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SET-VALUED SET THEORY: PART TWO

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3* Development of the Elementary Theory First note that the following concepts were defined in section **2** in the process of developing the axioms: subset, null set, strong pair, weak pair, strong unit set, weak unit set, ordered pair, function, domain, range, into, onto, standard union, union, strong power set, power set, strong Cartesian product, and Cartesian product. In addition, the two concepts degree and standard, which have no classical counterparts, were introduced. In this section, we first need to define the other concepts common in elementary classical set theory and then to verify that the sets given by the various definitions have the usual properties, including existence. (The reader is reminded that $\exists!y$ means that there exists a unique y: there is such a y and any two are equal.)

Thm. 6: $(\forall x)[\mathrm{Std}(x) \supset (\exists!y)(\forall z)(\forall w)(\epsilon(z,y,w) \equiv (\forall t)(\exists t')(\epsilon(t,x,t') \wedge \sim(t' = \emptyset)) \supset \epsilon(z,t,w))]$.

Def. 25: The set y of Theorem 6 is denoted by $\bigcap_s x$. (Strong Intersection)

Proof of Thm. 6: The uniqueness of the strong intersection of a given standard x follows by the usual extensionality argument, since membership in that intersection is defined by an equivalence. The existence of the strong intersection follows from the Axiom of Separation (listed in section **2** as a consequence of the Axiom of Replacement) and the Axiom of Unions, since the intersection is that subset of the union $\bigcup_s x = \bigcup x$ satisfying the condition given on the right hand side of the equivalence sign in the statement of Theorem 6. QED

Thm. 7: $(\forall x)(\exists!y)((\forall v)(\forall w)[\epsilon(v,y,w) \equiv (\forall t)[(\exists t')(\epsilon(t,x,t') \wedge |\sim(t' = \emptyset)) \supset (\epsilon(t,x,w) \wedge \epsilon(v,t,w))]])$.

Def. 26: The set y of Theorem 7 is denoted by $\bigcap y$. (Intersection)

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Proof of Thm. 7: This follows precisely in the same way that Theorem 6 did, since the membership relation is again given by an equivalence and the set in question is a subset of $\bigcup x$. QED

To determine whether we should place v in the intersection of x at least to the degree w, we examine all of the elements t that belong to some non-$\emptyset$ degree and then require that both of the following conditions be satisfied for all such t: a) t belongs to x at least to the degree w; b) v belongs to t at least to the degree w. As in the case of unions, we have the following consequence of the definitions.

Thm. 8: $(\forall x)\left[\mathrm{Std}(x) \supset \left(\bigcap_s x = \bigcap x\right)\right]$. QED

Def. 27: $x \cap y$ is $\bigcap\{x, y\}$.

Def. 28: We say that x and y are *disjoint* if $x \cap y = \emptyset$. (Disjoint)

From this point on, for sets whose definition insures their uniqueness by extensionality and whose existence is true because of a simple application of the Axiom of Replacement (or its consequences, the Axiom of Separation) to a set that we already know to exist, we will simply list the definition without writing out the formal proof that justifies that definition. For example, for the first definition below we need to apply the Axiom of Replacement of section 2 with u the set $x \cap y$ and A the formula that states that its second argument is the set of all degrees w of membership that its first argument has with respect to the set $x \cap y$.

Def. 29: Given x and y, suppose that

$(\forall z)(\forall w)\,[\epsilon(z, x \cap y, w) \supset (\exists v)(\forall v')(\forall v'')\,\epsilon(v', v, v'') \equiv [\epsilon(z, x \cap y, v') \wedge \mathrm{D}(v'')]]$:

then $\mathrm{DS}(x, y)$ is the set given by the condition

$(\forall z)(\forall w)\,[\epsilon(z, \mathrm{DS}(x, y), w) \equiv (\exists t)(\epsilon(t, x \cap y, z) \wedge (\exists t')(\epsilon(t, x \cap y, t') \wedge w \subseteq t'))]$.

(Degree of Separation)

Thus, if for every z, the collection of degrees with which z belongs to $x \cap y$ forms a standard set, the degree of separation of x and y is the set whose elements to some non-$\emptyset$ degree are the degrees with which the elements of $x \cap y$ belong to $x \cap y$. For any such degree z, z belongs to $\mathrm{DS}(x, y)$ at least to the degree w provided that some t which belongs to $x \cap y$ with degree z at least also belongs to $x \cap y$ with degree at least w. (This seems to be the maximal degree to which the Axiom of Replacement allows us to place elements in $\mathrm{DS}(x, y)$.) Again, the process of making $\mathrm{DS}(x, y)$ the collection of all suitable degrees is our substitute for picking out the "maximal" degree which may or may not exist. Since we know that $\{\emptyset\}$ is a set, the definition has the following corollary.

