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SOME EXTENSIONS OF S3

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Sobociński [4] obtained modal systems S3.02, S3.03, S3.04, by adding to S3 the respective axioms

Ł1 $((p ⥽ \mathsf{L}p) ⥽ p) ⥽ (\mathsf{LML}p \supset p)$
Ł2 $((p ⥽ \mathsf{L}p) ⥽ p) ⥽ (\mathsf{LML}p ⥽ p)$
L1 $\mathsf{LML}p ⥽ (p \supset \mathsf{L}p)$.

This note (in the notation of [2]) clarifies the relationships of these systems to one another, and to other extensions of S3.

It is easy to test ([2], p. 279f.) that

(1) $\vdash_{S3} (\mathsf{L}p ⥽ (\mathsf{L}q \supset r)) ⥽ (\mathsf{L}p ⥽ (\mathsf{L}q ⥽ r))$.

A substitution instance of this is **Ł1** ⥽ **Ł2**, so S3.02 = S3.03. It is also easy to test ([2], p. 284f.) that $\vdash_{S3.5}$**Ł1** and $\vdash_{S3.5}$**L1**. Hence both S3.02 and S3.04 are contained in S3.5. Moreover, both S3.02 and S3.04 contain the system 16s of [3]. For:

S3: (2) $((\mathsf{L}p \supset q) ⥽ \mathsf{L}r) ⥽ (\mathsf{L}s \supset \mathsf{LL}s)$

Ł2$[p/(\mathsf{L}p \supset \mathsf{LL}p)]$, (2) $[q/\mathsf{LL}p, r/(\mathsf{L}p \supset \mathsf{LL}p), s/p]$:

(3) $\mathsf{LML}(\mathsf{L}p \supset \mathsf{LL}p) ⥽ (\mathsf{L}p \supset \mathsf{LL}p)$

(3), (1): (4) $\mathsf{LML}(\mathsf{L}p \supset \mathsf{LL}p) ⥽ (\mathsf{L}p ⥽ \mathsf{LL}p)$

S3: (5) $\mathsf{LML}(\mathsf{LML}p \supset \mathsf{L}p)$

(4)$[p/\mathsf{MLLM}p]$, (5) $[p/\mathsf{LM}p]$, S3: (6) $\mathsf{LMLLM}p ⥽ \mathsf{LLM}p$

Hence ([3], p. 275) S3.02 contains 16s. Also:

S3: (7) $\mathsf{LMLLM}p ⥽ \mathsf{LM}p$

L1 $p/\mathsf{LM}p$, (7), S2: (8) $\mathsf{LMLLM}p ⥽ \mathsf{LLM}p$

Hence S3.04 contains 16s.

Another system between S3.5 and 16s is 14r ([3], p. 273). But Table 2.2 ([3], p. 274) (i.e., Lewis Group II) readily shows that neither S3.02 nor S3.04 contains 14r, so these systems have 16s modalities. Sobociński has pointed out ([4], p. 417) that S3.02 does not contain S3.04. Also, by Table 3.2

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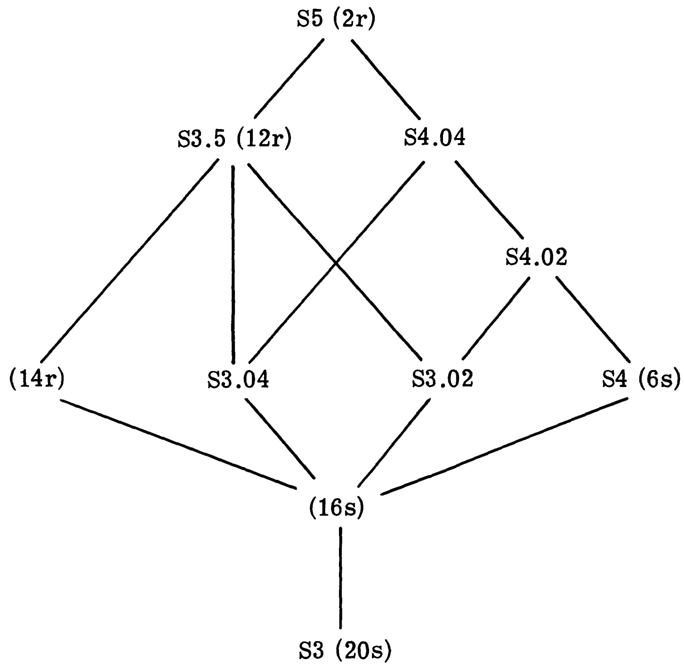
([3], p. 274), S3.02 is contained in neither S3.04 nor 14r. Finally, S3.04 is not contained in 14r; as is shown by the algebra with atoms a, b, c, d, with elements containing a designated, and with L and M specified by $\mathsf{M}0 = c \cup d$, $\mathsf{M}a = \mathsf{M}d = a \cup c \cup d$, and $\mathsf{M}b = \mathsf{M}c = 1$. (This was suggested by the 14r semantics of [1], p. 80.)

Thus the only inclusions among the systems mentioned in this note and their joins with S4, are those shown in the following diagram. (A line between two systems indicates that the upper contains the lower.)



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