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NOTE ON DEFINITIONAL REDUCTIONS

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In his theory of definitions in formal systems, Curry[2] has given two different forms of the rule Rd. They are both restrictions of

(1) $$X \mathsf{D} Y \ \& \ \Phi(A_1, \ldots, A_m) \mathsf{D} Z \Rightarrow X \mathsf{D} Y',$$

where Y' is formed from Y by replacing one occurrence of $\Phi(A_1, \ldots, A_m)$ by Z. The older restriction, used in [CLg] and [DFS] is that Z be a basic ob. I will call the rule with this restriction Rd'. The other restriction, used in [FML], is that $\Phi(A_1, \ldots, A_m) \mathsf{D} Z$ be one of the defining axioms. I will call the rule with this latter restriction Rd*. The equivalence of these two rules was apparently taken for granted in Curry's work. The purpose of this note is to verify this equivalence. It turns out that Rd* is slightly more general than Rd', but that the two are precisely equivalent for reductions to ultimate definienda. My basic notation is that of Curry in the papers referred to above. I will use '$\mathfrak{A}$' and '$\mathfrak{B}$' to stand for sequences of basic obs.

Lemma. Suppose

(2) $$\frac{X \mathsf{D} Y \quad \Phi_1(\mathfrak{A}_1) \mathsf{D} B}{X \mathsf{D} Y'}$$

is a contraction by Rd', *and suppose that the minor premise is the conclusion of a reduction* $\mathfrak{D}$ *using* Rd*. *Then there is a reduction*

(3) $$X \mathsf{D} Y \Rightarrow X \mathsf{D} Y'$$

using only Rd* *and having exactly one step more than* $\mathfrak{D}$.

Proof: There is no loss of generality in supposing that $\mathfrak{D}$ be standard (see [FML], p. 108). Then $\mathfrak{D}$ must be of the form

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$$\text{(4)}\qquad \begin{array}{c} \dfrac{\Phi_1(\mathfrak{A}_1) \supset Z_1 \quad \Phi_2(\mathfrak{A}_2) \supset Z_2}{\Phi_1(\mathfrak{A}_1) \supset U_2 \quad \Phi_3(\mathfrak{A}_3) \supset Z_3} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\Phi_1(\mathfrak{A}_{n-1}) \supset U_{n-1} \quad \Phi(\mathfrak{A}_n) \supset Z_n}{\Phi_1(\mathfrak{A}_1) \supset U_n,} \end{array}$$

where U_n is B, and if U_1 is defined to be Z_1, then for $i = 1, \ldots, n - 1$, U_{i+1} is obtained from U_i by replacing an occurrence of $\Phi_{i+1}(\mathfrak{A}_{i+1})$ by Z_{i+1}, and where for each i, $\Phi_i(\mathfrak{A}_i) \supset Z_i$ is a defining axiom. Now define the sequence $Y_1, \ldots, Y_n$ as follows: Y_1 is obtained from Y by replacing the occurrence of $\Phi_1(\mathfrak{A}_1)$ that was replaced by B in (2) by U_1, and each Y_{i+1} is obtained from Y_i by replacing the U_i introduced in the previous step by U_{i+1} for each $i = 1, \ldots, n - 1$. Then Y_n is Y', and further, the derivation

$$\begin{array}{c} \dfrac{X \supset Y \quad \Phi_1(\mathfrak{A}_1) \supset Z_1}{X \supset Y_1 \quad \Phi_2(\mathfrak{A}_2) \supset Z_2} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{X \supset Y_{n-1} \quad \Phi_n(\mathfrak{A}_n) \supset Z_n}{X \supset Y_n} \end{array}$$

is just the derivation (3), since, by the definition of the Y_i's, each Y_{i+1} is formed from Y_i by replacing an occurrence of $\Phi_{i+1}(\mathfrak{A}_{i+1})$ by Z_{i+1}. Further, this derivation has exactly one more step than the derivation $\mathfrak{D}$.

Theorem 1. *Every reduction by* Rd' *can be transformed into a reduction by* Rd*.

Proof. Begin at the top of the right-hand branch of the original reduction and apply the transformation specified in the proof of the lemma. After each application of this transformation, determine the place at which to carry out the next transformation as follows: if the major premise of the reduction (i.e., the major premise of (2) before the transformation is carried out) is the result of a reduction by Rd', then begin with the top of the rightmost branch leading out of it (which will be the rightmost branch of the original tree that has not yet been transformed), if the major premise is the result of a reduction by Rd*, then it will, after the transformation, be followed by a reduction by Rd*, and the conclusion of that reduction will either be the minor premise of a contraction by Rd' (in which case perform the transformation here) or else will be the conclusion of the original reduction, in which case the transformation of the entire original tree is completed.

Now consider a reduction by Rd*. Again, without any loss of generality, assume the reduction in standard. Then it is possible to partition the reduction into smaller reductions, each of which takes a component of the right-hand side of the major premise and reduces it as far as it is reduced in the original reduction. Clearly, if this reduction does not result in the replacement of this component by a basic ob, then the reduction cannot be transformed into a reduction using Rd'. On the other hand, if each component that is contracted at all in the reduction is eventually reduced to a basic ob, then the transformation carried out in Theorem 1 can be reversed (i.e., a transformation is carried out using the reverse of the transformation of the lemma and in the opposite order to that of Theorem 1). This proves

Theorem 2. *A reduction using* Rd* *can be transformed into a reduction by* Rd' *if and only if each component that is reduced at all in the reduction is reduced to a basic ob.*

Corollary. Every reduction to an ultimate definiens by Rd* *can be transformed into a reduction to the same ultimate definiens by* Rd'.

Proof: Any new ob that is not reduced in the reduction to a basic ob will have to appear in the conclusion on the right, since only basic obs can appear in the argument places of the left side of the minor premise, and hence any new ob that is not further reduced cannot be eliminated in the reduction.

This shows that the rules Rd* and Rd' are equivalent in the sense that any operation that can be defined in a definitional extension using one of these rules can be defined in a definitional extension using the other.

FOOTNOTES

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2. See [CLg], pp. 62-74, [DFS], and [FML], pp. 106-111. For an explanation of the letters in brackets, see the bibliography.

BIBLIOGRAPHY

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