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NORMAL FORMS IN MODAL LOGIC

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There are two main methods of completeness proof in modal logic. One may use maximally consistent theories or their algebraic counterparts, on the one hand, or semantic tableaux and their variants, on the other hand. The former method is elegant but not constructive, the latter method is constructive but not elegant.

Normal forms have been comparatively neglected in the study of modal sentential logic. Their champions include Carnap [3], von Wright [10], Anderson [1] and Cresswell [4]. However, normal forms can provide elegant and constructive proofs of many standard results. They can also provide proofs of results that are not readily proved by standard means.

Section **1** presents preliminaries. Sections **2** and **3** establish a reduction to normal form and a consequent construction of models. Section **4** contains a general completeness result. Finally, section **5** provides normal formings for the logics T and K4.

1 Preliminaries *Formulas* are constructed in the usual way from the following items: the set $Sl = \{p_0, p_1, \ldots\}$ of sentence letters; truth-functional operators, say $\vee$ and $-$; the modal operator $\Diamond$; and the brackets (and). We follow standard conventions concerning abbreviations, bracketing and use-mention. In particular, we use $\mathbf{T}$ for $p_0 \vee -p_0$ and $\bot$ for $-\mathbf{T}$.

The *minimal logic* **K** is the set of formulas derivable from the following postulates:

Axioms 1. All tautologous formulas
2. $\Diamond\bot \equiv \bot$
3. $\Diamond(p_0 \vee p_1) \equiv \Diamond p_0 \vee \Diamond p_1$

Rules 4. $A, A \supset B/B$
5. $A \equiv B/C \equiv (C^A/B)$
6. $A/(A^{p_i}/B)$

(C^A/B) is the result of substituting B for A in C; similarly for (A^{p_i}/B).

We refer to postulates 1 and 4 together as **PC**. A *logic* is a set of formulas that contains **K** and is closed under the same rules as **K**. Given a

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logic L and a set of formulas X, $\mathsf{L}X$ is the smallest logic to contain X. We write $\mathsf{L}A$ for $\mathsf{L}\{A\}$. A *model* is a triple $\langle W, R, \phi\rangle$ with W non-empty, $R \subseteq W^2$ and $\phi \subseteq W \times Sl$. The truth-relation, $\langle \mathfrak{A}, w\rangle \models A$, is defined in the usual way with $\langle \mathfrak{A}, w\rangle \models \Diamond B \Longleftrightarrow (\exists v)(wRv \ \& \ \langle \mathfrak{A}, v\rangle \models B)$. We may write $w \models A$ for $\langle \mathfrak{A}, w\rangle \models A$ if $\mathfrak{A}$ is understood. A *frame* $\mathfrak{F}$ is a pair $\langle W, R\rangle$ with W non-empty and $R \subseteq W^2$. A formula A is *valid in* a frame $\mathfrak{F} = \langle W, R\rangle$ if for all models $\mathfrak{A} = \langle W, R, \phi\rangle$ and for all $w \in W$, $\langle \mathfrak{A}, w\rangle \models A$. A logic L is *sound* (*sufficient*, *complete* respectively) for a class of frames X if, for any formula A, $A \in \mathsf{L} \Rightarrow (\Leftarrow, \Leftrightarrow$ respectively) $(\forall \mathfrak{F} \in X)$ (A is valid in $\mathfrak{F}$). A logic is *complete* if it is complete for some class of frames and it has the *finite model property* (fmp) if it is complete for some class of finite frames.

2 Reduction to Normal Form We fix, throughout the whole paper, on a finite set $Q = \{q_0, q_1, \ldots, q_h\}$ of sentence letters. For convenience, we suppose that all formulas are constructed from Q and that there is a standard ordering of all such formulas.