Col. to Def. 29: *If x and y are disjoint,* $\mathrm{DS}(x, y) = \emptyset$. QED

Def. 30: Given two sets x and y, x-y is the set given by the condition

$(\forall z)(\forall w)\,[(\epsilon(z, x\text{-}y, w) \wedge \sim(w = \emptyset)) \equiv [((\forall t)(\epsilon(z, y, t) \supset t = \emptyset) \wedge \epsilon(z, x, w)) \wedge \sim(w = \emptyset)]]$. (Difference)

The difference x-y (or relative complement) of the sets x and y may thus be described as follows. If z belongs to x only to the degree $\emptyset$, z belongs to x-y only to the degree $\emptyset$. If z belongs to y with any non-$\emptyset$ degree, then z belongs to x-y only to the degree $\emptyset$. Finally, if z belongs to x at least to some non-$\emptyset$ degree w, but belongs to y only to the degree $\emptyset$, then z belongs to x at least to the degree w. This definition of difference is rather strong in that it cuts the degree of membership of any element z of x which happens "really" to belong to y down to the minimal degree $\emptyset$ when z is considered as a possible element of x-y.

Def. 31: $x\theta y$ is $(x\text{-}y) \cup (y\text{-}x)$. (Symmetric Difference)

This looks like the usual definition of symmetric difference. Its interpretation is that for z's that belong to either both x and y or to neither x nor y with some non-$\emptyset$ degree, the degree of membership of z in $x\theta y$ is $\emptyset$, whereas for z's which belong to x to some non-$\emptyset$ degree but to y only to the degree $\emptyset$, the degrees of membership of z in x in $x\theta y$ are the same as the degrees of membership of z in x (and similarly for z's belonging to y to some non-$\emptyset$ degree but belonging to x only to the degree $\emptyset$).

Note: As usual in axiomatic set theory, there is no notion of (absolute) complement defined, since the collection given by the usual proposed definition for such a complement cannot in general consistently be assumed to be a set.

Thm. 9: *Let x be any set and a, b, and c subsets of x. Then the following identities hold:*

1. $a \cup b = b \cup a$ — 1′. $a \cap b = b \cap a$
2. $a \cup (b \cup c) = (a \cup b) \cup c$ — 2′. $a \cap (b \cap c) = (a \cap b) \cap c$
3. $a \cup \emptyset = a$ — 3′. $a \cap \emptyset = \emptyset$
4. — 4′. $a \cap (x\text{-}a) = \emptyset$
5. $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ — 5′. $a \cup (b \cap c) = (a \cup b) \cap (a \cap c)$

That is, for any given set x, the collection of all subsets of x fails to be Boolean algebra with respect to union, intersection and difference only in that the law $a \cup (x\text{-}a) = x$ need not hold.

Proof: It is immediate from the definitions of union, intersection, complement, and subset that all of the sets in question are subsets of x. 1 and 1′ are also immediate since, for example,

$$a \cup b = \bigcup\{a, b\} = \bigcup\{b, a\} = b \cup a.$$

2 and 2′ are almost as immediate. By definition, we are to show that, for example,

$$\bigcap\left\{a, \bigcap\{b, c\}\right\} = \bigcap\left\{\bigcap\{a, b\}, c\right\}.$$

Call this $p = q$. By the definition of intersection, $\epsilon(z, p, w)$ is true if and only if $\epsilon(z, a, w)$ and $\epsilon\left(z, \bigcap\{b, c\}, w\right)$, i.e., if and only if $\epsilon(z, a, w)$ and $\epsilon(z, b, w)$ and $\epsilon(z, c, w)$. (Remember, that the unordered pairs in question are standard sets.) But $\epsilon(z, q, w)$ is true if and only if $\epsilon\left(z, \bigcap\{a, b\}, w\right)$ and $\epsilon(z, c, w)$, i.e., if and only if $\epsilon(z, a, w)$ and $\epsilon(z, b, w)$ and $\epsilon(z, c, w)$. Hence $\epsilon(z, p, w)$ if and only if $\epsilon(z, q, w)$, so that $p = q$.

3 follows from the definition of union and the facts that $(\forall z)(\forall w)$ $\epsilon(z, \emptyset, v) \supset v = \emptyset$ and $(\forall a)(\forall z)(\epsilon(z, a, \emptyset))$. Similarly, 3′ follows from the definition of intersection and the fact that $a \subseteq x$.

