Notre Dame Journal of Formal Logic
Volume IX, Number 1, January 1968

COMPLETENESS WITHOUT THE BARCAN FORMULA

M. J. CRESSWELL

In [1] we proved the completeness of the modal predicate calculus T (S4,S5) with the *Barcan* formula. It is known[1] that in quantificational T and S4 without the Barcan formula as axiom it is not a theorem. Further there have been objections, mainly of a philosophical nature[2] raised against this formula when given its intended interpretation. In the semantics for systems with the Barcan formula (v.[1]) we assumed a single domain of individuals, which is unchanging in every world. To falsify the Barcan formula it would seem sufficient to give a semantics in which, for any given world there is a separate domain of individuals which 'exist' in that world. Such a semantics is given by Kripke in [7] (pp. 86-87). Briefly it adds to the notion of a quantificational T-model a function (which we shall call Q) which selects from the domain of individuals for any given world x_i a set of individuals $Q(x_i)$ which is the set of everything which exists in x_i. Unfortunately this semantics fails to verify the axiom; $(x)\alpha \supset \beta$ (where β differs from α only in having some individual symbol a wherever α has free x provided that no x free in α occurs within the scope of (a)). This axiom fails in a world x_i where a is assigned some individual not in $Q(x_i)$[3].

Kripke's solution ([7] p. 89) is to alter the quantification basis by universally closing every quantificational axiom. For it is clear that $(x)[(x)\alpha \supset \alpha]$ is universally valid on this semantics. However we can then no longer prove the Barcan formula even in S5 (v.[7] pp. 87, 88) and so the semantics of [7] does not reflect the omission of the Barcan formula from the standard T, S4, S5[4]. While this may shew[5] that these systems are inadequate it still leaves open the problem of finding a semantics which does characterize them exactly.

We define a quantificational T(S4,S5)-model (cf.[1] and [4]) as a quin- $<VWRQD>$ where W is a set of objects (worlds), R a reflexive relation over W, D a set of objects (individuals), Q a function taking as argument a member of W and as value of subset of D, such that for any x_jRx_i, $Q(x_i) \subseteq Q(x_j)$. (For an S4 model R is also transitive and for an S5 model an equivalence relation.) V is an assignment giving a formula α the value 1 or 0, or being undefined in any given world in W as follows:

Received March 22, 1966

i) If p is a propositional variable then for every $x_i \varepsilon W$, $V(px_i) = 1$ or 0
ii) Every individual variable a is assigned some member of D. (i.e. for some $u \varepsilon D$, $V(a) = u$)
iii) For n-adic predicate variable ϕ and n-tuple of individual variables $<a_1, \ldots, a_n>$ then provided $V(a_1), \ldots, V(a_n) \varepsilon Q(x_i)$ $V(\phi(a_1, \ldots, a_n)x_i) = 1$ or 0, otherwise $V(\phi(a_1, \ldots, a_n)x_i)$ is undefined.
iv) For any wff α and any $x_i \varepsilon W$, $V(\sim\alpha x_i) = 1$ iff $V(\alpha x_i) = 0$ and $V(\sim\alpha x_i) = 0$ iff $V(\alpha x_i) = 1$.
v) For any wffs α and β and any $x_i \varepsilon W$, $V(\alpha \vee \beta)x_i)$ is defined iff both $V(\alpha\, x_i)$ and $V(\alpha x_i)$ are defined and = 1 iff either $V(\alpha x_i) = 1$ or $V(\alpha x_i) = 1$, otherwise 0.
vi) If α is a wff and $x_i \varepsilon W$ then $V((x)\alpha\, x_i) = 1$ iff $V(\alpha' x_i) = 1$ for every α' differing from α only in having some variable a such that $V(a) \varepsilon Q(x_i)$ everywhere in place of free x in α. $V((x)\alpha x_i) = 0$ iff for some such α' $V(\alpha' x_i) = 0$, otherwise undefined.
vii) If α is a wff and $x_i \varepsilon W$ then $V(L\alpha x_i) = 1$ iff $V(\alpha x_j) = 1$ for every x_jRx_i, otherwise 0. (Since $Q(x_i) \subseteq Q(x_j)$ and since R is reflexive, $V(L\alpha x_i)$ will be defined iff $V(\alpha x_j)$ is defined for every x_jRx_i).[6]

Clearly for these requirements for a formula α with free variables $V(\alpha x_i)$ will be defined iff V assigns to each individual variable in α a member of $Q(x_i)$.

