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WITTGENSTEIN ON RUSSELL'S THEORY OF TYPES

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Although the principal concern of this paper* will be to examine Wittgenstein's criticisms of Russell's theory of types, I will argue that the criticisms would apply to any theory of types, given the metaphysics and the theory of logic in the *Tractatus*. Choosing between a theory of types and an approach similar to the *Tractatus* may have significant consequences for one's philosophical theories. If successful, this paper will delineate at least one of these consequences.

Russell constructed the theory of types to handle a number of paradoxes among which were (1) Russell's paradox, (2) Burali-Forti paradox, (3) "The Liar," and (4) Richard's paradox. Speaking about the paradoxes, Russell says:

> In each contradiction something is said about *all* cases of some kind, and from what is said a new case seems to be generated, which both is and is not of the same kind as the case of which *all* were concerned in what was said.[1]

Since Wittgenstein only considers Russell's paradox in the *Tractatus*, the remarks here will be restricted to it. The resolution of the paradox revolves on the construction of types, which are defined as "the range of significance of a propositional function."[2] With the rule, "Whatever involves all of a collection must not be one of the collection,"[3] Russell constructs his levels of language or hierarchy of types.

Here, perhaps a rehearsal of the developments which gave rise to Russell's paradox may be in order. We may recall that Russell's paradox is a result of the Laws, Definitions, and Rules which Frege set forth in the *Grundgesetze*. Frege was duly proud of his achievement as he states:

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> If anyone should find anything defective, he must be able to state precisely where according to him, the error lies: in the Basic Laws, in the Definitions, in the Rules, or in the application of the Rules at a definite point.[4]

Further, he goes on to say:

> As a refutation in this, I can only recognize someone's actually demonstrating . . . that my principles lead to manifestly false conclusions. But no one will be able to do that.[5]

Frege's achievements, namely the systematization of arithmetic and its reduction to logic may best be understood in light of the status of proof in his time.

In arithmetic he contested the intuitionists; and in logic he battled against psychologism. In regard to the former, consider the derived rule of conjunction: If 'p' is a theorem and 'q' is a theorem, then '$p \cdot q$' is a theorem. Intuitionists of the time might suggest that one can immediately see the correctness of this derived rule. Furthermore, this intuition constitutes its proof. Yet, in axiomatic systems developed after Frege's work, the proof requires several steps. By any stretch of the imagination, intuiting all these steps is a severe demand on most people's intellect. The intuitionists could not survive Frege's battle for rigorous proofs. Psychologism also suffered a fatal blow at the hands of Frege. This view, which held that the laws of thought operate on the mind, maintained that the function of logic was to teach one to think correctly. Frege argued, however, that 'laws' in 'laws of thought' was subject to an equivocal meaning: (1) descriptive and (2) prescriptive.[6] This equivocation does not permit one to distinguish what is thought about 'p' from the truth of 'p'. The equivocation lends validity to the inference. 'I think that 'p' is true; therefore, 'p' is true.' Contrariwise, Frege states:

> There is no contradiction in something's being true which everybody takes to be false. . . . If being true is thus independent of being acknowledged by somebody or other, then the laws of truth are not psychological laws: they are boundary stones set in eternal foundation, which thought can overflow, but never displace.[7]

The effects of Frege were to counter the muddy murk of thought.

Turning to Russell's paradox, it develops from Basic Law V: $\vdash [\epsilon' f(\epsilon) = \alpha' g(\alpha)] = [\smile^{\alpha} f(\alpha) = g(\alpha)]$.[8] In modern notation the law might be stated as $[\hat{x}(Fx) = \hat{x}(Gx)] \equiv (x)(Fx \equiv Gx)$. From this Law, the rules of inference and the Definitions in the *Grundgesetze*, the following theorem may be proven: $(f\alpha) = \alpha \cap \epsilon' f(\epsilon)$,[9] which may be rendered in modern notation as $Fy \equiv y \epsilon \hat{x}(Fx)$. From this theorem, Russell's paradox may be generated.

