Notre Dame Journal of Formal Logic
Volume XVI, Number 1, January 1975
NDJFAM

A COMPENDIUM OF C. S. PEIRCE'S 1866-1885 WORK

BENJAMIN S. HAWKINS

0 *Introduction* A short time ago Richard Beatty examined the evolution of Peirce's development of quantifiers. Beatty's treatment of the subject-matter is generally commendable and, as he suggests,[1] needed. Still, he makes a number of important historical and theoretical omissions. It is my purpose here to complete Beatty's work by making good his omissions.

1 *General Observations* Beatty nowhere mentions that Peirce (1885) and Frege (1879) were the first to publish systems of quantification theory.[2] Peirce's and Frege's development of quantification is well-known but worth mentioning in connection with Beatty's paper, since it is possible to trace Peirce's development of quantifiers and not Frege's.[3] This is especially so in their respective treatments of I and 0 sentence forms.[4]

Beatty states that "Charles Peirce had a metaphysical interest in logic."[5] Beatty adduces the evolution of Peirce's notation to be an accommodation to changes in his metaphysical categories.[6] The passage he cites[7] as an example concerns in fact Peirce's distinction between *logic* and *mathematics*. Peirce, admittedly, did not curry one systemic reduction of his views but propounded several, and each coincides with a period of his research in logic.[8] Beatty, however, never takes into account Peirce's repeated distinction between the methods (such as hypothesis and deduction) employed in mathematics and those (such as observation and interpretation) employed in logic; therefore between L_α as *mathematics* and as *logic*. Peirce, for example, rejects logicism by means of this distinction.[9] Furthermore, Peirce regards metaphysical concepts as "adopted from those of formal logic"[10] but views the transposal adoption of such concepts as a "vicious order of thought."[11]

A reader is apt to believe from Beatty's paper that Peirce's first article in logic was in 1867.[12] Actually, Peirce published four papers in logic that year.[13] The first article in logic that Peirce produced was in 1866 and in it he recognized the principles in an inference to be distinct from those in its transformation (reduction).[14] Beatty mentions that Peirce held a subject-copula-predicate theory of wffs during the period 1867-1884,[15] but he fails to mention that during this period Peirce's interpretation

Received September 26, 1972

of a copula sign was as a sign of *relation*, which led him to the important step of reducing *illation* to a sign of *relation*.[16]

2 *Peirce's Work* Beatty correctly suggests that Peirce developed quantifiers partly from a need to adequately render I and 0 sentences.[17] Beatty cites Peirce's 1870 treatment of I and 0 sentences in L_{P2}[18] but not a related example of 1867 in L_{P1}. Peirce in the latter more or less observes Boole's conventions for A and E sentences, but he uses logical subtraction in the analysis of I and 0 sentences. For example, if 'i' and 'a' denote classes overlapping for some one individual, then 'a' in $\ulcorner a -_{,} i \urcorner$ represents 'some a'.[19] Peirce also demonstrates in L_{P1} the (now familiar) exact parallels between logical addition and multiplication of idempotence, commutativity, associativity, and distributivity.[20] Not generally recognized are combinatory devices such as '$F \subseteq G$' to eliminate 'x' in $\ulcorner(x)(Fx \supset Gx)\urcorner$, '$F|G$' in $\ulcorner(F|G)yz\urcorner$ to eliminate 'x' in $\ulcorner(\exists x)(Fyx \,.\, Gxz)\urcorner$ and others figure in L_{P2}, antedating Schönfinkel's work by fifty-four years.[21]

In "Upon the Logic of Mathematics" of 1867, Peirce employs 'Σ' in $\ulcorner x =_{,} \Sigma(A, B)\urcorner$ as an *existential quantifier*, though within a limited range of values. Here, however, $\ulcorner\Sigma(A, B)\urcorner$ either is an analysis of a general term 'x' as a (limited) logical sum of individuals 'A' and 'B' denoted by 'x', or it represents the quantified expression 'some x' as a logical sum of 'A' and 'B' of the class 'x'.[22] Beatty mentions this 1867 paper only in connection with Peirce's ultimate use of 'Σ' as an existential quantifier.[23] He also presents Peirce's ambiguity of 'x' and 'some x' as though it began in L_{P2}.[24] These lacunae diminish the historical significance of Beatty's example of Peirce's 1880 use of subscripts (indices) in L_{P4}. That is, in $\ulcorner l = \Sigma_i(L_i : M_i)\urcorner$, the relative '$l$' is the sum of the pairs of individuals in the range of 'L' and 'M' picked out by 'i'.[25] Here Peirce, in effect, adds subscripts (indices) to his 1867 notation and treats a relative term as identical with a limitless logical sum (or product) of all the individual pairs.[26] This resolves the ambiguity of both "Upon the Logic of Mathematics" and L_{P2}, as it removes the restriction on individuals denoted by a general term.[27]

