be helpful in the sequel to stack these tables on top of one another thus:

p	$\otimes_1\ p$
t	t
f	f
t	f
f	t

Clearly a singulary S5 function F is a solution to the singulary-function-generation problem if and only if, for each pair (α, β) of truth values, there

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is an S5 value assignment Σ to p such that $(p, F(p))$ come out respectively (α, β) under Σ. (Concerning S5 value assignments, see Kripke [3] or Massey [4] or [5].) But inspection of the above chart shows that, for each pair (α, β) of truth values, there is an S5 value assignment to p under which $(p, \otimes_1 p)$ come out (α, β) respectively. This proves that $\otimes_1$ is a solution to the singulary-problem. Moreover, any permutation of the rows of the above chart which leaves the first column unchanged defines an S5 function $F(p)$ which is also a (distinct) solution to the singulary-function-generation problem. Since there are four such permutations (and no other solutions), there are four solutions to the singulary-problem, viz. $\otimes_1 p$, $\otimes_2 p$, $\otimes_7 p$ and $\otimes_8 p$ which are respectively equivalent to *ALpKNpMp*, *ALNpKpMNp*, *ALpLNp*, and *KNLpMp*. (This corrects the claim made in Canty and Scharle [1] that there are only two solutions to the singularly-problem.) For example, one of the aforementioned permutations yields the chart:

p	$\otimes_2\ p$
t	f
f	t
t	t
f	f

which makes it manifest that $\otimes_2$ is a solution to the singulary-problem.

2. *Generating the Binary* S5 *Functions*. In [1] Canty and Scharle leave open the general problem of generating the n-ary S5 functions via the m-ary truth functions, where $m = 2^n + n - 1$. The general problem can be put thus. Consider the schema:

$$T(p_1, \ldots, p_n, F_1(p_1, \ldots, p_n), \ldots, F_k(p_1, \ldots, p_n))$$

where $k = m - n$. Does there exist a k-tuple $(F_1, \ldots, F_k)$ of n-ary S5 functions such that, as T runs through the m-ary truth functions, the above expression generates all the n-ary S5 functions? Let us refer to such a k-tuple $(F_1, \ldots, F\ )$ of n-ary S5 functions as a *solution to the n-ary-function-generation problem*. I will show that such solutions always exist (when $n > 1$, their number is astronomical) and will give effective instructions for finding them.

Clarity is perhaps best served by treating the binary-problem before tackling the general problem. The binary-problem asks for a triple (F_1, F_2, F_3) of binary S5 functions such that, as T runs through the quinary truth functions, the expression:

$$T(p, q, F_1(p, q), F_2(p, q), F_3(p, q))$$

generates all the binary S5 functions. To solve the problem, we begin by constructing the following chart formed by entering beneath $(p, q, F_1(p, q), F_2(p, q), F_3(p, q))$ each of the 32 quintuples of truth values:

p	q	$F_1(p, q)$	$F_2(p, q)$	$F_3(p, q)$
t	t	t	t	t
t	f	f	f	f
f	t	f	f	f
f	f	f	f	f
t	t	t	t	f
t	f	t	t	t
t	t	t	f	t
f	t	t	t	t
t	t	t	f	f
f	f	t	t	t
t	f	f	t	t
f	t	f	t	t
t	f	f	t	f
f	f	f	t	t
f	t	f	f	t
f	f	f	f	t
t	t	f	t	t
t	f	t	t	f
f	t	t	t	f
t	t	f	t	f
t	f	t	f	t
f	f	t	t	f
t	t	f	f	t
f	t	t	f	t
f	f	t	f	t
t	f	f	f	t
f	t	f	t	f
f	f	f	t	f
t	t	f	f	f
t	f	t	f	f
f	t	t	f	f
f	f	t	f	f

The value assignment portions of a complete set of binary truth tables contain 32 entries, each pair of truth values occurring the same number of times, viz. eight. Confining our attention to the first two columns of the above chart, we see that each pair of truth values occurs eight times. Hence it is possible to group those 32 pairs of truth values into the fifteen groups which make up the value assignment portions of a complete set of binary truth tables; in the above chart, this grouping is effected by the horizontal lines. (When constructing such a chart, one will find it expedient to take care to list the quintuples of truth values in such an order that the requisite grouping can be effected by means of horizontal lines.) Hence, when taken together with the first two columns, the third column of the above chart defines the semantics of a binary S5 function F_1. Similarly, columns four and five define binary S5 functions F_2 and F_3 respectively.

