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S1 ≠ S0.9

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In both [1] and [3] it was conjectured that S0.9 is weaker than S1, but there was no proof that this is so. In what follows we see that this is so using Hintikka's model set model system semantics (see [2]).

Consider the systems defined in terms of the following axiom schemata and rules as in [4].

A1: $A \supset (B \supset A)$
A2: $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
A3: $(\sim A \supset \sim B) \supset (B \supset A)$
A4: $\Box A \supset A$
A5: $\Box(A \supset B) \supset (\Box(B \supset C) \supset \Box(A \supset C))$
A6: $\Box(A \supset B) \supset (\Box A \supset \Box B)$

R1: $\dfrac{A, A \supset B}{B}$ R2: $\dfrac{\Box(A \supset B) \;\&\; \Box(B \supset A)}{\Box(\Box A \supset \Box B) \;\&\; \Box(\Box B \supset \Box A)}$ R3: $\dfrac{\Box(A \supset B)}{\Box(\Box A \supset \Box B)}$

We use the standard definitions of $\Diamond$, &, v, and ≡, and we use $\Box$Ai($1 \leqslant i \leqslant 6$) for schema resulting from schema Ai by prefixing the symbol $\Box$ before the whole of Ai in brackets.

We define four modal systems:

S0.5 = {A4, A6, □A1 - □A3; R1}
S0.9 = {A4, □A1 - □A4, □A6; R1, R2}
S1 = {A4, □A1 - □A5; R1, R2}
S2 = {A4, □A1 - □A4, □A6; R1, R3}

It has been shown that S0.5 is included in S0.9, and S0.9 is included in S1, and both S0.9 and S1 are included in S2 (see [3]).

We now construct a Hintikka type model $\langle \Omega, C_S \rangle$ where Ω is a model system of model sets $\Omega = \{\mu_1, \mu_2, \ldots, \mu_n, \ldots\}$ ($n \geqslant 1$), and where C_S is a set of consistency conditions, for some system S, for deciding which formulae of the system S can be included (or imbedded) in any μ_n. The membership of C_S is drawn from:

1. If μ_n contains an atomic formula it does not contain its negation.
2. If $(A \supset B) \in \mu_n$ then $\sim A \in \mu_n$ or $B \in \mu_n$ or both.

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3. If $\sim(A \supset B) \in \mu_n$ then $A \in \mu_n$ and $\sim B \in \mu_n$.
4. If $\Box A \in \mu_n$ then $A \in \mu_n$.
5. If $\Diamond A \in \mu_n$ then there is in Ω at least one alternative to μ_n (such as μ_j) which contains A, provided that μ_n is not an alternative (is non-alternate) to any member of Ω or that there is at least one formula of the form $\Box B$ in μ_n (*cf.* [2], p. 124n).
6. If $\Box A \in \mu_n$ then there is in Ω at least one alternative to μ_n (such as μ_j) which contains A, provided that μ_n is not an alternative (is non-alternate) to any member of Ω.
7. If $\Diamond A \in \mu_n$, then provided that μ_n is not an alternative (non-alternate) to any member of Ω, there is in Ω at least one alternative to μ_n (such as μ_j) which contains A; but if μ_n is an alternative to some member of Ω, either $A \in \mu_n$ or there is in Ω at least one alternative to μ_n (such as μ_k) which contains $\sim A$, and μ_k cannot be an alternative to more than one set.
8. If $\Diamond A \in \mu_n$, then provided that μ_n is not an alternative (non-alternate) to any member of Ω, there is in Ω at least one alternative to μ_n (such as μ_j) which contains A; but if μ_n is an alternative to some member of Ω, and contains a formula of the form $\Box B$ such that either all the atomic parts of A are atomic parts of B, or C is a well-formed part of B and $A \equiv C \in \mu_n$, then either $A \in \mu_n$ or there is in Ω at least one alternative to μ_n (such as μ_k) which contains $\sim A$ and μ_k cannot be an alternative to more than one set.
9. If $\Box A \in \mu_n$ and if μ_j is an alternative to μ_n in Ω then $A \in \mu_j$.
10. If $\Box A \in \mu_n$ and μ_n is non-alternate to any member of Ω and if μ_j is an alternative to μ_n in Ω then $\Box A \in \mu_j$.
11. If $\Box A \in \mu_n$ and μ_n is non-alternate to any member of Ω and if μ_j is an alternative to μ_n in Ω, then $A \in \mu_j$; but if μ_n is an alternative to some member of Ω, then if μ_k is an alternative to μ_n in Ω, then $\sim A \in \mu_k$.
12. If $\Box A \in \mu_n$ and μ_n is non-alternate to any member of Ω, then $A \in \mu_n$.