It is worth mention that Peirce gives the full disjunctive normal form and its dual the full conjunctive normal form in L_{P4}.[28] Peirce also formulates L_{P3} or P_{P1} in 1880, a systemic fragment of sentential variables and the single primitive connective of *joint denial*.[29] The following year, he sets out in a paper many of the essentials for the formal development of natural numbers, and states the recursion equations for addition and multiplication.[30] It is interesting to note that although Peirce appraised "Upon the Logic of Mathematics" in 1904 as "the worst" paper he "ever published,"[31] in this 1867 paper he proposes a definition of *cardinal number* which anticipates that given by Whitehead and Russell in the *Principia Mathematica*.[32]

In 1885, Peirce publishes L_{P5}. This system comprises P_{P2}, a *sentential calculus*,[33] and F_{P1}, a *functional calculus with identity*.[34] It is in this system that Peirce abandons a subject-copula-predicate theory of wffs.

He does not employ '$-\!\!<$' in L_{P5} as a *copula*, as in previous systems, but like Whitehead's and Russell's '$\supset$' in the *Principia Mathematica*. In other words, Peirce uses '$-\!\!<$' for *philonian implication*, which, unlike Frege (1879), he connects with Philo and the Scholastics.[35] The following is Peirce's set of axioms (icons) for L_{P5}[36]:

1. $x -\!\!< x$
2. $[x -\!\!< (y -\!\!< z)] -\!\!< [y -\!\!< (x -\!\!< z)]$
3. $(x -\!\!< y) -\!\!< (y -\!\!< z) -\!\!< x -\!\!< z$
4. $\alpha -\!\!< x$
5. $[(x -\!\!< y) -\!\!< x] -\!\!< x$
6. $\Sigma_i \Pi_j x_{ij} -\!\!< \Pi_j \Sigma_i x_{ij}$
7. $\Sigma_h \Pi_i \Sigma_j \Pi_k (\alpha_{hik} + s_{jk}\ l_{ji})$
8. $\Sigma_u \Sigma_v \Pi_x \Pi_y (\varepsilon_{uyx} + \bar{s}_{yv} b_{vx})$
9. $\Pi_i \Sigma_k \Pi_j q_{ki}(\bar{q}_{kj} + 1_{ij})$
10. $\Pi_l \Sigma_k \Pi_i (q_{li}\bar{q}_{ki} + \bar{q}_{li} q_{ki})$
11. $\Pi_l \Pi_m \Sigma_k \Pi_i (q_{li} q_{ki} + q_{mi} q_{ki} + \bar{q}_{li}\bar{q}_{mi}\bar{q}_{ki})$
12. $\Pi_l \Pi_m \Pi_i \Sigma_k \Pi_j [q_{li}\bar{q}_{mi} + q_{ki}(q_{kj} + \bar{q}_{lj})]$

Philonian implication is involved in axioms 1-6, negation in 4, and identity in 1 and 9, though 1 is *not independent*.[37] 2 is the *principle of transposition*, 3 the *principle of syllogism* and 5 *Peirce's law*, which is required to *classically complete* the positive implication calculus.[38]

Peirce uses '-' and '$-\!\!<$' as *primitives* of L_{P5}, '=' for *logical equivalence*, '+' for *logical addition*, '-' for *logical subtraction* and juxtaposed wffs of L_{P5} to indicate *logical multiplication*. He also incorporates lower case Latin and Greek letters as indices of general symbols (tokens) and indices of negation (the operation denoted by '-') of general symbols of L_{P5} respectively.[39] In fact, Peirce introduces *negation* by way of construing 'x' in 4 as a formula of any value.[40] Thus, he anticipates Russell's 1903 definition of '$\sim p$' as '$p \supset_r r$'.[41] Peirce uses '1' as an *identify functor*, as in 9; and his definition of '1' is the first statement of the *Leibniz principle of interchange* not to confuse the use and mention of a sign.[42] In deriving $\ulcorner x -\!\!< (y -\!\!< x) \urcorner$ from 1 and 2, however, Peirce tacitly supplements substitution, interchange and detachment with a rule that if $\vdash \Delta$ of L_{P5}, then $\vdash \Gamma \rightarrow \Delta$ of L_{P5}.[43]