We define F_n, the set of *normal forms of degree n*, by induction on n:

(i) $n = 0$. F_0 is the set of formulas $\pi_0 q_0 \wedge \pi_1 q_1 \wedge \ldots \pi_h q_h$ where each π_i, $i \leqslant h$, is blank or -;
(ii) $n > 0$. Suppose that $A_0, A_1, \ldots, A_k$ are the members of F_{n-1} in the standard order. Then F_n is the set of formulas $B \wedge \pi_0 \Diamond A_0 \wedge \pi_1 \Diamond A_1 \wedge \ldots \pi_k \Diamond A_k$ where $B \in F_0$ and each π_i, $i \leqslant K$, is blank or -;
(iii) F, the set of *normal forms*, is $\bigcup_{n=0}^{\infty} F_n$.

The normal forms (nf's) of degree 0 are the state-descriptions of classical sentential logic. The normal forms of degree $n > 0$ are the state-descriptions in the sentence letters and the formulas of the form $\Diamond B$ with B an nf of degree $(n - 1)$. Distinct normal forms of the same degree *disagree*, i.e., one contains a conjunct that negates the corresponding conjunct of the other. These normal forms are the modal analogue to Hintikka's distributive normal forms [6]. They are essentially conjunctions of constituents in the sense of Anderson [1] or von Wright [10]. The corresponding model-theoretic notion is discussed in [5].

The (*modal*) *degree* $\mathsf{d}(A)$ *of* a formula A is the length of the longest string of nested $\Diamond$-occurrences in A: $\mathsf{d}(q_i) = 0$; $\mathsf{d}(-B) = \mathsf{d}(B)$; $\mathsf{d}(B \vee C) = \max(\mathsf{d}(B), \mathsf{d}(C))$; and $\mathsf{d}(\Diamond B) = \mathsf{d}(B) + 1$. *A implies* (*is equivalent to, is incompatible with*) *B in* logic L if $A \supset B$ ($A \equiv B$, $A \supset - B$) is a theorem of L.

Theorem 1 (Reduction to Normal Form) *Any formula A of degree* $\leqslant n$ *is equivalent in* K *to* $\perp$ *or a disjunction of normal forms of degree n.*

Proof: By induction on n.

$n = 0$. By **PC** and the disjunctive normal form theorem for classical sentential logic.
$n > 0$. A is a truth-functional compound of sentence letters and formulas of the form $\Diamond B$ where $\mathsf{d}(B) < n$. By **IH**, B is equivalent to $\perp$ or a disjunction

$B_0 \vee B_1 \vee \ldots \vee B_k$ of nf's of degree $(n - 1)$. In the former case, $\Diamond B$ is equivalent to $\perp$ by Postulate 2; and in the latter case, $\Diamond B$ is equivalent to $\Diamond B_0 \vee \Diamond B_1 \vee \ldots \vee \Diamond B_k$ by **PC** and repeated applications of Postulate 3. Therefore by **PC** and the theorem on disjunctive normal forms, A is equivalent to $\perp$ or a disjunction of nf's of degree n.

3 Model Construction For each nf A, let the *leading term* A_f of A be that nf of degree 0 which is a conjunct of A. Thus, for $\mathrm{d}(A) = 0$, A_f is A and, for $\mathrm{d}(A) > 0$, A_f is the first conjunct of A. For any nf's A and B, we say $A > B$ if $\Diamond B$ is a conjunct of A. It should be clear that for any subset Δ of F_n and $B \in F_0$ there is exactly one $A \in F_{n+1}$ such that $A_f = B$ and $(\forall C \in F_n)(A > C \Leftrightarrow C \in \Delta)$. In other words, an nf is uniquely determined by its leading terms and immediate successors.

For each $n \geq 0$, let $\mathfrak{A}_n$ be the model $\langle W, R, \phi\rangle$ such that:

$$W = \bigcup_{i \leqslant n} F_i;$$
$$R = \{\langle A, B\rangle \in W^2 : A > B\};$$
$$\phi = \{\langle A, p\rangle \in W \times Sl : p \text{ is a conjunct of } A_f\}.$$

Also, let $\mathfrak{F}_n = \langle W, R\rangle$.