To prove 4′, it is perhaps easiest to consider three cases separately. If w is such that

$$(\forall v)(\epsilon(w, x, v) \supset v = \emptyset),$$

so that also

$$(\forall v)(\epsilon(w, a, v) \supset v = \emptyset),$$

then by the definition of difference,

$$(\forall v)(\epsilon(w, x\text{-}a, v) \supset v = \emptyset),$$

so that

$$(\forall v)(\epsilon(w, a \cap (x\text{-}a), v) \supset v = \emptyset)$$

by the definition of intersection. If w is such that

$$(\exists v)(\epsilon(w, x, v) \wedge \sim(v = \emptyset))$$

but

$$(\forall v)(\epsilon(w, a, v) \supset v = \emptyset).$$

Then by the definition of difference

$$(\forall v)(\epsilon(w, x\text{-}a, v) \equiv \epsilon(w, x, v))$$

so that

$$(\forall v)(\epsilon(a \cap (x\text{-}a), v) \supset v = \emptyset),$$

by the definition of intersection. Finally, if

$$(\exists v)(\epsilon(w, a, v) \supset (v = \emptyset)),$$

so that

$$(\exists v)(\epsilon(w, x, v) \supset (v = \emptyset)).$$

Since $a \subseteq x$, then by the definition of difference

$$(\forall v)(\epsilon(w, x\text{-}a, v) \supset v = \emptyset),$$

so that

$$(\forall v)(\epsilon(w, a \cap (x\text{-}a), v) \supset v = \emptyset).$$

Hence,

$$(\forall w)(\forall v)(\epsilon(w, a \cap (x\text{-}a), v) \supset v = \emptyset).$$

Hence

$$a \cap (x\text{-}a) = \emptyset.$$

The final two identities 5 and 5′ have proofs rather like those of 3 and 3′ since all of the unordered pairs implied in their statements are standard. For example, for 5, we need to verify that

$$\bigcap\left\{a, \bigcup\{b, c\}\right\} = \bigcup\left\{\bigcap\{a, b\}, \bigcap\{a, c\}\right\}.$$

Call this $p = q$. Then we must show that

$$(\forall z)(\forall w)(\epsilon(z, p, w) \equiv \epsilon(z, q, w)).$$

So suppose $\epsilon(z, p, w)$. Then

$$\epsilon(z, a, w) \wedge \epsilon\left(z, \bigcup\{b, c\}, w\right),$$

so that

$$\epsilon(z, a, w) \wedge (\epsilon(z, b, w) \wedge \epsilon(z, c, w)).$$

In the former case, we have $\epsilon\left(z, \bigcap\{a, b\}, w\right)$ and in the latter $\epsilon\left(z, \bigcap\{a, c\}, w\right)$, so that in any case, $\epsilon(z, q, w)$. Conversely, if $\epsilon(z, q, w)$, then

$$\epsilon\left(z, \bigcap\{a, b\}, w\right) \vee \epsilon\left(z, \bigcap\{a, c\}, w\right),$$

i.e., either

$$\epsilon(z, a, w) \wedge \epsilon(z, b, w)$$

or else

$$\epsilon(z, a, w) \wedge \epsilon(z, c, w).$$

So in any case, we have

$$\epsilon(z, a, w) \wedge (\epsilon(z, b, w) \vee \epsilon(z, c, w)),$$

i.e.,

$$\epsilon(z, a, w) \wedge \epsilon(z, b \cup c, w),$$

so that we can conclude $\epsilon(z, p, w)$. Hence,

$$(\forall z)(\forall w)\, \epsilon(z, p, w) \equiv \epsilon(z, q, w). \qquad \text{QED}$$

The following observations should make it clear why we do not claim an identity 4 ($a \cup (x\text{-}a) = x$) parallel to 4′. Let a be a subset of x such that for some z in a, there exists two non-$\emptyset$ degrees v and w with the following properties:

$$\epsilon(z, x, v),\ \epsilon(z, x, v'),\ \epsilon(z, a, v),\ \sim\epsilon(z, a, v').$$

Then the definitions of union and difference show that the following hold:

$$\epsilon(z, a \cup (x\text{-}a), v),\ \sim\epsilon(z, a \cup (x\text{-}a), v').$$

Hence it would not be true that $a \cup (x\text{-}a) = x$ in this case; the problem seems to lie with those non-$\emptyset$ degrees to which some w belongs to x but not to a.

Def. 32: $\mathrm{Std}(x, y) \equiv [x \subseteq y \wedge (\forall z)(\forall w)((\exists w')(\epsilon(z, x, w') \wedge \sim(w' = \emptyset)) \supset (\epsilon(z, x, w) \equiv \epsilon(z, y, w)))]$. [Standard Subset]

We say that x is a standard subset of y if x is a subset of y and for any z, if z is an element of x to some non-$\emptyset$ degree, then z is an element of x precisely to the same degrees that z is an element of y. One can view standard sets as sets which are standard subsets of any sets of which they are subsets.

Thm. 10: $\mathrm{Std}(x, y) \equiv [x \cup (y-x) = y]$.