Peirce introduces 'Σ' and 'Π' as *existential* and *universal quantifiers* in L_{P5}.[44] He also introduces the constants **v** and **f** for *true* and *false*,[45] where "to find whether a formula is necessarily true substitute **f** and **v** for the letters and see whether it can be supposed false by any such assignment of values."[46] This is not only the first explicit use of the two truth-values and more *formalistic* than Frege's six years later, but Peirce develops it as the first statement of the *truth-table method* as a general decision procedure, which Łukasiewicz, Post and Wittgenstein popularized only in the 1920's.[47] Peirce inaugurates use of the prenex normal form in L_{P5}, and posits definitions of *infinite* and *finite classes* equivalent to what Dedekind introduces three years later.[48] Indeed, Dedekind's independence of conception is suspect here, for Peirce sent his work to Dedekind who never acknowledged it.[49]

3 *Conclusion* Peirce liked to refer to himself as a "logician"[50] and he, as Prior states, "perhaps had a keener eye for essentials than any other logician before or since."[51] His technical contributions are significant and wide; together with Boole's and Frege's work they are unquestionably the basis of mathematical logic.[52] Though De Morgan initiated modern relation theory, Peirce developed it. Peirce also conceived of a notation adequate for all of logic and virtually created all of the Boolean algebra of classes. This latter work of his extends Boole's work to dimension two and, as codified by Schröder, links up with Löwenheim's and Skolem's work. It is indeed superseded by Whitehead's and Russell's work, but their work is based largely upon Peirce's.[53] Peirce was aware of the value of his work and he considered it superior to that of De Morgan, Dedekind, Schröder, Peano, Russell, and others "to such a degree as to remind one of the difference between a pencil sketch of a scene and a photograph of it."[54] Perhaps such a claim may strike as conceited. Yet it has a large basis in facts.

NOTES

1. [1], p. 64.
2. Frege's and Peirce's results were evidently achieved in ignorance of the other's work. It is usually assumed unlikely that they came later into contact with even secondary sources of the other's work. Peirce, for example, was not always able "to procure necessary books" ([4], 4.118) and his writings contain no reference to Frege ([11], pp. 241-242). Yet not only did Peirce learn of Russell's paradox around 1910 from a review of [17] but he himself reviewed [14] in 1903, though his briefness suggests that his reading was selective and that he missed all reference to Frege ([4], 8.171, n. 1; *cf.* [11], pp. 241-242). Frege, on the other hand, reviewed the first volume of [15] in 1895 ([6], pp. 193-210), which suggests his acquaintance with Peirce's work as discussed by Schröder (e.g., [15], vol. 1, §§ 5, 27). I am, however, unable to find any reference to Peirce in Frege's published work.
3. *Vid.* [5], pp. 19-24; [6], pp. 1-91.
4. *Cf.* [1], pp. 65-66; [5], p. 24.
5. [1], p. 64.
6. *Ibid.*, pp. 64-65.
7. [4], 3.322; *cf.* [1], p. 64.
8. [11], pp. 1-5.
9. Peirce treats deduction and hypothesis under *mathematics* as involving L_α only with respect to Syn(L_α), whereas he treats L_α under *logic* as subject to Sem(L_α). Peirce's treatment of L_α as *mathematics* is then a *formalism* (*cf.* [4], 1.53-54, 66, 242, 247, 559, 2.191, 197, 266, 537-546, 712, 3.154-172, 363-364, 558, 560, 4.224-249, 481, 531, 5.8, 126, 145-148).
10. [4], 1.625, *cf.* 1.487, 624, 2.36-38, 121.

11. *Ibid.*, 2.38.

12. [1], p. 64.

13. [4], 2.391-426, 461-516, 3.1-44.

14. *Ibid.*, 2.792-807, *cf.* 4.1-3.

15. [1], pp. 65-66, 73.

16. [4], 3.175, 4.3; *cf.* [11], pp. 61-65.

17. [1], pp. 66, 73.

18. I.e., the systemic paper in [4], 3.45-148. Hereafter, other of Peirce's papers referred to by 'Lp_1', 'Lp_3', 'Lp_4', and 'Lp_5' shall correspond to [4], 3.1-19, 4.12-20, 3.154-251 and 3.359-403M respectively.

19. [4], 3.18, *cf.* 4.4.

20. *Ibid.*, 3.1-4; *cf.* [8], pp. 422-423; [9], p. 82.

21. [7], p. 355.

22. [4], 3.21.

23. [1], p. 67.

24. *Ibid.*, pp. 66-68; *cf.* [4], 3.69, 83.

25. [4], 3.247; *cf.* [1], pp. 68-69, 73.

26. Peirce ([4], 3.217; *cf.* [1], pp. 68, 73) also treats a *term* as identical with a limitless logical sum (or product) of all individuals of its range of values.

27. An arbitrary term (as noted in [1], pp. 68-69) will always be identical with the logical sum in its range. But, because every term applied to a range of values is identical with the same limitless logical sum, it is formally undecidable to distinguish any such term from another.

28. [4], 3.204-208; *cf.* [3], p. 166.

29. [4], 4.12-20, *cf.* 4.264-265; also [3], pp. 133-134.

