Notre Dame Journal of Formal Logic
Volume XVI, Number 4, October 1975
NDJFAM

ARITHMETIC AS A STUDY OF FORMAL SYSTEMS

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The notion of formal system appearing in the works of Curry seems to have been somewhat misunderstood. The purpose of this paper* is to help clarify this notion. I shall do this by considering elementary, almost trivial examples of formal systems of arithmetic and by showing how these systems can be considered a natural result of the increasing need, as the theory of arithmetic progressed, to find procedures for answering new kinds of questions. I shall also discuss some details of some of these examples, for example the difference between two kinds of metatheoretic implication relative to a formal system.

1 *Informal arithmetic and the need for increasing rigor* The earliest numbers were marks of some sort: notches in a stick or strokes on an animal skin. The idea of numbers in the modern sense arose when it was observed that no matter how many marks one has, it is always possible to add another one, so that there are infinitely many (natural) numbers.

With the growth of the theory about these numbers, a need arose for a more complete description of the nature of these numbers. This was especially true when, in the last century, the theory of classical analysis was developed entirely from the properties of natural numbers. Hence, at the end of the century, several such descriptions appeared, of which the best known is the set of axioms used by Peano. These axioms, in which the primitive ideas are 0 (an obvious modification can be made to begin with 1 instead of 0), representing the empty sequence of marks, |, representing the operation of adding one more mark, and =, a relation between numbers meaning that the marks of the two numbers can be paired in a 1 - 1 manner, are the following:

*During part of the work on this paper, the author was a U.S. National Science Foundation Graduate Fellow, and during another part of the work he was a temporary Lecturer at the University College of Swansea, Wales. The author wishes to thank Prof. H. B. Curry for reading several versions of the paper and making a number of helpful suggestions.

Received August 11, 1972

AXIOM P1. *0 is a number.*
AXIOM P2. *If X is a number, then there is a unique number $X|$.* (The uniqueness means that if $X = Y$, then $X| = Y|$.)
AXIOM P3. *If $X| = Y|$, then $X = Y$.*
AXIOM P4. *There is no number X such that $X| = 0$.*
AXIOM P5. *If $\boldsymbol{A}(X)$ is a statement whenever X is a number, and if*

(i) $\boldsymbol{A}(0)$ *is true, and*
(ii) *for each number X, if $\boldsymbol{A}(X)$ is true then $\boldsymbol{A}(X|)$ is true,*

then $\boldsymbol{A}(X)$ is true for every number X.

These axioms were supposed to be sufficient for arithmetic in the sense that they served as a definition of number, or picked out of the set of all things those that are numbers, and also in the sense that all true theorems about arithmetic could be derived from them by means of logic. However, the paradoxes of logic and set theory discovered at the very end of the century showed (1) that the idea of a set of all things can lead to difficulties and should be avoided, and (2) that the intuitive conception of logic of nineteenth century mathematicians was faulty, and that, therefore, the idea of deriving theorems from these axioms "by means of logic" is not a sufficiently precise definition of a mathematical proof.

Thus, although the Peano axioms are a sufficient description of numbers for some purposes, it is desirable to have a description that is still more precise. In particular, we want one that will answer the following questions:

Q1. What exactly is a proof?
Q2. How can we tell, for a given object, if it is one with which the theory deals?

Furthermore, we want the answers to these questions to be *effective*; i.e., we want a process which can be carried out by a machine (assuming no limitations of space and time; the fact that such a machine might require more space than the volume of our galaxy and more time than the life expectancy of the universe is, for pure mathematics, only a minor inconvenience) which will be able to distinguish a valid proof from an invalid one (i.e., that will be able to tell, for any supposed proof, whether or not it is valid), and which will be able to recognize the objects with which the theory deals. In this paper, I shall present some descriptions of arithmetic of a kind called *formal systems*, which answers these questions.

2 *Formal numbers* Let us begin with a simple case, which shows all the principles involved, by considering statements of the form

$$(1) \qquad X \text{ is a number.}$$

Axioms P1 and P2 suggest the following axiom and rule:

AXIOM. *0 is a number.*
RULE. *From "X is a number" to obtain "$X|$ is a number".*

A *proof* is then a sequence of statements each of which is either an axiom or else is obtained from a previous statement by the rule. Then it is clear that there is an effective process for deciding, for a given sequence of statements (each of the form (1)), whether or not it is a proof. A statement (of the form (1)) is a *theorem* if there is a proof of which it is the last statement, and in this case the proof is said to be a *proof of* the theorem. For example, the sequence

0 is a number, 0| is a number, 0|| is a number

is a proof of

0|| is a number,

but

0 is a number, 0|| is a number

is not a proof at all. Thus, we have an answer for Q1.