Theorem 2 (Model Construction) *For* $\mathfrak{A}_n = \langle W, R, \phi\rangle$ *and* $A \in W$, $\langle \mathfrak{A}_n, A\rangle \vDash A$.

Proof: By induction on the degree n of A.

$n = 0$. Then $A = \pi_0 q_0 \wedge \pi_1 q_1 \wedge \ldots \wedge \pi_h q_h$ where each π_i, $i \leqslant h$, is blank or -. Now $\langle \mathfrak{A}_n, A\rangle \vDash A \Leftrightarrow (\forall i \leqslant h)\,(A \vDash q_i \Leftrightarrow \pi_i$ is blank$) \Leftrightarrow (\forall i \leqslant h)\,(\phi\, A q_i \Leftrightarrow \pi_i$ is blank$) \Leftrightarrow (\forall i \leqslant h)$ (q_i is a conjunct of $A \Leftrightarrow \pi_i$ is blank), which is clearly true. $n > 0$. We show that $\langle \mathfrak{A}_n, A\rangle \vDash C$ for each conjunct C of A. (a) $C = A_f$. As for the case $n = 0$. (b) $C = \Diamond B$. Then $A > B$, i.e., ARB. By **IH**, $\langle \mathfrak{A}_n, B\rangle \vDash B$. Therefore $\langle \mathfrak{A}_n, A\rangle \vDash \Diamond B = C$. (c) $C = -\Diamond B$. Suppose ARD, i.e., $A > D$ (to show $\langle \mathfrak{A}_n, D\rangle \nvDash B$). By **IH**, $\langle \mathfrak{A}_n, D\rangle \vDash D$. But D is distinct from and so disagrees with B. Therefore $\langle \mathfrak{A}_n, D\rangle \nvDash B$.

We can now prove completeness, fmp and decidability for the minimal modal logic.

Theorem 3 (Completeness) **K** *is complete for the class of all frames.*

Proof: Soundness, straightforward. *Sufficiency*, suppose that A is not a theorem of **K**. By **PC**, -**A** is consistent in **K**, i.e., not equivalent to $\perp$. By Theorem 1, $-A$ is equivalent to a disjunction of normal forms of degree $n = \mathrm{d}(A)$. Let B be the first disjunct of the disjunction. Then by Theorem 2, $\langle \mathfrak{A}_n, B\rangle \vDash B$. So finally, by soundness, $\langle \mathfrak{A}_n, B\rangle \vDash -A$ and A is not valid in the frame $\mathfrak{F}_n$.

Corollary 1 **K** *has* fmp.

Proof: From the proof of Theorem 3 and the fact that $\mathfrak{A}_n$ is finite.

Corollary 2 **K** *is decidable.*

Proof: From Corollary 1 or, more practicably, by reduction to normal form.

4 A General Completeness Result The set U_n, $n \geqslant 0$, of *uniform formulas of* degree n is defined by induction on n:

$U_0 = \{A : A \text{ does not contain } \Diamond\}$;
$U_{n+1} = \{A : A \text{ is a truth-functional compound of formulas } \Diamond B \text{ with } B \in U_n\}$.

$$\mathbf{U}\text{, the set of } \textit{uniform} \text{ formulas, is } \bigcup_{n=0}^{\infty} U_n.$$

Any two maximal strings of nested $\Diamond$-occurrences are of the same length in a uniform formula. For example, $\Diamond(-\Diamond p \wedge \Diamond q) \vee \Diamond\Diamond t$ is uniform, whereas $\Diamond(-p \wedge \Diamond q) \vee \Diamond\Diamond t$ is not.