Proof: We noted above that if x is not a standard subset of y and x is a subset of y, then $x \cup (y-x)$ and y are not identical. On the other hand, if it is not a case that x is a subset of y, then

$$(\exists z)(\exists w)(\epsilon(z, x, w) \wedge \sim \epsilon(z, y, w)).$$

But then $\epsilon(z, x \cup (y-x), w)$ so that $\sim(x \cup (y-x) = x)$. Now suppose $\mathrm{Std}(x, y)$. Again the consideration of three cases seems easiest. If w is such that $\epsilon(w, y, z)$ holds only for $z = \emptyset$ (so that $\epsilon(w, x, z)$ holds only for $z = \emptyset$), then $\epsilon(w, y-x, w)$ holds only for $z = \emptyset$. Hence $\epsilon(w, x \cup (y-x), w)$ holds only for $w = \emptyset$, by the definition of union. If w is such that $\epsilon(w, y, z)$ holds for some non-$\emptyset$ z, but $\epsilon(w, x, z)$ hold only for $z = \emptyset$, then $\epsilon(w, y-x, z)$ holds precisely when $\epsilon(w, y, z)$ holds, by the definition of difference, so that $\epsilon(w, x \cup (y-x), z)$ holds precisely for those z such that $\epsilon(w, y, z)$ holds, by the definition of union. Finally, if w is such that for some non-$\emptyset$ z, $\epsilon(w, x, z)$ holds, then some x is a standard subset of y, $\epsilon(w, x, z)$ holds for precisely those z's such that $\epsilon(w, y, z)$ holds. Also, by the definition of difference, $\epsilon(w, y-x, z)$ holds only for $z = \emptyset$, so that by the definition of union, $\epsilon(w, x, \cup (y-x), z)$ holds if and only if $\epsilon(w, x, z)$ holds if and only if $\epsilon(w, y, z)$ holds. Hence

$$(\forall w)(\forall z)[\epsilon(w, x \cup (y-x), z) \equiv \epsilon(w, y, z)],$$

so that

$$x \cup (y-x) = y. \qquad \text{QED}$$

Thm. 11: *For any set y, the collection of all x such that* $\mathrm{Std}(x, y)$ *forms a Boolean algebra under union, intersection, and difference.*

Proof: Since the union (intersection) of two standard subsets of y and the difference between y and any standard subset x of y is a standard subset of y, we need only verify the Boolean properties. But all of these except one are given by Theorem 9 and that one is given by Theorem 10. QED

Note: One must be careful to remember that $\mathrm{Std}(x, y)$ implies neither $\mathrm{Std}(x)$ nor $\mathrm{Std}(y)$.

Now that it is clear that not all of the classical set theoretical identities are satisfied by all of our set, it becomes profitable to determine which of the commonly used identities do hold in our theory.

Thm. 12: *The operators of union and intersection satisfy all of the identities of a distributive lattice;* i.e., identities 1, 1′, 2, 2′, 5, and 5′, together with the following identities:

6. $a \cup a = a$ 6′. $a \cap a = a$
7. $a \cap (a \cup b) = a$ 7′. $a \cup (a \cap b) = a$

Proof: All of the identities of Theorem 9 mentioned above hold for arbitrary sets. The proofs given in the proof of Theorem 9 did not depend on the assumption that the sets in question were subsets of some set x. Identities 6 and 6′ above both follow immediately from the definition of the union (intersection) of a strong set: for any v and w, $\epsilon(v, a \cup a, w)$ holds if and only if $\epsilon(v, a, w)$ or $\epsilon(v, a, w)$ holds, etc. To verify 7, we note that if $\epsilon(w, a \cap (a \cup b), z)$, then $\epsilon(w, a, z)$ (and $\epsilon(w, a \cup b, z)$). On the other hand, if $\epsilon(w, a, z)$, then $\epsilon(w, a \cup b, z)$, so that $\epsilon(w, a \cap (a \cup b), z)$. Hence

$$(\forall w)(\forall z)[\epsilon(w, a, z) \equiv \epsilon(w, a \cap (a \cup b), z)],$$

so $a = a \cap (a \cup b)$. The proof of 7′ is similar. QED

Col. to Thm. 12: *The relation subset is the order relation corresponding to the above lattice operations*, i.e.,

$$8.\ a \subseteq b \equiv (a \cup b = b) \qquad 8'.\ a \subseteq b \equiv (a \cap b = a).$$

Further, the order relation has the following usual relationships with the lattice operations union and intersection:

9. $a \subseteq a \cup b$ 9′. $a \cap b \subseteq b$
10. $((a \subseteq b) \wedge (b \subseteq c)) \supset a \subseteq c$
11. $((a \subseteq b) \wedge (c \subseteq d)) \supset ((a \cup c) \subseteq (b \cup d))$
11′. $((a \subseteq b) \wedge (c \subseteq d)) \supset ((a \cap c) \subseteq (b \cap d))$.

In addition, the following laws relating subset and difference hold:

12. $(a \subseteq b) \supset (b\text{-}a = \emptyset)$
13. $(a \subseteq b) \supset (c\text{-}b \subseteq c\text{-}a)$.