We assume:

$$\begin{aligned} C_{S0.5} &= \{1 - 3,\ 12,\ 6,\ 9\} \\ C_{S0.9} &= \{1 - 4,\ 8,\ 10,\ 11\} \\ C_{S1} &= \{1 - 4,\ 7,\ 11\} \\ C_{S2} &= \{1 - 4,\ 5,\ 9,\ 10\} \end{aligned}$$

The satisfiability (consistency) of a set of formulae, λ, is defined as its imbeddability in a non-alternate member of Ω, i.e.,

$$\text{Satisfiable } (A) \ .\!\equiv\!. \ (\exists\Omega)(\exists n)(A \in \mu_n \ \&\ \mu_n \in \Omega \ \&\ \text{non-alternate } \mu_n)$$

A formula is said to be valid or self-sustaining if the unit set of its negation is not satisfiable, i.e.,

$$\text{Valid } (A) \ .\!\equiv\!. \ (n)\,(\text{non-alternate } \mu_n \supset \sim A \notin \mu_n)$$

For S0.5 we have the model $\langle \Omega, C_{S0.5} \rangle$. Using the conditions in $C_{S0.5}$ for *reductio ad absurdum* proofs in accordance with our definitions of validity, and as set out below, we see that all the axioms of S0.5 are valid.

$\Box$A1 is valid:

i. $\sim\Box(A \supset (B \supset A))$ $\qquad \in \mu_1$ assumption

ii. $\sim(A \supset (B \supset A))$ $\quad \epsilon\ \mu_2$ condition 6 from i
iii. A and $\sim A$ $\quad \epsilon\ \mu_2$ conditions 1-3 from ii

so by *reductio* $\Box$A1 is valid.

Similarly for $\Box$A2 and $\Box$A3. A4 is valid:

i. $\sim(\Box A \supset A)$ $\quad \epsilon\ \mu_1$ assumption
ii. $\Box A$ and $\sim A$ $\quad \epsilon\ \mu_1$ conditions 1-3 from i
iii. A $\quad \epsilon\ \mu_1$ condition 12 from ii

so by *reductio* A4 is valid.

Similarly for A6. But $\Box$A4 is not valid:

i. $\sim\Box(\Box A \supset A)$ $\quad \epsilon\ \mu_1$ assumption
ii. $\sim A$ and $\Box A$ $\quad \epsilon\ \mu_2$ condition 6 from i

resulting in no contradiction, since condition 12 does not allow for $A\ \epsilon\ \mu_2$ from ii. Similarly $\Box$A6 is not valid.

Using the condition in $C_{S0.9}$ in an S0.9 model in the manner of the above proofs we can show that all the axioms of S0.9 are valid. In particular:

$\Box$A4 is valid:

i. $\sim\Box(\Box A \supset A)$ $\quad \epsilon\ \mu_1$ assumption
ii. $\Box A$ and $\sim A$ $\quad \epsilon\ \mu_2$ condition 8 from i
iii. A $\quad \epsilon\ \mu_2$ condition 4 from ii

so by *reductio* $\Box$A4 is valid.

$\Box$A6 is valid:

i. $\sim\Box(\Box(A \supset B) \supset (\Box A \supset \Box B))$ $\quad \epsilon\ \mu_1$ assumption
ii. $\Box(A \supset B)$ and $\Box A$ and $\Diamond\sim B$ $\quad \epsilon\ \mu_2$ condition 8 from i
iii. $A \supset B$ and A $\quad \epsilon\ \mu_2$ condition 4 from ii
iv. B $\quad \epsilon\ \mu_2$ from iii

and then either (a):

v. $\sim B$ $\quad \epsilon\ \mu_2$ from ii by condition 8 since the atomic parts of $\sim B$ will be in $\Box(A \supset B)$ and μ_2 is an alternative to μ_1

or (b):

vi. B $\quad \epsilon\ \mu_3$ from ii by rule 8
vii. $\sim(A \supset B)$ $\quad \epsilon\ \mu_3$ from ii by rule 11
viii. A and $\sim B$ $\quad \epsilon\ \mu_3$ from vii by rules 1-3

so, as there is a contradiction in both (a) and (b), by *reductio* $\Box$A6 is valid.