To answer Q2, we want an effective procedure for recognizing objects which can be substituted for X in (1). Let us call the objects which can be substituted for X *formal objects*. Then an examination of the Axiom and Rule suggests the following conditions:

(i) 0 is a formal object;

and

(ii) If X is a formal object, then so is $X|$.

Clearly, any definition of formal objects that can be used with this Axiom and Rule must satisfy these two conditions. We may as well begin with the most restricted such definition; this amounts to adding to the above conditions the following:

(iii) Nothing is a formal object unless its being one follows from conditions (i) and (ii).

If this restricted definition is adopted, then it follows that X is a formal object if and only if there is a sequence of things of which the last is X and such that each thing of the sequence is either 0 or else is obtained from a previous thing in the sequence by adding a | (on the right). Such a sequence is called a *construction of X from* 0 *by* | or, when there is no confusion, a *construction of* X, and X is said to be *constructed from* 0 *by* |. As for proofs defined above, there is an effective procedure for deciding whether any sequence of things (of a suitable universe) is a construction.

Now, suppose we consider only those constructions in which there are no repetitions. Thus, we consider the construction

0, 0|, 0||, 0|||, 0||||,

but not

0, 0|, 0||, 0, 0|, 0|||, 0||||.

Then, for each formal object, there is exactly one construction of it. Then we can identify the formal objects with their constructions, and hence regard as distinct any formal objects with distinct constructions. Since there is an effective procedure for deciding whether or not a sequence is a construction (and it is easy to see that there is an effective procedure for picking out the restricted constructions considered here), we have an effective process for picking out the formal objects.

We can now define a formal system, called $\mathcal{N}_0$, which puts all this systemmatically. It consists of a set E of *elementary statements*, which are all those of form (1) where X is a formal object, and a subset $T \subseteq E$ of *theorems*, which are those defined above. For convenience, I will write

$$\boldsymbol{N}(X)$$

in place of (1). Furthermore, if X and Y are the same formal object, i.e., if they have the same construction, then I will write

$$X \equiv Y.$$

(I use the word "set" here in a purely informal way to denote collections of things that we already have. Hence, no higher set theory is assumed. See [FML, §2A5] under the heading "Conceptual Classes". For an explanation of the letters in brackets, see the Bibliography.)

Now it is easy to see that if the informal relation = of Axioms P2-P4 above is identified with the relation ≡ defined above, then these axioms are satisfied. For if X and Y have the same construction, then by adding one step we get a construction of both $X|$ and $Y|$; conversely, if $X|$ and $Y|$ have the same construction, then dropping the last step results in a construction of both X and Y. Furthermore, no formal object $X|$ has the same construction as 0, since the latter construction has only one step whereas the former has more than one.

As for Axiom P5, it is a special case of the general principle that if anything is true of the initial elements of a construction and if its truth is preserved by the construction operations, then it is true of anything constructed from these initial elements by these operations. This principle is part of the definition of a construction.

I noted above that it is decidable whether a given thing is a formal object. In many formal systems, it is not decidable whether or not a given elementary statement is a theorem, but in $\mathcal{N}_0$, this is decidable.

THEOREM 1. $\boldsymbol{N}(X)$ *is a theorem of* $\mathcal{N}_0$ *if and only if* X *is a formal object of* $\mathcal{N}_0$.

Proof: Suppose X is a formal object of $\mathcal{N}_0$, and consider its construction. Replace each step U of this construction by the statement $\boldsymbol{N}(U)$. The result is a proof of $\boldsymbol{N}(X)$. Conversely, given a proof of $\boldsymbol{N}(X)$, each step is a statement of the form $\boldsymbol{N}(U)$, and if each of these steps is replaced by U, the result is a construction of X.

This theorem shows that the system $\mathcal{N}_0$ is rather trivial, for there are too few formal objects for the predicate, $\boldsymbol{N}$, to pick anything out. This triviality can be avoided by introducing some new formal objects into the system.

One interesting way of introducing these new formal objects is to change the definition of formal object from that given above to the following; a formal object is a word (finite string of letters) on the alphabet $\{0, |\}$. That is, instead of clauses (i)-(iii) which were used above to define a formal object, we use the following clauses:

(i) 0 and | are formal objects;
(ii) If X and Y are formal objects, then XY is a formal object;

and

(iii) Nothing is a formal object unless its being one follows from conditions (i) and (ii).

Call the system with this definition of formal object $\mathcal{N}_0'$. Then Theorem 1 must be replaced by the following theorem.

THEOREM 1′. ***N***(X) *is a theorem of* $\mathcal{N}_0'$ *if and only if* X *is a formal object of* $\mathcal{N}_0'$.

Furthermore, this theorem fails, because $0|0$ is a formal object of $\mathcal{N}_0'$ but ***N***$(0|0)$ is not a theorem.