Let $\mathbf{D} = \mathbf{K}(\Diamond\mathbf{T})$. Then a logic $\mathbf{L}$ is *uniform* if $\mathbf{L} = \mathbf{D}\Delta$ for some $\Delta \subseteq \mathbf{U}$. In this section, we show that all uniform logics are complete to the point of possessing fmp. First, we prove three results for $\mathbf{D}$. We define the $\mathbf{D}$-*suitable* nf's by induction on degree:

$n = 0$. Any nf of degree 0 is $\mathbf{D}$-suitable.
$n > 0$. An nf $A \in F_n$ is $\mathbf{D}$-suitable if $(\forall B)(A > B \Rightarrow B$ is $\mathbf{D}$-suitable) and $(\exists B)(A > B)$.

Lemma 1 *Any formula A of degree n is equivalent in $\mathbf{D}$ to $\perp$ or a disjunction of $\mathbf{D}$-suitable nf's of degree n.*

Proof: As for Theorem 1. For the inductive step, show that the disjunction of all formulas $\Diamond B$, $B \in F_{n-1}$, is a theorem of the logic $\mathbf{D}$.

For any nf A, let $\mathfrak{A}_A$ be the model $\langle W, R, \phi\rangle$ such that:

$$W = \{B : (\exists n \geqslant 0)(A >^n B)\} \cup \{\mathbf{T}\};$$
$$R = \{\langle A, B\rangle \in W^2 : A > B \text{ or } B = \mathbf{T} \;\&\; \mathrm{d}(A) = 0\};$$
$$\phi = \{\langle A, p\rangle \in W \times Sl : A \in F \text{ and } p \text{ is a conjunct of } A\}.$$

Also, let $\mathfrak{F}_A = \langle W, R\rangle$. Thus $\mathfrak{F}_A$ differs from $\mathfrak{F}_n$ in that it is tied to A and contains the end-point $\mathbf{T}$.

Lemma 2 *For $A \in F$, $\mathfrak{A}_A = \langle W, R, \phi\rangle$ and $B \in W$, $\langle \mathfrak{A}_A, B\rangle \models B$.*

Proof: As for Theorem 1.2. The case of $B = \mathbf{T}$ is trivial.

We say that a formula is *closed* if it is constructed from $\mathbf{T}$ (in place of sentence letters).

Lemma 3 *Each closed formula A is equivalent in $\mathbf{D}$ to $\mathbf{T}$ or $\perp$.*

Proof: By induction on the construction of A. The main case is $A = \Diamond B$. (a) B equivalent to $\mathbf{T}$. Then $\Diamond B$ is equivalent to $\mathbf{T}$ by $\mathbf{PC}$, Postulate 5 and the axiom $\Diamond\mathbf{T}$. (b) B equivalent to $\perp$. Then $\Diamond B$ is equivalent to $\perp$ by $\mathbf{PC}$ and Postulates 2 and 5.

Secondly, we prove two lemmas on uniformity. The second of these is the crucial part of the whole proof. Given two models $\mathfrak{A} = \langle W, R, \phi\rangle$ and $\mathfrak{B} = \langle W, R, \psi\rangle$ and a point $w \in W$, we say $\langle \mathfrak{A}, w\rangle \equiv_n \langle \mathfrak{B}, w\rangle$ if $(\forall v \in W)(\forall p \in Sl)$ $(wR^n v \Rightarrow (\phi v p \Leftrightarrow \psi v p))$. Thus two models are n-equivalent, in the required sense, if their respective valuations agree on the n'th successors of a given point. The model-theoretic import of uniform formulas is contained in the following result.

Lemma 4 *Suppose* $\langle \mathfrak{A}, w\rangle \equiv_n \langle \mathfrak{B}, w\rangle$ *and* $A \in U_n$. *Then* $\langle \mathfrak{A}, w\rangle \vDash A \Longleftrightarrow \langle \mathfrak{B}, w\rangle \vDash A$.

Proof: By induction on the construction of A.