Proof: Most of these facts are consequences of 8, together with Theorems 9 and 12. To verify 8, suppose that $a \subseteq b$. Then

$$\epsilon(z, a, w) \supset \epsilon(z, b, w),$$

so that

$$\epsilon(z, a \cup b, w) \equiv \epsilon(z, b, w).$$

Hence, $a \cup b = b$. On the other hand, if $a \cup b = b$ so that

$$\epsilon(z, a \cup b, w) \equiv \epsilon(z, b, w),$$

then since

$$\epsilon(z, a, w) \supset \epsilon(z, a \cup b, w),$$

we have

$$\epsilon(z, a, w) \supset \epsilon(z, b, w),$$

i.e., $a \subseteq b$. The verification of 8′ is similar. Relations 9, and 9′, and 10 follow from the definition of subset. A translation of the classical proof of 11 and 11′ is sufficient to verify these implications.

To check 12, $a \subseteq b$ tells us that if $\epsilon(z, a, w)$, then $\epsilon(z, b, w)$, so that, in particular, if $\sim(w = \emptyset)$, then $\sim(\epsilon(z, b\text{-}a, w))$ so that

$$\epsilon(z, b\text{-}a, w) \supset w = \emptyset,$$

as desired. In 13, assume

$$\epsilon(z, a, w) \supset \epsilon(z, b, w),$$

so that, in particular

$$(\forall z)[(\exists w)(\epsilon(z, a, w) \wedge \sim(w = \emptyset)) \supset (\exists w)(\epsilon(z, b, w) \wedge \sim(w = \emptyset))].$$

From this we can conclude, using the definition of difference

$$(\forall z)(\forall w)[(\epsilon(z, c\text{-}b, w) \wedge \sim(w = \emptyset)) \supset \epsilon(z, c\text{-}a, w)]$$

i.e., $c\text{-}b \subseteq c\text{-}a$. QED

Observe that we could not expect the converses of 12 and 13 to hold in general. $b\text{-}a = \emptyset$ tells us that any z which belongs to b to some non-$\emptyset$ degree w also belongs to a with some non-$\emptyset$ degree w', but this does not give us the necessary ordering relationship between w and w' needed to insure that $a \subseteq b$. The converse of 13 is false classically.

The above theorems combine to tell us that our operations of union and intersection and our relation of subset all behave in the classical ways and that we expect classical facts about them to be true. Further, the fact that the converse of 12 does not hold gives more evidence that it is the definition of difference that does not coincide in its properties with the classical definition. This is to be expected since we have not assumed that our collection of degrees is even a set, much less that there is a maximal degree. Below we investigate how difference and symmetric difference behave with respect to the other operations.

Thm. 13: *The following equalities and inclusions all hold. When an inclusion is stated, the reverse inclusion need not be true in general.*

14. $(a\text{-}b) \cap (a\text{-}c) = a\text{-}(b \cup c)$ (De Morgan)
14′. $(a\text{-}b) \cup (a\text{-}c) \subseteq a\text{-}(b \cap c)$ (De Morgan)
15. $(a \cup b) - c = (a\text{-}c) \cup (b\text{-}c)$
15′. $(a \cap b) - c = (a\text{-}c) \cap (b\text{-}c)$
16. $a\text{-}(b \cup c) = (a\text{-}b)\text{-}c$
16′. $(a\text{-}b) \cup (a \cap c) \subseteq a\text{-}(b\text{-}c)$
17. $a \cap (b\text{-}c) = (a \cap b)\text{-}c$
18. $a\text{-}b \subseteq a\text{-}(a \cap b)$.

Proof: The inclusion and equalities all follow by the usual classical proofs, using the definitions of union, intersection, complement, and subset. Here we only note why the converse equalities need not hold in general in 14′, 16′, and 18. In 14′, the problem is that

(i) $$(\exists w)(\epsilon(z, b, w) \wedge \sim(w = \emptyset))$$

together with

(ii) $$(\exists w')(\epsilon(z, c, w') \wedge \sim(w = \emptyset))$$

need not imply

(iii) $$(\exists w'')(\epsilon(z, b \cap c, w'') \wedge \sim(w'' = \emptyset)).$$

Suppose w and w'' are disjoint, so that they have no common subsets to non-$\emptyset$ degrees except $\emptyset$. This was the property of Brown's intersection mentioned in section 1 of this paper. Given a z such that (i) and (ii) are satisfied, but (iii) is not and such that

$$(\exists v)(\epsilon(z, a, v) \wedge \sim(v = \emptyset)),$$

then z belongs to $(a-b) \cup (a-c)$ only to the degree $\emptyset$, but to $a-(b \cap c)$ at least to the degree v. In 16′, the problem is with z such that $(\exists w)(\epsilon(z, a, w) \wedge \sim(w = \emptyset) \wedge \sim\epsilon(z, c, w))$, $(\exists w')(\epsilon(z, b, w') \wedge \sim(w' = \emptyset))$ and $(\exists w'')(\epsilon(z, c, w'') \wedge \sim(w = \emptyset))$. In this case we have $\epsilon(z, a-(b-c), w)$ but not $\epsilon(z, (a-b) \cup (a \cap c), w)$, so that