A formula α is T(S4,S5)-valid iff for every T(S4,S5)-model $<V\,W R\,Q\,D>$ and every $x_i \varepsilon W$, $V(\alpha x_i) = 1$ wherever $V(\alpha x_i)$ is defined. We shew that every axiom is valid and that the rules are validity-preserving. First consider substitution instances of **PC** tautologies.[7] If β is a **PC** tautology and β' is obtained from β by replacing each distinct propositional variable in β by some wff α then if $V(\beta' x_i) = 0$ for some $x_i \varepsilon W$ let V' be the assignment such that where α is the wff in β' replacing the variable p in β, $V'(px_i) = V(\alpha x_i)$. Since β' is a truth function we need only consider the assignment in x_i, hence $V'(\beta x_i) = 0$ contrary to the tautologousness of β. For $L\alpha \supset \alpha$ (LA1) suppose that for some $<V\,W R\,Q\,D>$ and some $x_i \varepsilon W$, $V((L\alpha \supset \alpha)x_i) = 0$. Then $V(\alpha x_j) = 1$ for every x_jRx_i (Since $Q(x_i) \subseteq Q(x_j)$ $V(\alpha x_j)$ is defined). Hence $V(\alpha x_i) = 1$ (R reflexive), hence $V((L\alpha \supset \alpha)x_i) = 1$ contrary to reductio hypothesis. Hence LA1 is valid. For $L(\alpha \supset \beta) \supset : L\alpha \supset L\beta$ (LA2) suppose that for some $<V\,W R\,Q\,D>$ and $x_i \varepsilon W$, $V[(L(\alpha \supset \beta \;\supset : L\alpha \supset L\beta)x_i] = 0$. Then $V(L\alpha x_i) = 1$ and $V(L\beta x_i) = 0$. Hence for some x_jRx_i $V(\beta x_j) = 0$. But $V(\alpha x_j) = 1$ (since x_jRx_i and since $Q(x_i) \subseteq Q(x_j)$ $V(\alpha x_j)$, is defined for every x_jRx_i). Hence $V((\alpha \supset \beta)x_j) = 0$, hence $V(L(\alpha \supset \beta)x_i) = 0$ contrary to reductio hypothesis. Hence LA2 is valid. If R is transitive then $L\alpha \supset LL\alpha$(LA3) is valid for suppose $V(LL\alpha x_i) = 0$, Then for some x_jRx_i, $V(L\alpha x_j) = 0$, hence for some x_kRx_j, $V(\alpha x_k) = 0$, hence since R is transitive, for some x_kRx_i, $V(\alpha x_k) = 0$, hence $V(L\alpha x_i) = 0$, contrary to reductio hypothesis. Hence LA3 is valid. For modus ponens if for some $<V\,W R\,Q\,D>$ and some $x_i \varepsilon W$: If $V(\beta\, x_i) = 0$ then for some V' making the same assignment to the free variables of β (which are all in $Q(x_i)$) as V but assigning members of $Q(x_i)$ to the other individual variables in α, $V'(\beta x_i) = 0$. But, if α is valid then $V'(\alpha x_i) = 1$. Hence $V'((\alpha \supset \beta)x_i) = 0$, contrary to the assumed validity of $(\alpha \supset \beta)$. For necessitation (LR1) if $V(\alpha x_i) = 1$ for every x_i for which $V(\alpha x_i)$ is defined then $V(\alpha x_j) = 1$ for every $x_j R x_i$, hence (since $V(L\alpha x_i)$ is defined iff $V(\alpha x_j)$ is defined for every $x_j R x_i$) $V(L\alpha x_i) = 1$ for every x_i for which $V(L\alpha x_i)$ is defined. Hence $L\alpha$ is valid. For $\forall_1$ if for any x_i, $V((x)\alpha \supset \alpha)x_i)$ is defined then every individual variable in $(x)\alpha \supset \alpha$ must be assigned some member of $Q(x_i)$. Hence if β differs from α only in having some variable a such that $V(a) \varepsilon Q(x_i)$ everywhere in place of free x in α (x not within the scope of (a)) then if $V((x)\alpha x_i) = 1$, $V(\beta x_i) = 1$. For $\forall_2$ if $V((\alpha \supset \beta)x_i) = 1$ for every assignment to x free in β but not in α of a member of $Q(x_i)$ then $V((\alpha \supset (x)\beta)x_i) = 1$. Hence if $(\alpha \supset \beta)$ is valid then $(\alpha \supset (x)\beta)$ is valid. Hence every theorem is valid.

