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A NOTE ON REFLEXIVENESS

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The job of working through the algebra of relations is not made any easier by the fact that authors frequently use key terms differently. This is to be regretted; however, the important thing is that you be able to keep these usages straight in your own mind. What follows then is an attempt to get straight on one particular aspect of the algebra of relations—the notion of reflexiveness.

The most studied properties of relations are relexivity, symmetry, and transitivity. Variances appear from the very start; for whereas all authors define "symmetric" and "asymmetric" as:

$$\mathsf{Sym}\ R =_{df} (x)(y)(xRy \rightarrow yRx)$$
$$\mathsf{Asym}\ R =_{df} (x)(y)(xRy \rightarrow \sim yRx),$$

some authors (Carnap, e.g.) define "nonsymmetry" simply as "$\sim \mathsf{Sym}\ R$", and others (Copi, e.g.) define it "$\sim \mathsf{Sym}\ R\ \&\ \sim \mathsf{Asym}\ R$". And there is an analogous variance in the transitivity triad. For whereas all authors define "transitive" and "intransitive" as:

$$\mathsf{Tra}\ R =_{df} (x)(y)(z)\ [(xRy\ \&\ yRz) \rightarrow xRz]$$
$$\mathsf{Intra}\ R =_{df} (x)(y)(z)\ [(xRy\ \&\ yRz) \rightarrow \sim xRz],$$

some authors (again Carnap) define "nontransitive" simply as "$\sim \mathsf{Tra}\ R$", and others (again Copi) define it "$\sim \mathsf{Tra}\ R\ \&\ \sim \mathsf{Intra}\ R$". The reasons for adopting one version of nonsymmetry (or nontransitivity) instead of the other are most likely pragmatic, but we shall let this pass.

The variance already noted, of course, has an analogy in the reflexivity group. Some authors define "nonreflexive" simply as "$\sim \mathsf{Refl}\ R$", while others define it "$\sim \mathsf{Refl}\ R\ \&\ \sim \mathsf{Irrefl}\ R$". Yet the map is smudged even more by variant definitions of "$\mathsf{Refl}\ R$". Here are the most frequently used definitions of reflexiveness: (giving just the definiens)

(A) $(x)xRx$
(B) $(x)(y)\,[xRy \rightarrow (xRx\ \&\ yRy)]$
(C) $(x)\,[(\exists y)(xRy \vee yRx) \rightarrow xRx]$.

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(D) $(x)(y)(xRy \rightarrow xRx)$
(E) $(x)(y)(xRy \rightarrow yRy)$

To my mind, it is pointless to ask 'Which definition is correct?' It is to the point, however, to enquire into the logical relationships among the definitions. Thus, first of all, it can be shown that definitions B (Quine) and C (Copi) are logically equivalent, viz.,

(T1) $(x)(y)[xRy \rightarrow (xRx \;\&\; yRy)] \dashv\vdash (x)[(\exists y)(xRy \vee yRx) \rightarrow xRx]$

Hence C will be eliminated in all that follows, and we are left then with four kinds (or, better, degrees) of reflexiveness. In order to keep them separate, they are now given distinct names and complete definitions:

(D1) Totrefl $R =_{df} (x)xRx$ (totally reflexive)
(D2) Birefl $R =_{df} (x)(y)[xRy \rightarrow (xRx \;\&\; yRy)]$ (bireflexive)
(D3) R-refl $R =_{df} (x)(y)(xRy \rightarrow yRy)$ (right reflexive)
(D4) L-refl $R =_{df} (x)(y)(xRy \rightarrow xRx)$ (left reflexive)

By the method of interpretation it can be shown that no two of these definitions are logically equivalent. Consider the following three interpretations:

(Ia) U: positive integers xRy: x is a prime factor of y
(Ib) U: positive integers xRy: x is prime factored by y
(Ic) U: sentences xRy: x entails y

According to Ia, D4 is true and the other D's are all false. According to Ib, D3 is true and both D2 and D1 are false. And according to Ic, D2 is true while D1 is false. Thus it is proved that no two of the definitions (D1-D4) are logically equivalent.