On the other hand, it is not true that each formal object of $\mathcal{N}_0'$ has only one construction; the formal object $0|0$ has two constructions, one obtained by forming first $0|$ and then adding 0 on the right, the other obtained by forming first $|0$ and then adding 0 on the left. Hence, in $\mathcal{N}_0'$, it is not possible to identify formal objects with constructions.

3 *Formal systems* We are now in a position to define a formal system in general. (For a more thorough treatment of this definition, see [FML, §3C].) A *formal system* is a set E of statements, called *elementary statements*, together with a subset $T \subseteq E$ of *theorems* such that the following conditions are fulfilled:

(i) The set T is *inductive*; i.e., it is generated from a *definite* set of *axioms* by a set of *rules* in such a way that there is an effective procedure for deciding, given a sequence of elementary statements (possibly with the name of an axiom or rule associated with each), whether the sequence is a proof. (A *definite* set is one for which there is an effective procedure for deciding whether or not a given object (of a suitable universe) is an element of the set. See [FML, §2A5] and [ETD, footnote 2, p. 14].)

(ii) The set E is formed from a definite set O of *formal objects* and a definite set of *elementary predicates* requiring various members of arguments as follows: a statement is in E if and only if it asserts that a predicate of n arguments applies to an n-tuple of formal objects.

In both $\mathcal{N}_0$ and $\mathcal{N}_0'$, there is only one elementary predicate, which we have called ***N*** and which means "is a number", and this predicate has one argument. Note that the set E is definite in both systems. In fact, it follows from the general definition that E is definite.

The only difference between $\mathcal{N}_0$ and $\mathcal{N}_0'$ is the difference in the set O of formal objects in each case. These different sets of formal objects correspond to different ways of looking at a formal system. In $\mathcal{N}_0'$, the set of formal objects consisted of the set of all words of the alphabet consisting of the symbols 0 and $|$. If all formal objects of a formal system are words of some alphabet, then the formal system is called a *syntactical system* ([FML, §2C2]). As mentioned above, it is possible for a formal object of a syntactical system to have more than one construction. In $\mathcal{N}_0$, in which each formal object has only one construction, we took the formal objects to be those objects which can be constructed from an *atom*, 0, by the *primitive operation*, $|$; it followed that every formal object has a unique construction. Furthermore, we did not care whether or not the formal objects also happened to be words on some alphabet. A formal system in which the set of formal objects is taken to be an inductive set, generated from a definite set of *atoms* by a definite set of *primitive operations* in such a way that formal objects with different constructions are regarded as distinct, is called an *ob system* ([FML, §2C3]), and the formal objects of such a system are called *obs*.

Most of the standard systems of symbolic logic can be considered as being both syntactical and ob systems: They are essentially syntactic because the formal objects are words on an alphabet; but those words which are formal objects, usually called *well formed formulas*, are constructed from a definite set of *atomic formulas* by the operations representing the logical connectives and quantifiers, and since each of these well formed formulas has a unique construction, these systems can be considered to be ob systems as well. Note that although the definitions of these systems specify that the well formed formulas are words on an alphabet, this fact is never used, and often the symbols of the alphabet are only referred to and not listed. In any case, these systems usually have only one elementary predicate, of one argument, denoted by '$\vdash$' or 'is provable' or else left unexpressed.

In working with formal systems, we rarely just derive theorems one after the other. Instead, we consider the system as a whole and make statements about it. These statements, which are not a part of the formal system (i.e., they are not elements of E), are called *metastatements*, and those which are true are called *metatheorems*. (I am using the prefix 'meta-' the way Curry now uses the prefix 'epi-' ([FML, chapter 3]); this is Curry's own original usage.) For example, the statements about $\mathcal{N}_0$ and $\mathcal{N}_0'$ in relation to Axioms P2-P5 are metastatements, and so is Theorem 1. The proofs of metatheorems are, of course, informal. Hence, to avoid the problems that arose in deriving theorems informally from Axioms P1-P5, we often restrict our methods of proof to those which are *constructive*. This means, essentially, that any proof should indicate an effective process for reaching the conclusion; most important, a proof that something exists must either exhibit it or else give an effective process for constructing it. I shall not try to completely define constructive methods of proof, but shall rather try to indicate what they are by using them. To help explain the idea, I will list some metatheoretical connectives and their meaning; in this list, $\boldsymbol{A}$ and $\boldsymbol{B}$ are statements constructed from the elementary statements by these connectives and any others that may be in use at the time (used as operations);

$\boldsymbol{A}$ & $\boldsymbol{B}$ means that there is a proof of $\boldsymbol{A}$ and a proof of $\boldsymbol{B}$;

$\boldsymbol{A}$ or $\boldsymbol{B}$ means that there is a proof of $\boldsymbol{A}$ or a proof of $\boldsymbol{B}$, and that there is an effective process for deciding which;

$\boldsymbol{A} \rightarrow \boldsymbol{B}$ means that there is an effective process for converting any proof of $\boldsymbol{A}$ into a proof of $\boldsymbol{B}$, and is regarded as vacuously true if it can be constructively shown that there is no proof of $\boldsymbol{A}$;

and

$\boldsymbol{A} \rightleftarrows \boldsymbol{B}$ means $(\boldsymbol{A} \rightarrow \boldsymbol{B})$ & $(\boldsymbol{B} \rightarrow \boldsymbol{A})$.