Russell's theory of types at the outset appears to be a successful treatment of the paradox. Given the restrictions Russell places on well-formed formulas, the paradox seems to be eliminated and mathematics is purged from the taint of contradiction. Yet, Wittgenstein rejects the theory. To examine why, let us first put down Wittgenstein's aphoristic argument:

(1) It must be possible to establish logical syntax without mentioning the meaning of a sign. . .[10]

(2) It can be seen that Russell must be wrong because he had to mention the meaning of signs when establishing the rules for them (3.331).

(3) No proposition can make a statement about itself, because a propositional sign cannot be contained in itself . . . (3.332).

(4) The reason why a function cannot be its own argument is that the sign for a function already contains the prototype of its argument, and it cannot contain itself (3.333a).

(5) That disposes of Russell's paradox (3.333d).

Russell developed the theory of types to handle the paradoxes mentioned earlier. If the paradoxes can be disposed via an alternative method and if the proposed method is simpler in some sense than the theory of types, then the alternative would be preferable. Both Wittgenstein and Russell appreciated the barber close shave of Ockam's razor.

Wittgenstein's criticism of Russell's theory of types at 3.331 is based on 3.33. The confusion of syntax and semantics is the charge which Wittgenstein hurls at Russell. Two points are at issue here. Russell, when constructing the theory of types, actually used such words as 'type', 'function', and 'number' in their technical sense before defining them. Note:

> By a "propositional function" we mean something which contains a variable *x*, and expresses a *proposition* as soon as a value is assigned to *x*. That is to say, it differs from a proposition solely by the fact that it is ambiguous: it contains a variable of which the value is unassigned.[11]

For Wittgenstein, the above is a case of circular definition. The employment of a sign in a formal language presupposes the rules for the use of said sign. What could possibly be one's guide except fallacious reasoning if, constructing semantical rules, one employs the proposed definition to define the sign.

The second point for Wittgenstein is that the signs such as 'number', and 'function' must be constantly reintroduced for each type level. This point divides into two problems. First, how is it possible to determine that 'function' at type level n means the same as 'function' at type level $n+3$? To maintain that such signs as 'function' are typically ambiguous is to beg the issue. Second, the constant reintroduction of terms violates the conditions for definitions which Wittgenstein adopted from Frege: In regard to definitions, Frege states:

> A definition of a concept (of a possible predicate) must be complete; it must unambiguously determine, as regards any object, whether or not it falls under the concept (whether or not the predicate is truly assertible of it).[12]

In addition to completeness, Frege argued that definitions must also avoid ambiguity and vagueness. Wittgenstein accepted these conditions, especially completeness. He states:

> If a primitive idea has been introduced, it must have been introduced in all combinations in which it ever occurs (5.451).

Expanding on this, Wittgenstein adds:

> The introduction of any new device into the symbolism of logic is necessarily a momentous event. In logic a new device should not be introduced in brackets or in a footnote with what one might call a completely innocent air. . . . But if the introduction of a new device has proved necessary at a certain point, we must ask ourselves, 'At what points is the employment of this device now *unavoidable*?' and its place in logic must be made clear (5.452).

Given the fact that such signs as 'function' must constantly be reintroduced, the completeness condition may not be attained. Because of this incompleteness, the role of such signs in logic is not clear. Since Wittgenstein does not admit the use of signs such as 'function' because of their lack of clarity, he discards the theory of types.

Since Russell flagrantly violated the distinctions between syntax and semantics, and since he has not followed the counsel of Frege on definitions, one might believe that these are the principal criticisms which Wittgenstein brings against the Russellian theory of types. However, suppose that a theory of types is constructed purely syntactically. That is, all the signs are left uninterpreted. Church attempts this project in his article, "A Formulation of the Simple Theory of Types."[13] If Church is successful, could the Wittgenstein of the *Tractatus* accept such a theory of types? If the answer is "yes," then one must interpret Wittgenstein as criticizing only Russell's theory of types. On the other hand, if the answer is "no," then one must discover other criticisms of the program for a theory of types than simply the confusion of syntax and semantics.