$A = p$. $\langle \mathfrak{A}, w\rangle \vDash p \Longleftrightarrow \phi wp \Longleftrightarrow \psi wp$ (since $n = 0$) $\Longleftrightarrow \langle \mathfrak{B}, w\rangle \vDash p$.
$A = -B$. By **IH** and the fact that $-B \in U_n \Longrightarrow B \in U_n$.
$A = B \vee C$. By **IH** and the fact that $B \vee C \in U_n \Longrightarrow B, C \in U_n$.
$A = \Diamond B$. $\langle \mathfrak{A}, w\rangle \vDash \Diamond B \Longleftrightarrow (\exists v \in W)(wRv \;\&\; \langle \mathfrak{A}, v\rangle \vDash B) \Longleftrightarrow (\exists v \in W)(wRv \;\&\; \langle \mathfrak{B}, v\rangle \vDash B)$ (by **IH** and the fact that $\langle \mathfrak{A}, v\rangle \equiv_{n-1} \langle \mathfrak{B}, v\rangle) \Longleftrightarrow \langle \mathfrak{B}, w\rangle \vDash \Diamond B$.

For a set Δ of formulas, a nf A is Δ-suitable if each formula of Δ is valid in $\mathfrak{F}_A$.

Lemma 5 *For* $\Delta \subseteq$ **U**, *any* **D**-*suitable* nf *that is not also* Δ-*suitable is inconsistent in* **D**.

Proof: Suppose that A is a **D**-suitable nf but not Δ-suitable. Then, for some $B \in \Delta$, B is not valid in $\mathfrak{F}_A = \langle W, R\rangle$, i.e., for some model $\mathfrak{A} = \langle W, R, \phi\rangle$ and $C \in W$, $\langle \mathfrak{A}, C\rangle \nvDash B$. If **DB** is inconsistent, i.e., is the set of all formulas, then A is not consistent in **DB**. Therefore we may suppose that **DB** is consistent. If C is inconsistent, then so is A. For either $C = \mathbf{T}$, in which case **DB** is inconsistent, or $\exists n \geqslant 0 : A >^n C$, in which case A implies $\Diamond^n C$ and so is inconsistent. Therefore we may suppose that $C = A$. Let $\Gamma = \{C \in W : AR^n C\}$; and for $i \geqslant 0$, let $\Gamma_i = \{C \in \Gamma : \phi C p_i\}$ and $D_i = \bigvee_{C \in \Gamma_i} C$, i.e., the disjunction (in standard order) of the formulas in Γ_i if Γ_i is non-empty and $\perp$ otherwise. Finally, let $\mathfrak{B} = \langle W, R, \psi\rangle$ where $\psi = \{\langle C, p_i\rangle \in W \times \mathbf{SI} : \langle \mathfrak{A}_A, C\rangle \vDash D_i\}$.

Now for all $C \in \Gamma$, $\psi C p_i \Longleftrightarrow \langle \mathfrak{A}_A, C\rangle \vDash D_i \Longleftrightarrow C$ is a disjunct of D_i (by Lemma 2) $\Longleftrightarrow C \in \Gamma_i \Longleftrightarrow \phi C q_i$. Therefore $\langle \mathfrak{A}, A\rangle \equiv_n \langle \mathfrak{B}, A\rangle$; and so by Lemma 4, $\langle \mathfrak{B}, A\rangle \nvDash B$.

Let B' be the result of substituting D_i for p_i in B, $i \geqslant 0$. Then by an easy induction on the construction of B, $\langle \mathfrak{A}_A, A\rangle \nvDash B'$. We now distinguish two cases:

(a) $\mathrm{d}(A) \leqslant n$. Then $\Gamma = \{\mathbf{T}\}$ and so B' is closed. By Lemma 3, and the consistency of **DB**, B' is equivalent to $\mathbf{T}$ in **D**. By A **D**-suitable, $(\forall w \in W)(\exists v \in W)(wRv)$ and so $\Diamond \mathbf{T}$ is valid in $\mathfrak{F}_A$. But then $\langle \mathfrak{A}_A, A\rangle \nvDash \mathbf{T}$. A contradiction.
(b) $\mathrm{d}(A) = m > n$. Then $\Gamma \subseteq F_{m-n}$ and so B' is of degree $n + (m - n) = m$. By Theorem 1, A implies B' or is incompatible with B' in **K**. In the first case, $\langle \mathfrak{A}_A, A\rangle \nvDash A$, contrary to Lemma 2; and in the second case, A is inconsistent in **DB**.