Nevertheless, these degrees of reflexiveness are bound together by several entailments. The two most basic are:

(T2) Totrefl $R \vdash$ Birefl R
(T3) Birefl $R \dashv\vdash$ R-refl R & L-refl R

Furthermore, should an arbitrary relation display one specific property, then for *that* relation the four degrees of reflexiveness collapse remarkably. And the collapse is complete if both the relation and its converse (denoted by R°) display this property. For lack of a better word I have named this property "density". It is defined as follows:

(D5) Den $R =_{df} (x)(\exists y)xRy$

If a relation or its converse is dense, then bireflexiveness collapses to total reflexiveness. That is,

(T4) Den $R \vee$ Den $R^{\circ} \vdash$ Totrefl $R \leftrightarrow$ Birefl R

If a relation is dense, then it is left reflexive only if it is right reflexive. That is,

(T5) Den $R \vdash$ L-refl $R \rightarrow$ R-refl R

If the converse of a relation is dense, then the relation itself is right reflexive only if it is left reflexive. That is,

(T6) $\text{Den } R^{\circ} \vdash \text{R-refl } R \rightarrow \text{L-refl } R$

Finally, if a relation and its converse are both dense, the collapse is complete. That is,

(T7) $\text{Den } R \;\&\; \text{Den } R^{\circ} \vdash (\text{L-refl } R \leftrightarrow \text{R-refl } R) \;\&$
$(\text{L-refl } R \leftrightarrow \text{Birefl } R) \;\&$
$(\text{L-refl } R \leftrightarrow \text{Totrefl } R) \;\&$
$(\text{R-refl } R \leftrightarrow \text{Birefl } R) \;\&$
$(\text{R-refl } R \leftrightarrow \text{Totrefl } R) \;\&$
$(\text{Birefl } R \leftrightarrow \text{Totrefl } R)$

In the universe of real numbers most relations (at least the typical relations with which I am familiar) and their converses are dense; that is, e.g., '>', '<', '=', 'precedes', 'is a prime factor of'. (An exception is 'is a factor of' which is not dense, though its converse is dense.) Thus mathematically inclined logicians (such as Tarski) are free to choose any one of these four definitions of '$\text{Refl } R$'. Quite naturally, Tarski uses D1 since it is the simplest. Quine, having in mind a broader universe, opts for D2. Anderson and Johnstone (perhaps wanting both scope and simplicity) choose D4, but unfortunately they fail to take note of D3.

The question "how ought one to define 'reflexiveness'?" is best answered "According to one's purposes". What I have done here is discuss the alternatives and map (begin to map) their connections.

Proofs: Natural Deduction Technique

Key:

(a) The extreme left column is assumption dependence, after Lemmon.
(b) Abbreviations: **A** (assumption), **QE** (quantifier exchange), Π**E** (universal quantifier elimination), Σ**E** (existential quantifier elimination), **CD** (conditionalization), **TF** (truth functional reasoning), **DF** (definition exchange), **RAA** (reductio ad absurdum).
(c) Note that in the proofs 'xRy' is written 'Rxy'.

(T1) $\Pi x \Pi y CRxyKRxxRyy \dashv\vdash \Pi x C \Sigma y ARxyRyxRxx$

(a) $\Pi x \Pi y CRxyRxxRyy \vdash \Pi x C \Sigma y ARxyRyxRxx$

1	(1)	$\Pi x \Pi y CRxyKRxxRyy$	A
2	(2)	$N \Pi x C \Sigma y ARxyRyxRxx$	A
2	(3)	$\Sigma x N C \Sigma y ARxyRyxRxx$	2 QE
2	(4)	$NC \Sigma y ARayRyaRaa$	3 ΣE
2	(5)	$\Sigma y ARayRya$	4 TF
2	(6)	$ARabRba$	5 ΣE
1	(7)	$CRabKRaaRbb$	1 ΠE (2)
1	(8)	$CRbaKRbbRaa$	1 ΠE (2)
1, 2	(9)	$KRaaNRaa$	4, 6, 7, 8 TF
1	(10)	$\Pi x C \Sigma y ARxyRyxRxx$	2-9 RAA

(b) $\Pi x C \Sigma y A R x y R y x R x x \vdash \Pi x \Pi y C R x y K R x x R y y$