Of these connectives, $\rightarrow$ is the most important. It is related to the idea of admissibility of Lorenzen ([BBD]), which is that if $\boldsymbol{A} \rightarrow \boldsymbol{B}$ is added to the system as a rule, no new theorems are introduced. If $\boldsymbol{A} \rightarrow \boldsymbol{B}$ can be constructively shown to be admissible, then it is true in the sense given above. (Thus, the answer to a question in [FML, p. 97] is now known.) As an example, we can prove both

(2) $$\boldsymbol{N}(X||) \rightarrow \boldsymbol{N}(X)$$

and

(3) $$\boldsymbol{N}(X) \rightarrow \boldsymbol{N}(X||)$$

in $\mathcal{N}_0$; for (2) follows by removing the last two steps from the proof of $\boldsymbol{N}(X||)$, and (3) follows by adding these same two steps to the proof of $\boldsymbol{N}(X)$.

Actually, (3) is an example of a stronger kind of implication, for we can actually start with $\boldsymbol{N}(X)$ and use the rule of the system to obtain $\boldsymbol{N}(X||)$. Thus, we will also consider as an implication the following:

$\boldsymbol{A} \Rightarrow \boldsymbol{B}$ means that if $\boldsymbol{A}$ is adjoined to the system as an axiom, then $\boldsymbol{B}$ is a theorem;

and

$\boldsymbol{A} \Leftrightarrow \boldsymbol{B}$ means $(\boldsymbol{A} \Rightarrow \boldsymbol{B})$ & $(\boldsymbol{B} \Rightarrow \boldsymbol{A})$.

Thus, considering the above examples, we have

$$\boldsymbol{N}(X) \Rightarrow \boldsymbol{N}(X||)$$

in $\mathcal{N}_0$, but not

$$\boldsymbol{N}(X||) \Rightarrow \boldsymbol{N}(X).$$

I will give further examples of the difference between these two kinds of implication below.

4 *Equality* Since Axioms P1-P5 are metatheorems of the system $\mathcal{N}_0$, this system is a sufficient system on which to base arithmetic. However, these

metatheorems, and many other statements of arithmetic in which we are interested, involve equality between numbers, or statements which we have interpreted as having the form

(4) $$X \equiv Y$$

and if arithmetic is based on $\mathcal{N}_0$, then the truth of these statements must be decided informally. In this section, I shall define a formal system in which there is a formal predicate corresponding to $\equiv$.

The system will be called $\mathcal{N}_1$. (This is the system introduced by Curry in [FML, p. 256]. I have numbered Rules 1-5 to correspond to Curry's rules. Many of the results proved here are stated there without proof.) Its obs are the same as those of $\mathcal{N}_0$; hence, by Theorem 1, we do not need the predicate **N**. The primitive predicate is a binary predicate, written '='. Hence, the elementary statements are all statements of the form

(5) $$X = Y.$$

The theorems of the system are obtained from the following axiom and rule:

AXIOM. $0 = 0$

RULE 1. $X = Y \Rightarrow X| = Y|$.

By induction, first on the set of obs, then on the set of theorems, we can prove

THEOREM 2. *$X = Y$ if and only if $X \equiv Y$.*

COROLLARY 2.1. *$\mathcal{N}_1$ is consistent and decidable.*

The following are metatheorems of $\mathcal{N}_1$:

RULE 2. $X = Y \rightarrow Y = X$.
RULE 3. $X = Y \;\&\; Y = Z \rightarrow X = Z$.
RULE 4. $X| = Y| \rightarrow X = Y$.
RULE 5. $0 = X| \rightarrow 0 = X$.

This shows that = has the important properties of $\equiv$. Rule 5 corresponds to Axiom P4; it looks different because there is no negation in this system. For a discussion between the form of Rule 5 and the general problem of negation, see [FML, pp. 256-7].

Rules 2-5 do not hold if the single arrow is replaced by the double one. This result is a corollary to the following theorem. First, note that X has more |'s than Y—i.e., $X \equiv Y|| \ldots |$—if and only if in any construction of X, Y occurs at a step preceding X. In this case, say that X follows Y, and call the |'s added to Y to construct X the *|'s of X left out of Y*.