$$a-(b-c) \subseteq (a-b) \cup (a-c)$$

is impossible. In 18, the problem again is with intersections. Suppose we have a z such that $(\exists w)(\epsilon(z, b, w) \wedge \sim(w = \emptyset))$ and $(\exists w')(\epsilon(z, w, w') \wedge \sim(w' = \emptyset))$, but $(\forall w'')(\epsilon(z, a \cap b, w'') \supset w'' = \emptyset)$. Then we have that $\epsilon(z, a-b, v)$ only for $v = \emptyset$, but $\epsilon(z, a-(a \cap b), w')$ where w' is the degree asserted to be non-$\emptyset$ above. Hence, we cannot have

$$a-(a \cap b) \subseteq a-b,$$

in this case. QED

Thm. 14: *The following properties of symmetric difference hold:*

19. $a\theta b = b\theta a$
20. $a\theta\emptyset = a$

20′. $a\theta a = \emptyset$

21. $a \cap b = \emptyset \equiv a\theta b = a \cup b$
22. $(a\theta b)\,\theta(a \cap b) \subseteq a \cup b$, *but not necessarily conversely.*

22′. $a\theta(b\theta(a \cap b)) \subseteq a \cup b$, *but not necessarily conversely.*

23. $a\theta b \subseteq a\theta(a \cap b)$, *but not necessarily conversely.*
24. $a \cap (b\theta c) \subseteq (a \cap b)\,\theta(a \cap c)$, *but not necessarily conversely.*
25. *Neither of* $(a\theta b)\theta c$ and $a\theta(b\theta c)$ *need be a subset of the other.*
26. *Neither of* $a\theta(a\theta b)$ *and* b *need be a subset of the other.*
27. *It is not the case that* $(a\theta b) = c$ *necessarily implies* $b = a\theta c$.

Proof: Again, we leave the direct proofs to the reader, and indicate here why the classical cases of 22 through 27 fail here. In 22, if

$$\epsilon(z, a, w) \wedge \epsilon(z, b, w')$$

and

$$\sim\epsilon(z, a, w') \wedge \sim\epsilon(z, b, w),$$

then

$$\epsilon(z, a \cup b, w) \wedge \epsilon(z, a \cup b, w'),$$

but

$$\sim \epsilon(z, (a\theta b)\ \theta(a \cap b), w),$$

and

$$\sim \epsilon(z, (a\theta b)\ \theta(a \cap b), w'),$$

so that

$$\sim ((a \cup b) \subseteq (a\theta b)\ \theta(a \cap b)).$$

In 22′, several possibilities exclude the necessity of

$$a \cup b \subseteq a\theta(b\theta(a \cap b)).$$

For example, suppose that $\epsilon(z, a, w) \wedge \sim(w = \emptyset)$, and $\epsilon(z, b, w')$ and $\sim(w' = \emptyset)$. Then we always have $\epsilon(z, a \cup b, w')$. But if $w \cap w' = \emptyset$, we have

$$\sim \epsilon(z, a\theta(b\theta(a \cap b)), w)$$

and

$$\sim \epsilon(z, a\theta(b\theta(a \cap b)), w'),$$

while if $\sim(w \cap w') = \emptyset$, we have

$$\epsilon(z, a\theta(b\theta(a \cap b)), w),$$

but still

$$\sim \epsilon(z, a\theta(b\theta(a \cap b)), w').$$

In 23, if $\epsilon(z, a, w) \wedge \sim(w = \emptyset)$ and $\epsilon(z, b, w') \wedge \sim(w' = \emptyset)$. Then

$$(\forall v)(\epsilon(z, a\theta b, v) \supset v = \emptyset).$$

But if

$$(\forall v)(\forall v')((\epsilon(z, a, v) \wedge \epsilon(z \cdot b \cdot v')) \supset v \cap v' = \emptyset),$$

then

$$\epsilon(z, a\theta(a \cap b), w) \wedge \sim(w = \emptyset).$$

In 24, we need to consider z such that we have $\epsilon(z, a, w) \wedge \sim(w = \emptyset)$, $\epsilon(z, b, w') \wedge \sim(w' = \emptyset)$ and $\epsilon(z, c, w'') \wedge \sim(w'' = \emptyset)$. For such z, we have

$$(\forall v)(\epsilon(z, a \cap (b\theta c), v) \supset v = \emptyset),$$

but if

$$w \cap w' = \emptyset \wedge \sim(w \cap w' = \emptyset),$$

then

$$\epsilon(z, (a \cap b)\ \theta(a \cap c),\ w \cap w'').$$

In 25, we may use 19 and reletter to see that if one inclusion can fail, so can the other. So consider the case that z is such that $\epsilon(z, a, w) \wedge \sim(w = \emptyset)$ and $\epsilon(z, b, w') \wedge \sim(w' = \emptyset)$, and $\epsilon(z, c, w') \wedge \sim(w' = \emptyset)$. Then

$$\epsilon(z, a\theta(b\theta c), w) \wedge \sim\epsilon(z,(a\theta b)\theta c, w),$$

and also

$$[\epsilon(z(a\theta b)\theta c, w'') \wedge \sim\epsilon(z, a\theta(b\theta c), w'')].$$