1	(1)	$\Pi x C \Sigma y A R x y R y x R x x$	A
2	(2)	$N \Pi x \Pi y C R x y K R x x R y y$	A
2	(3)	$\Sigma x \Sigma y N C R x y K R x x R y y$	2 QE (2)
2	(4)	$N C R a b K R a a R b b$	3 ΣE (2)
1	(5)	$C \Sigma y A R a y R y a R a a$	1 ΠE
1	(6)	$C \Sigma y A R b y R y b R b b$	1 ΠE
1	(7)	$N \Sigma y A R a y R y a$	A
7	(8)	$\Pi y N A R a y R y a$	7 QE
7	(9)	$N A R a b R b a$	8 ΠE
	(10)	$C(7)(9)$	CD
11	(11)	$N \Sigma y A R b y R y b$	A
11	(12)	$\Pi y N A R b y R y b$	11 QE
11	(13)	$N A R b a R a b$	12 ΠE
	(14)	$C(11)(13)$	CD
1, 2	(15)	$K R a b N R a b$	4, 5, 6, 10, 14 TF
1	(16)	$\Pi x \Pi y C R x y K R x x R b b$	2-15 RAA

(T2) Totrefl $R \vdash$ Birefl R

1	(1)	Totrefl R	A
2	(2)	$N \Pi x \Pi y C R x y K R x x R y y$	A
2	(3)	$\Sigma x \Sigma y N C R x y K R x x R y y$	2 QE (2)
2	(4)	$N C R a b K R a a R b b$	3 ΣE (2)
1	(5)	$\Pi x R x x$	1 DF (D1)
1	(6)	$R a a$	5 ΠE
1	(7)	$R b b$	5 ΠE
1, 2	(8)	$K K R a a R b b N K R a a R b b$	4, 6, 7 TF
1	(9)	$\Pi x \Pi y C R x y K R x x R y y$	2-8 RAA
1	(10)	Birefl R	9 DF (D2)

(T3) Birefl $R \dashv\vdash K$(R-refl R)(L-refl R)

(a) K(R-refl R)(L-refl R) $\vdash$ Birefl R

1	(1)	K(R-refl R)(L-refl R)	A
2	(2)	$N \Pi x \Pi y C R x y K R x x R y y$	A
2	(3)	$\Sigma x \Sigma y N C R x y K R x x R y y$	2 QE (2)
2	(4)	$N C R a b K R a a R b b$	3 ΣE (2)
1	(5)	$K \Pi x \Pi y C R x y R y y \Pi x \Pi y C R x y R x x$	1 DF (D3 & D4)
1	(6)	$\Pi x \Pi y C R x y R y y$	5 TF
1	(7)	$\Pi x \Pi y C R x y R x x$	5 TF
1	(8)	$C R a b R b b$	6 ΠE (2)
1	(9)	$C R a b R a a$	7 ΠE (2)
1, 2	(10)	$K K R a a R b b N K R a a R b b$	4, 8, 9 TF
1	(11)	$\Pi x \Pi y C R x y K R x x R y y$	2-10 RAA
1	(12)	Birefl R	11 DF (D2)

(b) Birefl $R \vdash K$(R-refl R)(L-refl R)

1	(1)	Birefl R	A

2	(2)	$N\Pi x\Pi yCRxyRyy$	A
2	(3)	$\Sigma x\Sigma yNCRxyRyy$	2 QE (2)
2	(4)	$NCRabRbb$	3 ΣE (2)
1	(5)	$\Pi x\Pi yCRxyKRxxRyy$	1 DF (D2)
1	(6)	$CRabKRabRbb$	5 ΠE (2)
1, 2	(7)	$KRbbNRbb$	4, 6 TF
1	(8)	$\Pi x\Pi yCRxyRyy$	2-7 RAA
1	(9)	R-refl R	8 DF (D3)
10	(10)	$N\Pi x\Pi yCRxyRxx$	A
10	(11)	$\Sigma x\Sigma yNCRxyRxx$	10 QE (2)
10	(12)	$NCRcdRcc$	11 ΣE (2)
1	(13)	$CRcdKRccRdd$	5 ΠE (2)
1, 10	(14)	$KRccNRcc$	12, 13 TF
1	(15)	$\Pi x\Pi yCRxyRxx$	10-14 RAA
1	(16)	L-refl R	15 DF (D4)
1	(17)	K(R-refl R)(L-refl R)	9, 16 TF

For T4 (also T6) we need to have the concept of *converse relation* stated formally. This is best done axiomatically:

Ax R° $\Pi x\Pi yER^{\circ}xyRyx$

(T4) A(Den R)(Den R°) $\vdash E$(Totrefl R)(Birefl R)

1	(1)	A(Den R)(Den R°)	A
	(2)	C(Totrefl R)(Birefl R)	From T2 directly
3	(3)	Birefl R	A
4	(4)	$N\Pi xRxx$	A
4	(5)	$\Sigma xNRxx$	4 QE
4	(6)	$NRaa$	5 ΣE
7	(7)	Den R	A
7	(8)	$\Pi x\Sigma yRxy$	7 DF (D5)
7	(9)	$\Sigma yRay$	8 ΠE
7	(10)	Rab	9 ΣE
3	(11)	$\Pi x\Pi yCRxyKRxxRyy$	3 DF (D2)
3	(12)	$CRabKRaaRbb$	11 ΠE (2)
3, 4, 7	(13)	$KRaaNRaa$	6, 10, 12, TF
3, 4	(14)	N(Den R)	7-13 RAA
1, 3, 4	(15)	Den R°	1, 14 TF
1, 3, 4	(16)	$\Pi x\Sigma yR^{\circ}xy$	15 DF (D5)
1, 3, 4	(17)	$\Sigma yR^{\circ}xy$	16 ΠE
1, 3, 4	(18)	$R^{\circ}ac$	17 ΣE
	(19)	$\Pi x\Pi yER^{\circ}xyRyx$	Ax R°
	(20)	$ER^{\circ}acRca$	19 ΠE (2)
3	(21)	$CRcaKRccRaa$	11 ΠE (2)
1, 3, 4	(22)	$KRaaNRaa$	7, 18, 20, 21 TF
1, 3	(23)	$\Pi xRxx$	4-22 RAA
1, 3	(24)	Totrefl R	23 DF (D1)
1	(25)	C(3)(24)	CD
1	(26)	E(Totrefl R)(Birefl R)	2, 25 TF

(T5) Den $R \vdash C($L-refl $R)($R-refl $R)$

1	(1)	Den R	A
2	(2)	L-refl R	A
1	(3)	$\Pi x \Sigma y Rxy$	1 DF (D5)
2	(4)	$\Pi x \Pi y CRxyRxx$	2 DF (D4)
5	(5)	$N \Pi x \Pi y CRxyRyy$	A
5	(6)	$\Sigma x \Sigma y NCRxyRyy$	5 QE (2)
5	(7)	$NCRabRbb$	6 ΣE (2)
1	(8)	$\Sigma y Rby$	3 ΠE
1	(9)	Rbc	8 ΣE
2	(10)	$CRbcRbb$	4 ΠE (2)
1, 2, 5	(11)	$KRbbNRbb$	7, 9, 10 TF
1, 2	(12)	$\Pi x \Pi y CRxyRyy$	5-11 RAA
1, 2	(13)	R-refl R	12 DF (D3)
1	(14)	$C(2)(13)$	CD

(T6) Den $R^\circ \vdash C($R-refl $R)($L-refl $R)$

1	(1)	Den R°	A
2	(2)	R-refl R	A
1	(3)	$\Pi x \Sigma y R^\circ xy$	1 DF (D5)
	(4)	$\Pi x \Pi y ER^\circ xyRyx$	Ax R°
2	(5)	$\Pi x \Pi y CRxyRyy$	2 DF (D3)
6	(6)	$N \Pi x \Pi y CRxyRxx$	A
6	(7)	$\Sigma x \Sigma y NCRxyRxx$	6 QE (2)
6	(8)	$NCRabRaa$	7 ΣE (2)
1	(9)	$\Sigma y R^\circ ay$	3 ΠE
1	(10)	$R^\circ ac$	9 ΣE
	(11)	$ER^\circ acRca$	4 ΠE (2)
2	(12)	$CRcaRaa$	5 ΠE (2)
1, 2, 6	(13)	$KRaaNRaa$	8, 10, 11, 12 TF
1, 2	(14)	$\Pi x \Pi y CRxyRxx$	6-13 RAA
1, 2	(15)	L-refl R	14 DF (D4)
1	(16)	$C(2)(15)$	CD

Curiously enough, T7 is implied by **TF** alone from the conjunction of T3, T4, T5, and T6.

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