THEOREM 3. *$X = Y \Rightarrow Z = W$ if and only if $Z = W$ or Z follows X and W follows Y and the |'s of Z left out of X can be paired with the |'s of W left out of Y.*

The proof is like that of Theorem 1.

5 *Order* In this section, I formalize the predicate "follows". A recursive definition of this predicate, symbolized by $<$, is given by Lorenzen ([BBD, p. 170]) as follows:

(6) $$X < X|,$$
(7) $$X < Y \Rightarrow X < Y|.$$

In what follows, suppose that $\mathcal{J}$ is any system with the same obs as $\mathcal{N}_0$. Then $\mathcal{J}(<)$ will be the system formed from $\mathcal{J}$ by adjoining $<$ as a binary predicate, (6) as an axiom scheme, and (7) as a rule. It is easy to prove

THEOREM 4. *$X < Y$ if and only if Y follows X.*

From this theorem, it follows that $\mathcal{J}(<)$ is consistent and, if $\mathcal{J}$ is decidable, so is $\mathcal{J}(<)$. In particular, $\mathcal{N}_0(<)$ and $\mathcal{N}_1(<)$ are decidable. In addition, the following hold in $\mathcal{J}(<)$:

(8) $X < Y \,\&\, Y < Z \rightarrow X < Z$,
(9) $X < Y \rightarrow X| < Y|$,
(10) $X| < Y| \rightarrow X < Y$.

But (8)-(10) do not hold if the single arrow is replaced by the double one (Lorenzen [BBD, p. 173] makes this point about (9)), as the following theorem proves.

THEOREM 5. *$X < Y \Rightarrow Z < W$ if and only if $Z < W$ or $X \equiv Z$ and $Y < W$.*

If $\mathcal{J}$ has the predicate $=$, then it is easy to define $\leqslant$. If $\mathcal{J}$ does not contain $=$, then either $X < Y|$ can be used to express $\leqslant$ or else $\leqslant$ can be defined as follows: let $\mathcal{J}(\leqslant)$ be the system formed by adjoining to $\mathcal{J}$ the binary predicate $\leqslant$, the axiom scheme

(11) $$X \leqslant X$$

and the rule

(12) $$X \leqslant Y \Rightarrow X \leqslant Y|.$$

$\mathcal{J}(\leqslant)$ is similar to $\mathcal{J}(<)$. Note that if $X < Y|$ is used for $X \leqslant Y$, then (11) is just (6) and (12) follows easily by (7). Lorenzen ([BBD]) proposes, instead of (12), the rule

$$X < Y \Rightarrow X \leqslant Y$$

(p. 171). This rule, with the above interpretation, is just (7) itself. However, in this case there is not a formulation of $\leqslant$ independent of that of $<$.

6 *Definitional extensions* The next step is to introduce the arithmetic operations, such as addition and multiplication. These operations cannot be taken simply as ordinary primitive operations because the result of applying one of these operations to a suitable number of arguments is to be identified with an ob of the old system: the sum of two numbers is again a

number. Hence, a means must be found for assigning to each sum (product, etc.) an ob already in the system.

Let us begin by considering the special case of addition. We introduce a new primitive operation +, so that for any obs X and Y, $X + Y$ will also be an ob. In order to keep things straight, let us use letters from the beginning of the alphabet, such as

$$A, B, C, \ldots,$$

for obs we already have (i.e., obs of the system $\mathcal{N}_0$), and let us call these obs *basic*. Obs which are not basic (i.e., obs in which the operation + actually occurs) will be called *new*. Then letters from the end of the alphabet, such as

$$X, Y, Z, \ldots,$$

will denote obs of the extended system (i.e., obs which are either old or new). The problem we have to solve in this extended system is finding a method of assigning to each ob of this extended system a basic ob, called its *value*.

Since + is a primitive recursive function, it is natural to take as this method of assignment the usual primitive recursive definition of + together with the usual machinery of recursive function theory. If we write

$$X \boldsymbol{D} Y \tag{13}$$

to express that Y is assigned to X, then the usual primitive recursive definition of addition consists of the statements

$$\begin{gathered} X + 0 \;\boldsymbol{D}\; X, \\ X + Y| \;\boldsymbol{D}\; (X + Y)|. \end{gathered} \tag{14}$$