We can now prove the general results on normal form reduction, model construction, and completeness.

Theorem 1 *For* $\Delta \subseteq \mathbf{U}$, *any formula* A *of degree* $\leqslant n$ *is equivalent in* $\mathfrak{D}\Delta$ *to* $\perp$ *or a disjunction of* **D**-*suitable and* Δ-*suitable* nf's *of degree* n.

Proof: From Lemmas 1 and 5.

Theorem 2 *For* $\Delta \subseteq \mathbf{U}$, *and for any* **D**-*suitable and* Δ-*suitable* nf A, $\langle \mathfrak{A}_A, A\rangle \models A$ *and each formula of* $\mathbf{D}\Delta$ *is valid in* $\mathfrak{F}_A$.

Proof: From Lemma 2 and the definitions of **D**-suitable and Δ-suitable.

Theorem 3 *Each uniform logic is complete with* fmp, *and decidable if finitely axiomatizable.*

Proof: From Theorems 1 and 2.

One case of Theorem 3 is of special interest. Let M be the axiom $\Box\Diamond p_0 \supset \Diamond\Box p_0$. Lemmon and Scott [7] proved the completeness of S4M (the logic S4.1 of McKinsey [8]), and Segerberg [9] and Bull [2] its decidability. But the questions of completeness and decidability for $\mathbf{K}M$ were left open. Now clearly M is uniform and $\Diamond\mathbf{T}$ is a theorem of $\mathbf{K}M$. So by the theorem above, $\mathbf{K}M$ is both complete and decidable.

5 The Logics T and K4 This section uses normal forms to establish the completeness of $T = K\ (p_0 \supset \Diamond p_0)$ and $K4 = K\ (\Diamond\Diamond p_0 \supset \Diamond p_0)$.

First we consider T. The *correlate* A' of an nf A of degree $n \geqslant 1$ is defined by induction on n:

$n = 1$. $A' = A_f$.
$n > 1$. A' is the nf B of degree $(n - 1)$ such that $B_f = A_f$ and $(\forall C)(B > C \Leftrightarrow (\exists D)(A > D \;\&\; D' = C))$.

Lemma 1 *For any* nf A *of degree* $\geqslant 1$, A *implies* A' *in* **K**.

Proof: By induction on the degree n of A:

$n = 1$. Since A implies $A_f = A'$.
$n > 1$. We show that A implies each conjunct C of A':

(a) $C = A_f$. By **PC**. (b) $C = \Diamond B$. Then $A' > B$. So $\exists D : A > D$ and $D' = B$. By **PC**, A implies $\Diamond D$. By **IH**, D implies B and so, by **K**, $\Diamond D$ implies $\Diamond B$. But then A implies $\Diamond B = C$. (c) $C = -\Diamond B$. Let $\Delta = \{D \in F_{n-1} : D' = B\}$. Then $-\Diamond D$ is a conjunct of A for each $D \in \Delta$. By **IH**, each E in $F_{n-1} - \Delta$ implies E', which is incompatible with B. So by Theorem 1.1, B implies $\bigvee_{D \in \Delta} D$. But then A implies $\bigwedge_{D\in\Delta} -\Diamond D$, which, by **K**, implies $-\Diamond B = C$.

We can now establish reduction to normal form. Define the **T**-*suitable* nf's by induction on degree n:

$n = 0$. Any nf of degree 0 is **T**-suitable.
$n > 0$. An nf A is **T**-suitable if $(\forall B)(A > B \Rightarrow B$ is **T**-suitable) and $A > A'$.