30. [4], 4.113-125; *cf.* [3], pp. 321-322.

31. [4], 4.333.

32. *Ibid.*, 3.43-44; *cf.* [17], vol. 2, section A.

33. [4], 3.365-391.

34. *Ibid.*, 3.392-403M.

35. *Ibid.*, 3.373-374, 389, *cf.* 3.175; also [2], pp. 22-23, 311, 318-319; [17], vol. 1, pp. 7-8.

36. [4], 3.376-377, 379, 381, 384, 396-397, 399; *cf.* [2], p. 242; [12], pp. 24, 303; [13].

37. [13].

38. [7], p. 416.

39. [4], 3.361-372, 385-398; *cf.* [13], p. 135.

40. [4], 3.381-385.

41. [14], p. 18; *cf.* [3], p. 151.

42. [4], 3.398, *cf.* 5.323; also [3], p. 300.

43. [4], 3.377-383; *cf.* [13], p. 135.

44. 'Quantification', 'quantifying' and 'quantifiers' are first used in this work and quantifiers are developed as today ([4], 3.393-403M; *cf.* [3], p. 288; [16], p. 1, n. 3). Beatty notes these contributions and identifies Mitchell's part in Peirce's development of quantifiers ([1], pp. 71-72, 74-75). He also points out ([1], p. 76, n. 7) the error in [8], p. 431, regarding Mitchell's contribution to Peirce.

45. [4], 3.366, *cf.* 4.250-263.

46. *Ibid.*, 3.387.

47. *Ibid.*, 3.386-388; *cf.* [2], p. 392; [3], pp. 25, 162; [8], p. 420.

48. [4], 3.401-403M, *cf.* 3.252-288, 505, 564, 4.331; also [3], pp. 292, 344; [8], p. 440.

49. [4], 3.564, 4.331, 5.178, 526, 7.209.

50. E.g., *ibid.*, 2.197, 4.239.

51. [12], p. 4.

52. *Cf.* [8], pp. 432, 435, 511.

53. *Vid.* [7], p. 228; [8], p. 432; [9], pp. 79, 85; [11], pp. 151-152; [14], p. 23.

54. [4], 5.147, *cf.* 4.617.

REFERENCES

[1] Beatty, R., "Peirce's development of quantifiers and of predicate logic," *Notre Dame Journal of Formal Logic*, vol. X (1969), pp. 64-76.

[2] Bocheński, I. M., *A History of Formal Logic*, ed. and tr. from the German by I. Thomas, University of Notre Dame, Notre Dame (1961).

[3] Church, A., *Introduction to Mathematical Logic*, Princeton University Press, Princeton (1961).

[4] *Collected Papers of Charles Sanders Peirce*. Vols. 1-6 ed. by C. Hartshorne and P. Weiss, and vols. 7-8 ed. by A. Burks, Harvard University Press , Cambridge, Mass., 1960-1966.

[5] Frege, G., *Begriffsschrift und Anders Aufsätze*, ed. by I. Angelelli, Georg Olms Verlagsbuchhandlung, Hildesheim (1964).

[6] Frege, G., *Kleine Schriften*, ed. by I. Angelelli, Georg Olms Verlagsbuchhandlung, Hildesheim (1967).

[7] *From Frege to Gödel*, ed. by J. van Heijenoort, Harvard University Press, Cambridge, Mass. (1967).

[8] Kneale, W. and M., *The Development of Logic*, The Clarendon Press, Oxford (1962).

[9] Lewis, C. I., *A Survey of Symbolic Logic*, Dover Publications, Inc., New York (1960).

[10] Łukasiewicz, J., *Selected Works*, ed. by L. Borkowski, North-Holland Publishing Company, Amsterdam (1970).

[11] Murphey, M. G., *The Development of Peirce's Philosophy*, Harvard University Press, Cambridge, Mass. (1961).

[12] Prior, A. N., *Formal Logic*, The Clarendon Press, Oxford (1962).

[13] Prior, A. N., "Peirce's axioms for propositional calculus," *The Journal of Symbolic Logic*, vol. 23 (1958), pp. 135-136.

[14] Russell, B. A. W., *The Principles of Mathematics*, George Allen and Unwin Ltd., London (1956).

[15] Schröder, E., *Vorlesungen über die Algebra der Logik*, Chelsea Publishing Co., Bronx, New York (1966), 3 vols.

[16] Tarski, A., *Logic, Semantics, Metamathematics*, ed. by H. Woodger, The Clarendon Press, Oxford (1956).

[17] Whitehead, A. N., and B. A. W. Russell, *Principia Mathematica*, The Clarendon Press, Cambridge, 1925-1927, 3 vols.

The University of Western Australia
Nedlands, Western Australia