To evaluate an ob X of the extended system, we take as axioms the statements (14) and

$$X \boldsymbol{D} X, \tag{$\rho\boldsymbol{D}$}$$

and then apply the rule

$$X \boldsymbol{D} Y \;\&\; A + B \boldsymbol{D} C \Rightarrow X \boldsymbol{D} Y', \tag{15}$$

where Y' is obtained from Y by replacing an occurrence of $A + B$ by C. (This system is equivalent to the system of Kleene ([IMM, p. 263f.]). Kleene's system uses variables and axioms instead of axiom schemes, and so he needs a rule of substituting basic obs for the variables; furthermore, Kleene's other rule, which corresponds to (15), permits replacement in X as well as Y, and he has not postulated ($\rho\boldsymbol{D}$), but since our concern is to pick out those of his statements (13) for which the right-hand side is a basic ob, this is not an important restriction.) For example, to evaluate $0| + 0||$ we can proceed as follows:

$$\dfrac{0| + 0|| \;\boldsymbol{D}\; (0| + 0|)| \qquad \dfrac{0| + 0| \;\boldsymbol{D}\; (0| + 0)| \qquad 0| + 0 \;\boldsymbol{D}\; 0|}{0| + 0| \;\boldsymbol{D}\; 0||}}{0| + 0|| \;\boldsymbol{D}\; 0|||.}$$

It now appears (and it can be proved) that for each ob X of the extended system there is a unique basic ob A, which we call the *value* of X, such that

$$X \boldsymbol{D} A. \tag{16}$$

The extended system introduced in the last paragraph is an example of what Curry has called a *definitional extension.* (See [CLg, pp. 62-74], [DFS], and [FML, pp. 106-111].) Since definitional extensions offer a generalization of recursive function theory to the case in which the basic obs are not numbers, I will define a definitional extension in general. First, however, let us consider a modification of the rule (15). Let us rewrite this rule as

$$X \boldsymbol{D} Y \,\&\, A + B \boldsymbol{D} Z \Rightarrow X \boldsymbol{D} Y', \tag{15$'$}$$

where $A + B \boldsymbol{D} Z$ is one of the statements (14) and Y' is obtained from Y by replacing an occurrence of $A + B$ by Z. The modification is that the ob replacing $A + B$ need not be basic but the second premise must be an axiom of the system. (Rule (15) is an example of the rule which Curry called Rd in [CLg] and [DFS]. In [FML], Curry changed the rule Rd to designate the rule called Rd below, of which (15) is an example. In my note [NDR], I show that although these two rules are not precisely equivalent, they give the same reductions to an ultimate definiens (value).) Then $0| + 0||$ is evaluated as follows:

$$\cfrac{\cfrac{0| + 0|| \boldsymbol{D} (0| + 0|)| \qquad 0| + 0| \boldsymbol{D} (0| + 0)|}{0| + 0|| \boldsymbol{D} (0| + 0)||} \qquad 0| + 0 \boldsymbol{D} 0|}{0| + 0|| \boldsymbol{D} 0|||.}$$

It is fairly easy to see that the rule (15)′ gives derivations that have more of an algorithmic character than rule (15), and that furthermore its derivations have a simpler structure, since their tree diagrams consist of one long branch on the left from each node of which, except the top one, there is a branch of length one node coming in from the right. Hence, rule (15)′ will be taken as the basis for the rule in the system of definitional extensions.

Now, to state the definition, suppose that $\mathscr{J}_0$ is any formal system. Call the atoms, operations, etc. of $\mathscr{J}_0$ *basic.* Let $\mathscr{J}_1$ be an extension of $\mathscr{J}_0$; call the atoms, etc. that are in $\mathscr{J}_1$ but not $\mathscr{J}_0$ *new*. I will use italic letters from the beginning of the alphabet, such as 'A', 'B', 'C', etc., possibly with subscripts, for basic obs; letters from the end of the alphabet, such as 'X', 'Y', and 'Z', possibly with subscripts, for arbitrary obs, basic or new. Then $\mathscr{J}_1$ is a *definitional extension* of $\mathscr{J}_0$ if and only if the following conditions are satisfied:

a. The obs of $\mathscr{J}_1$ are those of $\mathscr{J}_0$ together with those formed by means of certain adjoined operations.

b. There is a new binary predicate $\boldsymbol{D}$. Thus, the new elementary statements are of the form (13). I will call X the *left side* (*definiendum*) and Y the *right side* (*definiens*).

c. The new axioms of $\mathcal{J}_1$ consist of the scheme

$$(\rho \boldsymbol{D}) \qquad X \boldsymbol{D} X$$

and a certain set E of *defining axioms*, each of which is of the form

$$\phi(A_1, \ldots, A_m) \boldsymbol{D} Z,$$

where ϕ is a new operation of m arguments. Note that the arguments of ϕ here must be basic obs.

d. There is one new rule, called the *rule of definitional reduction*, viz.

$$\text{Rd} \qquad X \boldsymbol{D} Y \;\&\; \phi(A_1, \ldots, A_m) \boldsymbol{D} Z \Rightarrow X \boldsymbol{D} Y',$$

where the second premise is in E and where Y' is obtained from Y by replacing an occurrence of $\phi(A_1, \ldots, A_m)$ by Z.