But consider the Barcan formula. Take the model $<V\,W R\,Q\,D>$[8] where $W = \{x_1 x_2\}$, $(x_1 \neq x_2)$, $D = \{u_1 u_2\}$ $(u_1 \neq u_2)$, $Q(x_1) = \{u_1\}$, $Q(x_2) = \{u_1 u_2\}$, $R = \{<x_1 x_1> <x_2 x_1> <x_2 x_2>\}$, $V(x) = u_1$, for any individual variable a other than x $V(a) = u_2$. $V(\phi x x_1) = 1$, $V(\phi x x_2) = 0$, for $a \neq x$, $V(\phi a x_2) = 0$. ($V(\phi a x_1)$ is undefined since $V(a) \not\varepsilon Q(x_1)$). Since $V(\phi x x_1) = 1$ and $V(\phi x x_2) = 1$ then $V(L\phi x x_1) = 1$ (since $x_2 R x_1$ and $x_1 R x_1$) but $Q(x_1) = \{u_1\}$, hence $V((x)L\phi x x_1) = 1$. But $V(\phi y x_2) = 0$, hence $V((x)\phi x x_2) = 0$, (since $V(y) \varepsilon Q(x_2)$) hence $V(L(x)\phi x x_1) = 0$, hence $V[((x)L\phi x \supset L(x)\phi x)x_1] = 0$. Clearly $<V\,W R\,Q\,D>$ is an S4-model, hence the Barcan formula is not S4-valid (and so not T valid.).[9] However from the definition of R it is not an S5 model since $x_1 R x_2$ does not hold in it. In fact we can shew that where R is an equivalence relation the Barcan formula is valid. We first note that where R is symetrical we have that if $x_i R x_j$ then $Q(x_i) = Q(x_j)$. Suppose $(x)L\alpha \supset L(x)\alpha$ were not valid in S5. Then for some $<V\,W R\,Q\,D>$ and some $x_i \varepsilon W$ $V[((x)L\alpha \supset L(x)\alpha)x_i] = 0$, i.e. $V((x)L\alpha x_i) = 1$ and $V(L(x)\alpha x_i) = 0$. Hence $V((x)\alpha x_j) = 0$ for some $x_j R x_i$, hence $V(\alpha' x_j) = 0$ for some α' differing from α only in having some variable a such that $V(a) \varepsilon Q(x_j)$ everywhere in place of free x in α. But $x_j R x_i$, hence $Q(x_i) = Q(x_j)$. Hence $V(\alpha' x_j) = 0$ for some α' differing from α only in having some variable a such that $V(a) \varepsilon Q(x_i)$ everywhere in place of free x in α. Hence $V(L\alpha' x_i) = 0$ for some α' having a variable a such that $V(a) \varepsilon Q(x_i)$ everywhere in place of x free in α. Hence $V((x)L\alpha x_i) = 0$, contrary to reductio hypothesis, hence the Barcan formula is valid in S5.

To prove completeness we shall as in [1] use Henkin's device ([8] p. 162) of maximal consistent sets of formulae but we shall dispense with the 'C-forms' of [1]. Given some consistent closed[10] formula H of model **LPC** we shew that H is satisfiable (i.e. can be assigned 1 in some appropriate model). We add to the symbols of the modal **LPC** sets of individual constants as follows;

$$\mathbf{D}_1 = \{u_1^1, u_2^1, \ldots, u_k^1, \ldots\}$$
$$\mathbf{D}_i = \{u_1^i, u_2^i, \ldots, u_k^i, \ldots\}$$

Call the system made up from the symbols of the **LPC** and the constants of $\mathbf{D}_1$, $\mathbf{S}_1$. We let Γ_1 be a maximal consistent set of cwffs of $\mathbf{S}_1$ containing H