Theorem 1 *Any formula of degree* $\leqslant n$ *is equivalent in the logic* T *to* $\perp$ *or a disjunction of* **T**-*suitable* nf's *of degree* n.

Proof: As for Theorem 1.1. For the inductive step, show that any consistent nf A is **T**-suitable. $(\forall B)(A > B \Rightarrow B$ is **T**-suitable) follows from **IH**. $A > A'$ follows from Lemma 1 and the axiom $p_0 \supset \Diamond p_0$.

For the theorem on model construction we require the following result:

Lemma 2 *For any* **T**-*suitable* nf A *there is a* **T**-*suitable* nf B *such that* $B' = A$.

Proof: By induction on the degree n of A:

$n = 0$. Choose B so that $B_f = A$ and $B > B'$.
$n > 0$. Let B be the nf of degree $(n + 1)$ such that $B_f = A_f$ and $(\forall C)(B > C \Leftrightarrow$ C is **T**-suitable & $A > C')$. Now $B' = A$. For $B'_f = B_f = A_f$; $B' > D \Rightarrow \exists C$: $B > C$ & $C' = D \Rightarrow A > C' = D$; and $A > D \Rightarrow \exists C : C$ is **T**-suitable & $C' = D$ (by **IH** and **D** **T**-suitable) $\Rightarrow B > C \Rightarrow B' > C' = D$. Also, $B > B' = A$. For A is **T**-suitable and $A > A'$ by A **T**-suitable.

For $n \geqslant 0$, let $\mathfrak{B}_n$ be the model $\langle W, R, \phi \rangle$ such that:

$$W = \{A \in F_n : A \text{ is } \mathbf{T}\text{-suitable}\};$$
$$R = \{\langle A, B\rangle \in W^2 : A > B'\};$$
$$\phi = \{\langle A, p\rangle \in W \times Sl : p \text{ is a conjunct of } A\}.$$

The models $\mathfrak{B}_n$ are *ungraded* in contradistinction to the *graded* models $\mathfrak{A}_n$ of section **2**. Each normal form in $\mathfrak{B}_n$ is of the same degree. Thus such models are equivalent to models that can be obtained by standard methods of filtration. For any nf A, let $A^0 = A$ and $A^m = (A')^{m-1}$, $0 < m \leqslant \mathrm{d}(A)$. Thus A^2 is the correlate of the correlate of A.

Theorem 2 *For* $\mathfrak{B}_n = \langle W, R, \phi\rangle$, $A \in W$, *and* $m \geqslant n$, $\langle \mathfrak{B}_n, A\rangle \models A^m$.

Proof: By induction on the degree $(n - m)$ of A^m:

$(n - m) = 0$. Since A^n is A_f.
$(n - m) > 0$. We show that $\langle \mathfrak{B}_n, A\rangle \models C$ for each conjunct C of A^m. (a) $C = A^m_f$. Since $A^m_f = A_f$. (b) $C = \Diamond B$. Then $\exists D : A > D$ and $D^m = B$. By Lemma 2, $\exists$ **T**-suitable $E : E' = D$. So ARE and, by **IH**, $\langle \mathfrak{B}_n, E\rangle \models E^{m+1} = (E')^m = D^m = B$. Therefore $\langle \mathfrak{B}_n, E\rangle \models \Diamond B = C$. (c) $C = -\Diamond B$. Suppose ARD, i.e., $A > D'$ (to show $\langle \mathfrak{B}_n, D\rangle \not\models B$). By **IH**, $\langle \mathfrak{B}_n, D\rangle \models D^{m+1}$. But D^{m+1} is distinct from but of the same degree as B and so disagrees with B. Therefore $\langle \mathfrak{B}_n, D\rangle \not\models B$.

Theorem 3 *The logic* T *is complete for the class of reflexive frames, has* fmp, *and is decidable.*

Proof: For any **T**-suitable nf $A, A > A'$, i.e., ARA. So the theorem follows from Theorems 1 and 2 in the usual way.