A demonstration from the new axioms by Rd alone is called a *definitional reduction.* It must begin with an instance of $(\rho \boldsymbol{D})$ or a defining axiom, and all the replacements are made on the right, the left side remaining unchanged. Thus, a definitional reduction can be represented by simply giving, in order, the right sides (and the left side first if the beginning of the reduction is a defining axiom). If a basic ob occurs in this sequence, then it must be the last element, for clearly (16) cannot be reduced further by Rd. If (16) holds, then A is called the *value* (*ultimate definiens*) of X.

It is not necessary that each new ob X have exactly one value: it may have no value, or it may have more than one. (Clearly, each basic ob has exactly one value.) How many values X has depends on the defining axioms. If the axioms are such that each new ob has at most one value, then I will call the new operation(s) defined by the axioms (*a*) (*partial recursive*) *function*(*s*). Curry has shown (see [FML, p. 108]) that if no two distinct defining axioms have the same left side, then the operation(s) defined is (are) function(s). An extension with this property is called *proper.* Finally, if each new ob X has at least one value, the operation(s) defined is (are) called *total.*

The relation $\boldsymbol{D}$ is clearly transitive; i.e.,

$$(\tau \boldsymbol{D}) \qquad X \boldsymbol{D} Y \;\&\; Y \boldsymbol{D} Z \rightarrow X \boldsymbol{D} Z.$$

(This fails if the double arrow is used.) However, it is not symmetric. It is therefore convenient to extend $\boldsymbol{D}$ to a symmetric relation $\equiv$. This relation has as axioms

$$(\rho \equiv) \qquad X \equiv X$$

and all the defining axioms with $\boldsymbol{D}$ replaced by $\equiv$; its rules are Rd with $\boldsymbol{D}$ replaced by $\equiv$ and

$$(\sigma \equiv) \qquad X \equiv Y \Rightarrow Y \equiv X,$$
$$(\tau \equiv) \qquad X \equiv Y \;\&\; Y \equiv Z \Rightarrow X \equiv Z.$$

Curry has proved (see [CLg, p. 67-8]) the following theorem.

THEOREM 6. *Let the definitional extension be proper and let*

$$X \equiv Y. \tag{17}$$

Then if either X or Y has a value, the other does also and the values are identical.

COROLLARY 6.1. *If the definitional extension is proper, then $X \equiv A \rightarrow X \mathbf{D} A$.*

COROLLARY 6.2. *If $A \equiv B$, then A and B are the same ob.*

This corollary justifies my previous use of $\equiv$.

COROLLARY 6.3. *If the definitional extension is proper and if, in addition, each new operation is total, then* (17) *holds if and only if X and Y have the same value.*

COROLLARY 6.4. *If all the new operations are total functions, then for each operation ϕ of m arguments*

$$X_1 \equiv Y_1 \;\&\; \ldots \;\&\; X_m \equiv Y_m \rightarrow \phi(X_1, \ldots, X_m) \equiv \phi(Y_1, \ldots, Y_m). \tag{18}$$

Thus, for total functions the predicate $\mathbf{D}$ can be eliminated. The defining axioms and rule Rd can be stated in terms of $\equiv$.

In general, it is not possible to tell for any given set of defining axioms whether or not the operations it defines are total. However, it is possible to prove for some specific kinds of sets E that they do define total functions.

THEOREM 7. *Suppose the atoms of $\mathscr{J}_0$ are $a_1, \ldots$ (possibly an infinite sequence), and suppose the basic operations are $\omega_1, \ldots$, where ω_i has n_i arguments. Then, if all the other operations are total functions, an operation ϕ defined by the* primitive recursive scheme, *viz.*

$$\phi(X_1, \ldots, X_m, a_i) \equiv \psi(X_1, \ldots, X_m, a_i),\; i = 1, 2, \ldots,$$

$$\phi(X_1, \ldots, X_m, \omega_k(Y_1, \ldots, Y_{n_k})) \equiv \chi(X_1, \ldots, X_m, Y_1, \ldots, Y_{n_k},$$
$$\phi(X_1, \ldots, X_m, Y_1), \ldots, \phi(X_1, \ldots, X_m, Y_{n_k})),\; k = 1, 2 \ldots,$$

where ψ and χ are either basic operations or new total functions, is a total function.