such that for every cwff $(\exists x)\alpha \ \varepsilon\ \Gamma_1$ there is also in Γ_1 a formula α' with some constant of $\mathbf{D}_1$ replacing free x everywhere in α.[11] Given a maximal consistent set of cwffs Γ_i we construct for every formula $M\alpha \ \varepsilon\ \Gamma_i$ a maximal consistent set Γ_j (containing α) of cwffs of $\mathbf{S}_i$ + all those made up using also the constants of $\mathbf{D}_j$ (call this system $\mathbf{S}_j$). Let the first member of Γ_j be α. α is consistent for if not $\vdash \sim\alpha$ hence $\vdash L \sim \alpha$, hence $\vdash \sim M\alpha$ contrary to the assumed consistency of Γ_i. Now for every cwff of the form $L\beta \ \varepsilon\ \Gamma_i$ add to Γ_j, β. Γ_j remains consistent for if not then for some $L\beta_1, \ldots, L\beta_k \ \varepsilon\ \Gamma_i$, $\vdash \sim (\beta_1, \ldots, \beta_k.\alpha)$, hence $\vdash \sim M(\beta_1, \ldots, \beta_k.\alpha)$, hence $\vdash \sim (L\beta_1, \ldots, L\beta_k.M\alpha)$ contrary to the assumed consistency of Γ_i. Then for every formula $(\exists x)\alpha \supset \alpha$ of $\mathbf{S}_j$ add to Γ_j a formula of the form $(\exists x)\alpha \supset \alpha'$ where α' differs from α only in having some constant of $\mathbf{D}_j$ (not already occurring in Γ_j) everywhere in place of x free in α. The set remains consistent since the members of $\mathbf{D}_j$ do not occur in any cwff of $\mathbf{S}_i$, in particular do not occur in any $L\beta \ \varepsilon\ \Gamma_i$ and so not in any $\beta \ \varepsilon\ \Gamma_j$. Further by construction the constant does not occur in any earlier $(\exists x)\ \alpha \supset \alpha'$. Finally increase Γ_j to a maximal consistent set of cwffs of $\mathbf{S}_j$.

We now define an assignment to **LPC** formulae (i.e. formulae without constants) which satisfies H. Let W be a set $\{x_1, x_2, \ldots, x_i, \ldots\}$. Let $\mathbf{D}$ be the set of all the individual constants in any $\mathbf{D}_i$ and let each x_i be associated with some Γ_i, let $Q(x_i)$ be the set of all individual constants in $\mathbf{S}_i$ and let x_iRx_j iff Γ_j is a subordinate of Γ_i (i.e. has been constructed from some α such that $M\alpha \ \varepsilon\ \Gamma_i$). Clearly by the construction of each Γ_i if x_jRx_i then $Q(x_i) \subset Q(x_j)$. Let $V(p x_i) = 1$ or 0 according as $p \ \varepsilon\ \Gamma_i$. Let V make some assignment to the individual variables of members of $\mathbf{D}$. Where $V(a_1) = u_1, \ldots, V(a_n) = u_n$, then let $V(\phi(a_1, \ldots, a_n)x_i) = 1$ or 0 according as $\phi(u_1, \ldots, u_n) \ \varepsilon\ \Gamma_i$. Quantification and truth-functions are evaluated as usual. Clearly by induction on the construction of α we may prove as in [1] that $V(\alpha\, x_i) = 1$ or 0 according as $\alpha' \ \varepsilon\ \Gamma_i$. Where α' is obtained from α by replacing every free variable in α by the constant assigned to it by V.[12] Since each Γ_i is a maximal consistent set of cwffs of $\mathbf{S}_i$ and since any formula containing a variable assigned an individual constant not in $\mathbf{S}_i$ will be undefined in x_i, then V is a complete assignment for wffs of the modal **LPC**. Hence $V(Hx_1) = 1$ and so H is satisfiable. Hence every valid formula is a theorem.

The proof though does depend on introducing, for each $Q(x_i)$ a new set $\mathbf{D}_j$ of individuals which are not in $Q(x_i)$ and thus we have $Q(x_i) \subset Q(x_j)$ which does not hold in an S5-model. However since in any connected S5-model $Q(x_i) = Q(x_j)$ for every $x_i, x_j \ \varepsilon\ W$ the proof used in [1] will suffice for S5.

NOTES

1. cf. [2].

2. e.g. [3], pp. 26-28, and [4]. For a somewhat different objection v. [5], p. 80. For a defence of the formula v. [6], pp. 88-90.

3. This is not strictly correct since one might choose (as in fact we do *infra*) to define validity as truth in every world x_i for all assignments to the variables of members of $Q(x_i)$. However in this case certain other rules and axioms fail if we make no other changes in the semantics and as Kripke points out (p. 89, n. 1) one would need to alter either quantification theory or modal logic. What we are in fact doing is to modify Kripke's semantics so that we neither close the axioms nor modify modal theory.

4. Kripke proves (p. 89) the even stronger result that the converse of the Barcan formula (i.e. $L(x)\alpha \supset (x)L\alpha$) is not valid and this formula is a theorem of ordinary quantificational T and S4.