This result could also have been obtained by modifying the graded models of section **3**. We now turn to Kr. An nf A is K4-*suited* if $(\forall B)(A >^2 B \Rightarrow (\exists C)(A > C \;\&\; C' = B))$. For A, $B \in F_n$, we say ASB if $(\forall C)(B > C \Rightarrow A > C)$. An nf is K4-*suitable* if $(\forall B, C)(A > B \;\&\; B > C \Rightarrow (\exists D)(A > D \;\&\; D' = C \;\&\; BSD))$. All K4-suitable nf's are K4-suited, but the converse need not be true.

Lemma 3 *Any formula* A *of degree* $\leqslant n$ *is equivalent in* K4 *to* $\bot$ *or a disjunction of* K4-*suited* nf's *of degree* n.

Proof: A simple modification of Theorem **5**.1.

Lemma 4 *If an* nf A *of degree* >0 *is* K4-*suited, then* A' *is* K4-*suitable.*

Proof: Suppose $A' > B$ & $B > C$. Then, for some D and E, $A > D$, $D > E$, $D' = B$ and $E' = C$. By A K4-suited, $\exists F: A > F$ and $F' = E$. So $A' > F'$ and $F^2 = C$. Also, BSF'. For suppose $F' > G$. Then, $D > E = F' > G$; and so by D K4-suited, $\exists H: D > H$ and $H' = G$. But then $B = D' > H' = G$.

Theorem 4 *Any formula of degree* $\leqslant n$ *is equivalent in* K4 *to* $\bot$ *or a disjunction of* K4-*suitable* nf's *of degree* n.

Proof: From Lemmas 1, 3 and 4.

For $n \geqslant 0$, let $\mathfrak{C}_n$ be the model $\langle W, R, \phi\rangle$ such that:

$$W = \{A \in F_n : A \text{ is K4-suitable}\};$$
$$R = \{\langle A, B\rangle \in W^2 : A > B' \ \&\ ASB\};$$
$$\phi = \{\langle A, p\rangle \in W \times Sl : p \text{ is a conjunct of } A\}.$$

Theorem 5 *For* $\mathfrak{C}_n = \langle W, R, \phi\rangle$, $A \in W$, *and* $m \leqslant n$, $\langle \mathfrak{B}_n, A\rangle \vDash A^m$.

Proof: By induction on the degree $(n - m)$ of A^m.

The cases $(n - m) = 0$ and $(n - m) > 0$, (a) and (c) are proved as for theorem. There remains (b) of the case $(n - m) > 0$, i.e., the conjunct $C = \Diamond B$. This requires that if K4-suitable $A > B$ then $\exists$ K4-suitable $C: C' = B$ and ASC. Let C be such that $C_f = B_f$ and $(\forall D)(C > D \Leftrightarrow A > D$ & BSD & $B > D')$. Now ASC by definition. Also, $C' = B$. For $C_f' = C_f = B_f$: $C' > D \Rightarrow \exists E: C > E$ & $E' = D \Rightarrow B > E' = D$; and $B > D \Rightarrow \exists E: A > E$ & BSE & $E' = D$ (by A K4-suitable) $\Rightarrow C > E \Rightarrow C' > E' = D$. Finally, C is K4-suitable. For suppose $C > D > E$. Then $A > D > E$ and BSD. By A K4-suitable, $\exists F: A > F$, $F' = E$ and DSF. Since BSD & DSF, BSF; since BSD & $D > E = F'$, $B > F'$; and since $A > F$, $C > F$.

Theorem K4 *is complete for the class of transitive frames has* fmp, *and is decidable.*

Proof: From Theorems 4 and 5 and the transitivity of $\mathfrak{C}_n$.

K4 is not complete for any class of finite antisymmetric frames. So the graded models of section **3** are not applicable in this case. The methods for T and K4 could be combined to prove completeness for S4. Also the method could be modified to prove completeness for other standard logics, including the Brouwersche system B and Montague's minimal logic E. I leave the details to the diligent reader.

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