But $\Box$A5 is not valid:

i. $\sim\Box(\Box(A \supset B) \supset (\Box(B \supset C) \supset \Box(A \supset C)))$
$\epsilon\ \mu_1$ assumption

ii. $\Box(A \supset B)$ and $\Box(B \supset C)$ and $\Diamond\sim(A \supset C)$
$\epsilon\ \mu_2$ from i by condition 8
iii. $A \supset B$ $\epsilon\ \mu_2$ from ii by condition 4
iv. $B \supset C$ $\epsilon\ \mu_2$ from ii by condition 4

but since there is no formula in μ_2 of the form $\Box D$ such that either (a)

both A and C are well-formed parts of D,

or (b)

there are in μ_2 formulae which show that A and C are materially equivalent to well-formed parts of D, there is no further alternate set of μ_2 under condition 8, and no contradiction follows.

Nevertheless A5 is valid:

i. $\sim(\Box(A \supset B) \supset (\Box(B \supset C) \supset \Box(A \supset C)))$
$\epsilon\ \mu_1$ assumption
ii. $A \supset B$ and $B \supset C$ and A and $\sim C$ $\epsilon\ \mu_2$ from i by condition 8
iii. C $\epsilon\ \mu_2$ from ii

so by *reductio* A5 is valid (*cf.* [3]).

Using the conditions for the S1 model we can show that all the axioms of S1 are valid. In particular:

$\Box$A5 is valid:

i. $\sim\Box(\Box(A \supset B) \supset (\Box(B \supset C) \supset \Box(A \supset C)))$
$\epsilon\ \mu_1$ assumption
ii. $\Box(A \supset B)$ and $\Box(B \supset C)$ and $\Diamond\sim(A \supset C)$
$\epsilon\ \mu_2$ from i by condition 7
iii. $A \supset B$ $\epsilon\ \mu_2$ from ii by condition 4
iv. $B \supset C$ $\epsilon\ \mu_2$ from ii by condition 4

and then either (a):

v. A and $\sim C$ $\epsilon\ \mu_2$ from ii by condition 7
vi. C $\epsilon\ \mu_2$ from v, iv, and iii

or (b):

vii. $A \supset C$ $\epsilon\ \mu_3$ from ii by condition 7
viii. A and $\sim B$ and B and $\sim C$ $\epsilon\ \mu_3$ from ii by condition 11

so, as there is a contradiction in both (a) and (b), by *reductio* $\Box$A5 is valid.

Also $\Box$A6 is valid and $\Box$A6 is a thesis of S1 (*cf.* [3]).

Using the conditions for the S2 model we can also show that all the axioms of S2 are valid.

Since we can show that the axioms of each system are valid in the models constructed for each, and that the sets of axioms are independent in the appropriate sense, we now turn to the rules of inference. Clearly, from conditions 1 to 3, R1 holds in all four models. It can be shown that R2 holds in the models for S0.9, S1 and S2, and that R3 holds in the model for S2. Our main interest will be to show that R2 holds for the S0.9 model. In order to show that R2 holds in $\langle \Omega, C_{S0.9} \rangle$ we show that if $\Box(A \supset B)$ & $\Box(B \supset A)$ is valid then $\Box(\Box A \supset \Box B)$ & $\Box(\Box B \supset \Box A)$ is valid.

First we define Ω^N, Ω^A, Ω^{AA}.

$$\mu_n \in \Omega^N \;.\equiv.\; \mu_n \in \Omega \;\&\; \text{non-alternate } \mu_n.$$

so $\Omega^N \subseteq \Omega$, and is the set of non-alternate model sets in a model system Ω.

$$\mu_n \in \Omega^A \;.\equiv.\; \mu_n \in \Omega \;\&\; \mu_m \in \Omega^N \;\&\; \mu_n \text{ is alternate to } \mu_m.$$

So $\Omega^A \subseteq \Omega$, and is the set of alternate model sets in a model system Ω which are alternate to non-alternate sets.

$$\mu_n \in \Omega^{AA} \;.\equiv.\; \mu_n \in \Omega \;\&\; \mu_n \text{ is alternate to } \mu_m \;\&\; \mu_m \in \Omega - \Omega^N.$$

So $\Omega^{AA} \subseteq \Omega$, and is the set of alternate model sets in a model system Ω which are alternate to alternate sets.