Proof: Since this scheme is obviously proper, the uniqueness of the value is proved, and it remains to prove that each new ob has a value. Since all the new operations other than ϕ are total, it is enough to prove that $\phi(X_1, \ldots, X_m, A)$ has a value for each basic ob A. If A is an atom, then this has a value by the first axiom of the scheme. So suppose A is $\omega_k(B_1, \ldots, B_{n_k})$. By the induction hypothesis, each $\phi(X_1, \ldots, X_m, B_j)$, for $j = 1, \ldots, n_k$, has a value, say C_j. Then, by the second axiom of the scheme, applying Rd repeatedly, we get

$$\phi(X_1, \ldots, X_m, A) \equiv \chi(X_1, \ldots, X_m, B_1, \ldots, B_{n_k}, C_1, \ldots, C_{n_k}),$$

and this has a value because χ is total.

7 *The operations of arithmetic* In what follows, I assume that $\mathscr{J}_0$ is $\mathscr{N}_1$. The basic obs will be called *numerals*.

The first thing to notice about this case is that an operation is a partial recursive function in this sense if and only if it is a partial recursive function in the usual sense. Thus, my use of the vocabulary of recursive function theory coincides with the standard use, and the theory of recursive functions applies to functions definable by a definitional extension.

Now let $\Phi = \{\phi_1, \ldots, \phi_k\}$ be a finite set of total functions; let ϕ_i have n_i arguments and defining axioms $\mathscr{E}_i$; and suppose that all these extensions are proper. Let $\mathscr{E} = \mathscr{E}_1 \cup \ldots \cup \mathscr{E}_k$. Let $\mathscr{N}_\Phi$ be the definitional extension of $\mathscr{N}_1$ formed by taking $\mathscr{E}$ as the set of defining axioms and adding as a rule

RULE 0 $$X \equiv Y \Rightarrow X = Y.$$

Since the axiom $0 = 0$ is a special case of this rule, it is now redundant. Thus, $\mathscr{N}_\Phi$ can be formulated as follows:

ATOM. 0.
OPERATIONS. $|$ and ϕ_i for $i = 1, \ldots, k$.
PREDICATES. $\equiv$ and $=$.
AXIOMS. The elements of $\mathscr{E}$ and $(\rho\equiv)$.
RULES. For $\equiv$, rules Rd, $(\rho\equiv)$, and $(\tau\equiv)$; for $=$, rules 0-1.

Then, the following theorem can be proved much as the previous ones.

THEOREM 8. $X = Y \rightarrow X \equiv Y$.

As a result of this theorem, we can eliminate the predicate $\equiv$ and Rule 0 and replace $\equiv$ by $=$ in all axioms and rules in which it occurs. Then the axioms become the elements of $\mathscr{E}$ and the scheme

$$X = X,$$

and the rules are Rd and Rules 1-3 (using the double arrow, of course). Hereafter, I assume that $\mathscr{N}_\Phi$ is formulated in this way.

As a result of Corollary 6.3,

THEOREM 9. *In $\mathscr{N}_\Phi$, $X = Y$ if and only if there is a numeral A such that $X = A$ and $Y = A$.*

It follows from this theorem that $\mathscr{N}_\Phi$ is consistent and decidable and that for each $i = 1, \ldots, k$, (18) holds with ϕ_k for ϕ and $=$ for $\equiv$.

All of the ordinary arithmetic operations can be defined by primitive recursive schemes, and so these theorems apply to them. Thus, for example, as we have seen, addition is defined by

$$X + 0 = X,$$
$$X + Y| = (X + Y)|,$$

and multiplication by

$$X \cdot 0 = 0,$$
$$X \cdot (Y|) = X \cdot Y + X.$$

Let $\mathcal{N}_2$ be $N_{\{+\}}$ and $\mathcal{N}_3$ be $\mathcal{N}_{\{+,\cdot\}}$, etc. All of these systems are consistent and decidable.

Curry, in [FML, p. 106], suggests as an alternative for + the defining equations

$$0 + Y = Y,$$
$$X| + Y = X + Y|.$$

It is easy to see that this definition is equivalent to the one we have taken, but it is not primitive recursive and so Theorem 7 does not apply.

By methods similar to those used above, it is possible to prove that + is repeated | in the following way:

THEOREM 10. *$A + B = C$ if and only if C follows A and the |'s of C left out of A can be paired with the |'s of B.*

In order to have my definition of $\mathcal{J}(<)$ and $\mathcal{J}(\leqslant)$ apply to $\mathcal{N}_2$, I modify the requirement that $\mathcal{J}$ have the same obs as $\mathcal{N}_0$ and instead allow the possibility that it is a definitional extension whose basic obs are those of $\mathcal{N}_0$.

COROLLARY 10.1. *In $\mathcal{N}_2$ $(<)$*

$$A < B \rightleftarrows A + C| = B,$$

and in $\mathcal{N}_2(\leqslant)$,

$$A \leqslant B \rightleftarrows A + C = B.$$

Furthermore, the C can be found effectively in each case.

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