Furthermore, the triple (F_1, F_2, F_3) so defined is a solution to the binary-function-generation problem since, given any quintuple $(\alpha_1, \dots, \alpha_5)$ of truth values, there is an S5 value assignment to (p, q) under which $(p, q, F_1(p, q), F_2(p, q), F_3(p, q))$ come out respectively $(\alpha_1, \dots, \alpha_5)$, as the above chart makes evident. Moreover, any permutation of the rows of the above chart which leaves the first two columns unchanged determines a distinct solution to the binary-problem. As there are $(8!)^4$ such permutations, there are $(8!)^4$ or approximately 2.66×10^{18} solutions to the binary-function-generation problem. Since there are approximately 7.9×10^{28} triples of binary S5 functions, this means that about one of every 3×10^{10} triples of binary S5 functions is a solution to the binary-function-generation problem. Where (F_1, F_2, F_3) is an arbitrary solution to the binary problem, it is left to the reader to verify that F_1, F_2 and F_3 are distinct from one another, that each is a modal function, and that none of them is a vacuous extension of a singulary modal function (in the sense that, for some singulary modal function G, $F_i(p, q)$ is equivalent to $G(p)$ or to $G(q)$). It is mildly surprising to note that there are 8! solutions to the binary-problem in which F_1, F_2, and F_3 are all uniform modal connectives (see Massey [4] regarding uniform modal connectives).

3. *The General Problem.* We are now ready to deal with the general problem of generating the n-ary S5 functions from the m-ary truth functions ($m = 2^n + n - 1$) via the schema:

$$T(p_1, \dots, p_n, F_1(p_1, \dots, p_n), \dots, F_k(p_1, \dots, p_n))$$

where $k = m - n$. There are 2^{2^m} m-ary truth functions, hence 2^{2^m} n-ary S5 functions. Now the number of n-ary S5 functions is given by 2^r, where r is the number of rows in the tables of a complete set of n-ary truth tables. Hence, there are 2^m rows in the tales of a complete set of n-ary truth tables. Construct a chart C similar to the one above by entering all the m-tuples of truth values beneath these headings:

$p_1 p_2, \dots, p_n$	$F_1(p_1, \dots, p_n)$	, . . . ,	$F_k(p_1, \dots, p_n)$

Thus C will contain 2^m rows. Confining our attention to the first n columns of C, we see that there are 2^m occurrences of n-tuples of truth values, each possible n-tuple occurring as often as any other possible n-tuple of truth values. But that is precisely the stuff of which the value assignment portions of a complete set of n-ary truth tables are made! Hence we can group the rows of C in such a way that these 2^m n-tuples of truth values are collected into the $2^{2^n} - 1$ groups which become the value assignment portions of the $2^{2^n} - 1$ tables in a complete sent of n-ary truth tables. The chart that results, call it C', defines the semantics of a k-tuple $(F_1, \dots, F_k)$ of n-ary S5 functions which is a solution to the n-ary-function-generation-problem. Moreover, every permutation of the rows of C' which leaves the first n columns of C' unchanged yields a distinct solution to the n-ary problem. As there are no other solutions, this entails that there are $(a!)^b$ solutions to the n-ary-function-generation-problem, where, $a = 2^{(2n-1)}$ and $b = 2^n$. Where $(F_1, \dots, F_k)$ is an arbitrary solution to the n-ary-function-generation problem, it should be evident that all k functions are modal S5 functions, that each is distinct from the rest, and that none of them is a vacuous extension of a function of lesser degree.

As suggested in Canty and Scharle [1], the results of this paper supply an elegant normal form representation for S5 functions. They also illustrate the fruitfulness of Kripkean truth tables for the study of modal connectives.

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Michigan State University
East Lansing, Michigan