In 26, consider a z such that $\epsilon(z, a, w) \wedge \sim(w = \emptyset)$, and $\epsilon(z, b, w') \wedge \sim(w' = \emptyset)$ and $\sim\epsilon(z, a, w') \wedge \sim\epsilon(z, b, w)$. Then

$$\epsilon(z, a\theta(a\theta b), w) \wedge \sim\epsilon(z, b, w)$$

and

$$\epsilon(z, b, w') \wedge \sim\epsilon(z, a\theta(a\theta b), w').$$

In 27, it is consistent with $a\theta b = c$ to assume that we have a z such that $(\forall w)(\epsilon(z, c, w) \supset w = \emptyset)$, $\epsilon(z, a, w') \wedge \sim(w' = \emptyset)$, and $\epsilon(z, b, w'') \wedge \sim(w'' = \emptyset)$, but $\sim\epsilon(z, b, w')$. Then $\epsilon(z, a\theta c, w')$ and $\sim\epsilon(z, a\theta c, w'')$, so that we can have

$$\sim(a\theta c \subseteq b) \wedge \sim(b \subseteq a\theta c)$$

in this case. QED

The following definitions will serve to complete the usual elementary vocabulary of set theory.

Def. 33: $\mathrm{BinR}(x) \equiv (\forall z)(\forall w)\,[(\epsilon(z, x, w) \wedge \sim(w = \emptyset)) \supset (\exists u)(\exists u')(\exists v)(v = u \times u' \wedge x \subseteq v)]$. (Binary Relation)

Thus, a binary relation is just a subset of some Cartesian product. The following weaker concept insists only that x be a collection of ordered pairs, all of whose first elements come from some given set u and whose second elements come from set v. Without the strong Axiom S12 of Products, these concepts may well differ.

Def. 34: $\mathrm{WBR}(x) \equiv [(\forall z)(\forall w)\,[(\epsilon(z, x, w) \wedge \sim(w = \emptyset)) \supset (\exists y)(\exists y')(z = \langle y, y'\rangle)] \wedge (\exists u)(\exists v)(\forall z')(\forall z'')\,[(\epsilon(z', x, z'') \wedge \sim(z'' = \emptyset)) \supset ((\forall w)(\forall w')(z = \langle w, w'\rangle) \supset (\exists t)(\exists t')(\epsilon(w, u, t) \wedge \epsilon(w', u', t') \wedge \sim(t = \emptyset) \wedge \sim(t' = \emptyset)))]]$. (Weak Binary Relation)

A comparison of the definitions will show that every binary relation is a weak binary relation. A function is a weak binary relation, since for example, because it is a set, by the Axiom of Replacement, both its domain and range are sets. (In this context, to say that something is a set just means that either we are assuming that we have already proved it to exist or else we know how to prove its existence.)

Thm. 15: $(\forall x)(\forall y)\,[\mathrm{WBR}(x) \wedge \mathrm{WBR}(y) \wedge [(\exists z)(\forall z')(\forall w)\,[(\epsilon(z', z, w) \wedge \sim(w = \emptyset)) \equiv (\exists u)(\exists u')(\exists v)(\exists v')(\exists t)(\exists t')(\exists t'')(\epsilon(u, x, v) \wedge \epsilon(u', y, v') \wedge \sim(v = \emptyset) \wedge \sim(w' = \emptyset) \wedge u = \langle t, t'\rangle \wedge v' = \langle t, t''\rangle \wedge z' = \langle t, t''\rangle \wedge w \subseteq v \wedge w \subseteq v')]] \supset \mathrm{WBR}(z)]$.

Def. 35: The z of Theorem 15 is denoted by $x \circ y$. (Composition)

Proof of Thm. 15: We have by hypothesis that all the z' that belong to z to some non-$\emptyset$ degree are ordered pairs and that the first elements of those pairs are members to the appropriate degrees to the same set of which the first elements of the pairs of x are members and that the second elements of those pairs are members to the appropriate degrees to the set of which the second elements of y are members. QED

As would be expected, we put a pair (t, t'') into the composition of two relations x and y to the "maximum" degree allowed for all t' such that (t, t') is in x to some non-$\emptyset$ degree and (t', t'') is in y to some non-$\emptyset$ degree, where, for each t', we allow the "minimum" of the respective degrees of membership of (t, t') and (t', t'') is x and y. Naturally, this same definition of composition applies to functions.