Secondly, let us consider under what conditions $\{\sim\Box(A \supset B) \vee \sim\Box(B \supset A)\}$ is imbeddable in μ_1 where $\mu_1 \in \Omega^N$. By definition of $\vee$ and condition 2, either $\sim\Box(A \supset B)$ and $\sim\Box(B \supset A) \in \mu_1$, or $\sim\Box(A \supset B) \in \mu_1$, or $\sim\Box(B \supset A) \in \mu_1$. Hence by condition 8, $\mu_2 \in \Omega^A$ and either $\{A, \sim B\} \subseteq \mu_2$ or $\{B, \sim A\} \subseteq \mu_2$. But if we had assumed that $(n)(\mu_n \in \Omega^A \;.\supset.\; A \equiv B \in \mu_n)$ then $A \equiv B \in \mu_2$, then it would have followed that $\Box(A \supset B)$ & $\Box(B \supset A)$ would be valid. Conversely, if we had assumed that $\sim(n)(\mu_n \in \Omega^A \;.\supset.\; A \equiv B \in \mu_n)$, then $\{\sim\Box(A \supset B) \vee \sim\Box(B \supset A)\}$ is imbeddable in some non-alternate set. From this it follows that $\Box(A \supset B)$ & $\Box(B \supset A)$ would not be valid. So, we can conclude that $\Box(A \supset B)$ & $\Box(B \supset A)$ is valid iff $(n)(\mu_n \in \Omega^A \;.\supset.\; A \equiv B \in \mu_n)$.

Finally, to prove our hypothesis we assume that $\Box(A \supset B)$ & $\Box(B \supset A)$ is valid but that $\Box(\Box A \supset \Box B)$ & $\Box(\Box B \supset \Box A)$ is not valid. I.e., $\sim(\Box(\Box A \supset \Box B) \;\&\; \Box(\Box B \supset \Box A))$ is satisfiable. Let $\sim(\Box(\Box A \supset \Box B) \;\&\; \Box(\Box B \supset \Box A)) \in \mu_1$ and $\mu_1 \in \Omega^N$ so, by condition 8, $\mu_2 \in \Omega^A$ and either

$$\text{i) } \{\Box A, \sim\Box B\} \subseteq \mu_2, \quad \text{or} \quad \text{ii) } \{\Box B, \sim\Box A\} \subseteq \mu_2,$$

so, since $\Box(A \supset B)$ & $\Box(B \supset A)$ is valid, $A \equiv B \in \mu_2$ and therefore, by condition 8, either

$$\text{iii) } \{A \equiv B, \Box A, \sim\Box B\} \subseteq \mu_2, \quad \text{or} \quad \text{iv) } \{A \equiv B, \sim\Box A, \Box B\} \subseteq \mu_2,$$

and so for iii) either

$$\{A \equiv B, \Box A, A, \sim B\} \subseteq \mu_2, \text{ or } \mu_3 \in \Omega^{AA} \text{ and } \{A, \sim A, B, \sim B\} \subseteq \mu_3;$$

(and similarly for iv)), all of which are contradictory, so by *reductio* our hypothesis is proved.

We show that R3 holds in $\langle \Omega, C_{S2} \rangle$ by showing that if $\Box(\Box A \supset \Box B)$ is not valid then $\Box(A \supset B)$ is not valid.

Proof: If we can construct an S2 model system in which $\{\sim\Box(\Box A \supset \Box B)\}$ is imbedded in a non-alternate set, then we can construct an S2 model system in which $\{\sim\Box(A \supset B)\}$ is imbedded in a non-alternate set also. Let Ω_1 be an S2 model system such that

$$\Omega_1 = \{\mu_1, \mu_2, \mu_3\}$$

where $\mu_1 = \{\sim\Box(\Box A \supset \Box B)\}$, so it will follow that:
$\mu_2 = \{\Box A, \sim\Box B\}$ (by condition 5, μ_2 is an alternative to μ_1)
and $\mu_3 = \{A, \sim B\}$ (by condition 5, μ_3 is an alternative to μ_2).

Now, although $\{\mu_3\} = \Omega_1^{A}$ there is nothing in any of the conditions in C_{S2} to prevent $\Omega_1^{AA} = \Omega_2^{A}$ where Ω_2 is an S2 model system such that

$$\Omega_2 = \{\mu_4, \mu_3\} \quad \text{and} \quad \mu_4 = \{\sim\Box(A \supset B)\}$$

so that μ_3 is an alternative to μ_4 in terms of condition 5. Since, if $\sim\Box(\Box A \supset \Box B)$ is satisfiable then $\sim\Box(A \supset B)$ is also satisfiable, then if $\Box(A \supset B)$ is valid then $\Box(\Box A \supset \Box B)$ is valid.

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