5. And by implication Kripke seems to think it does shew this. (v. p. 89, n. 1).

6. This provision is necesasry since otherwise either $V(L\alpha x_i)$ may fail to be defined even though all its variables are assigned members of $Q(x_i)$ or else may have a different truth value depending on whether $V(\alpha x_i)$ is defined in some particular x_jRx_i. For suppose we define $V(L\alpha x_i) = 1$ iff for every x_jRx_i for which $V(\alpha x_j) = 1$ and $V(L\alpha x_i) = 0$ iff $V(\alpha x_j) = 0$ for some x_jRx_i. Here $\check{V}(L\alpha x_i)$ will certainly be defined iff $V(\alpha x_i)$ is defined but in this case a substitution instance of a T axiom can be falsified. Consider $L(\alpha \supset \beta) \supset : L\alpha \supset L\beta$. $(L(\sim \phi x \supset \sim (x)\phi x) \supset : L\sim \phi x \supset L(x)\phi x$ is false in the following model $<V W R Q D>$; $W = \{x_1 x_2\}$. $D = \{u_1 u_2\}$ x_iRx_j for every $x_i, x_j \varepsilon W$, $Q(x_1) = \{u_1\}, Q(x_2) = \{u_2\}$, $V(x) = u_1$, for individual variable a other than x $V(a) = u_2$, $V(\phi x\, x_1) = 0$, $V(\phi a\, x_2) = 1$ (all other cases will be undefined). Hence $V((\sim\phi x \supset \sim(x)\phi x)x_1) = 1$, $V((\sim\phi x \supset \sim(x)\phi x)x_2)$ is undefined, hence $V(L(\sim\phi x \supset \sim(x)\phi x)x_1) = 1$. Now $V(\sim\phi x\, x_1) = 1$ and $V(\sim\phi x\, x_2)$ is undefined, hence $V(L\sim\phi x\, x_1) = 1$. But $V(\phi a\, x_2) = 1$ for every a such that $V(a)\ \varepsilon\ Q(x_2)$ hence $V((x)\phi x\, x_2) = 1$, hence $V(\sim (x)\phi x\, x_2) = 0$, hence $V(L\sim(x)\phi x\, x_1) = 0$. Hence $V[(L(\sim\phi x \supset \sim(x)\phi x) \supset : L\sim\phi x \supset L\sim(x)\phi x\, x\ ] = 0$.

   Under interpretation an intuitive justification for $Q(x_i) \subseteq Q(x_j)$ could perhaps proceed by claiming that anything which actually exists in a given world possibly exists in that world, i.e. exists in every world possible relative to that world.

7. In view of the falsity in some world of a substitution instance of LA2 on the definition given in footnote 6 it behoves us to prove the validity of each axiom schema of the propositional basis with the range of the wffs explicitly intended to include quantificational formulae.

8. This is adapted from the model used in [7] p. 87 by changing slightly the definition of R and, of course, allowing for cases where a formula with free variables is not assigned a truth value.

9. This constitutes a semantic independence proof of the Barcan formula in T and S4. The models of [7] do not verify all theorems and so cannot be used for this purpose.

10. It is convenient to consider only closed formulae although this is not strictly necessary since for a given assignment a formula with variables may be regarded as constant. However the restriction does not matter since universal generalization is both valid and provable.

11. For the construction of such a set v. [1] or [8].

12. And as in [1] where each set is maximal consistent in S4 then the model can be one in which R is transitive.

REFERENCES

[1] Cresswell, M. J., A Henkin Completeness Theorem for T, *Notre Dame Journal of Formal Logic*, Vol 8 (1967), pp. 186-190.

[2] Lemmon, E. J., Quantified S4 and the Barcan Formula, *The Journal of Symbolic Logic*, vol 24 (1960), pp. 391-392.

[3] Prior, A. N., *Time and Modality*, Oxford 1957.

[4] Hintikka, K. J. J., Modality and Quantification, *Theoria*, vol 27 (1961), pp. 110-128.

[5] Myhill, J. R., Problems arising in the Formalization of Intensional Logic, *Logique et Analyse* 1 (1958), pp. 74-83.

[6] Barcan-Marcus, Ruth, Modal Logics I, Modalities and Intensional Languages, *Boston Studies In the Philosophy of Science*, Proceedings of the Colloquium of 1961/62, pp. 97-104.

[7] Kripke, Saul A., Semantical Considerations on Modal Logics, *Acta Philosophica Fennica*, fasc. 16 (1963) (*Modal and Many-Valued Logic*), pp. 83-94.

[8] Henkin, Leon, The Completeness of the First-Order Functional Calculus, *The Journal of Symbolic Logic*, vol 14 (1949), pp. 159-166.

Victoria University of Wellington
Wellington, New Zealand