Applying the Axiom of Replacement to a weak binary relation in the usual manner with $A_n(x, y) = (\exists z)(x = \langle y, z\rangle)$, etc., constructs both its "domain" and "range." Hence we have the following theorem.

Thm. 16: $\mathsf{WBR}(x) \supset (\exists! y)(\exists! z)(\forall v)(\forall v')(\forall w)(\forall w')[[\epsilon(v, y, w) \equiv (\exists t)(\exists t')(\epsilon(t, x, w) \wedge t = \langle v, t'\rangle)] \wedge [\epsilon(v', z, w') \equiv (\exists u)(\exists u')(\epsilon(u, x, w) \wedge u = \langle u', v'\rangle)]]$.

Def. 36: The y of Theorem 16 is denoted $\mathsf{Dom}(x)$. (Domain)

Def. 37: The z of Theorem 16 is denoted $\mathsf{Ran}(x)$. (Range)

Col. to Thm. 16: *In* Za^+, $(\mathsf{WBR}(x) \wedge \mathsf{WBR}(y)) \supset (\exists! z)(z = x \circ y)$.

Proof: In Za^+, given $\mathsf{WBR}(x)$ and $\mathsf{WBR}(y)$, we can form $\mathsf{Dom}(x) \times_s \mathsf{Ran}(y)$. We then specify the unique collection $x \circ y$ by the Axiom of Separation. QED

Note that these definitions are consistent with the corresponding definitions (10 and 11) given in section 2 for functions. The interpretation given these as to the degrees that various elements belong to a domain or range also applies here. Note also that we could now weaken the definition of a weak binary relation by dropping the requirement that all the first elements belong to some fixed set to non-$\emptyset$ degrees and by dropping the requirement that all second elements belong to some fixed set to non-$\emptyset$ degrees. Finally, we define a few concepts that are helpful in comparing the axiomatized theory with the original less formal versions of Zadeh, Brown, and others.

Def. 38: $(x, y; z)$ is the set $(x-(x-z)) \cup (y-z)$. (Convex Combination)

Following Zadeh ([3], p. 345), it is easy to verify that the following theorem holds.

Thm. 17: $(\forall x)(\forall y)(\forall z)(x \cap y \subseteq (x, y; z) \subseteq x \cup y)$. QED

However, the corresponding theorem about representing all sets between $x \cap y$ and $x \cup y$ as some $(x, y; z)$ does not carry over. For example, if x' is an element of x to several different non-$\emptyset$ degrees but an element of y only to the degree $\emptyset$, then no matter what z we choose, $(x, y; z)$ will contain x' either only to the degree $\emptyset$ or to all of the degrees that x' belongs to x: no intermediate possibilities are allowed.

Def. 39: $x \cup_w y$ is the set $(x-y) \cup (x \cap y) \cup (y-x)$. (Weak Union)

Using the methods employed earlier in this section, the reader can verify that $x \cup_w y \subseteq x \cup y$, but not necessarily conversely: consider elements belonging both to x and to y to distinct non-$\emptyset$ degrees.

Def. 40: $x \cap_s y$ is the set defined by $(\forall z)(\forall w)[(\epsilon(z, x \cap_s y, w) \wedge \sim(w = \emptyset)) \equiv (\exists w')(\exists w'')(\epsilon(z, x, w') \wedge \sim(w' = \emptyset) \wedge \epsilon(z, x, w) \vee \epsilon(z, y, w))]$. (Strong Intersection)

As above we can verify that $x \cap y \subseteq x \cap_s y$ but not necessarily conversely. The interest in sets of the type $x \cap_s y$ and $x \cup_w y$ is that they satisfy identities of the type that fail for the usual operations. For example, it is easy to verify that the following weak forms of 28, 28′ of De Morgan's Law hold, although, as was noted in Theorem 13 above, the corresponding relation does not hold universally for the ordinary union, intersection, and complement.

Thm. 18: *The following identities hold, although their counterparts in* Theorems 13 *and* 14 *failed.*

27. $(\forall x)(\forall y)(((x \cup y) - (((x \cup y) - x) \cup ((x \cup y) - y))) = (x \cap_s y))$
28′. $(\forall x)(\forall y)(((x \cup_w y) - (((x \cup_w y) - x) \cup_w ((x \cup_w y) - y))) = (x \cap y))$
29. $(a-b) \cup_w (a \cap_s c) = a - (b-c)$
30. $a-b = a - (a \cap_s b)$
31. $(a\theta b)\, \theta(a \cap_s b) = a \cup b$
32. $a-b = a\theta(a \cap_s b)$
33. $a \cap_s (b\theta c) = (a \cap_s c)$ QED

Hence we have operations that satisfy most of the classical identities of elementary set theory. In the next section, we will use the various operations we have defined here to develop the theory of natural numbers, ordinals, and cardinals.

REFERENCES

Items [1] - [3] were given in Part I of this paper. See *Notre Dame Journal of Formal Logic*, vol. XV (1974), p. 634.

To be continued

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