Indicating his concern only for syntax, Church states:

> Of course the matter of interpretation is in any case irrelevant to the abstract construction of the theory.[14]

Although Church refrains from interpreting the signs and thus avoids the semantical issues, he does give rules for well-formed formulas:

> (1) a formula consisting of a single proper symbol is well-formed and has the type indicated by a subscript; (2) if x_β is a variable with the subscript β, and M_α is a well-formed formula of type α, then $(\lambda x_\beta M_\alpha)$ is a well-formed formula having type $\alpha\beta$; (3) if $F_{\alpha\beta}$ and A_β are well-formed formula of type $\alpha\beta$ and β respectively, then $(F_{\alpha\beta}A_\beta)$ is a well-formed formula having type α.[15]

Without being unfair to Church, the above mentioned rules may be considered as rules of logical syntax. Yet, Wittgenstein becomes upset at such talk:

> The rules of logical syntax must go without saying, once we know how each sign signifies (3.334).

For Wittgenstein, what a sign signifies is *shown* in the symbolism. Likewise, the rules of logical syntax are *shown* in the symbolism. Wittgenstein states:

> These rules are equivalent to the symbols; and in them their sense is mirrored (5.414).

Thus, the rules of logical syntax can never be stated or said. Therefore, Wittgenstein must also reject as untenable even a theory of types constructed syntactically.

However, now we see that Wittgenstein has more criticisms of Russell's theory of types than simply 3.33. Any theory which invokes or gives comfort to a notion of a meta-language is incompatible with the Wittgenstein of the *Tractatus*. Before concluding that Wittgenstein misled us in regard to the criticisms of the theory of types, let us reexamine the arguments in the 3.33's.

Wittgenstein closes 3.333 announcing the deposition of Russell's paradox. How has Wittgenstein attained this feat? If, indeed, Russell's paradox may be disposed without employing a theory of types, then the theory is superfluous. The situation to avoid is '$F(F(fx))$'. Treating the 'F' in each of the two occurrences as having the same meaning yields unacceptable results. But, according to Wittgenstein, the meanings of the signs are not the same because the forms are different (*Cf.* 3.333). $\psi[\phi(fx)]$ is markedly different from $\phi(fx)$. Wittgenstein states at 3.321:

> So one and the same sign (written or spoken) can be common to two different symbols in which case they will signify in different ways.

In effect, "the modes of signification" of the 'F's' are different. Hence, the 'F' does not have the same meaning in the two occurrences. Failure to recognize differences in the modes of signification leads to problems. Wittgenstein says:

> In this way the most fundamental confusions are easily produced (the whole of philosophy is full of them) (3.324).

Wittgenstein has demonstrated what would happen if a function could be its own argument—namely the form is different from the original and the signs do not have the same meaning. However, Wittgenstein denies that a function could even take itself as an argument:

> The sign for a function already contains the prototype of its argument (3.333a).

In other words, the function is not even a possible value of the variable.

In regard to the values of the variable, Wittgenstein states:

> To stipulate values for a propositional function is to give the propositions whose common characteristic the variable is (3.317a).

This common characteristic to which Wittgenstein refers is logical form. The form of the variable and the form of the function are not the same. Hence, the function is not a possible value of the variable. Russell's paradox, thence, crumbles. Wittgenstein's alternative treatment of the paradox indicates that a theory of types is unnecessary.

Given the metaphysics and the theory of logic in the *Tractatus*, Wittgenstein would have to begin anew on his dissertation if he were to accept a theory of types. First, the propositions of the *Tractatus* would become second-order propositions instead of nonsensical. Although there are problems with Wittgenstein's theory of logic, (for example, the account of quantification theory is inadequate), the introduction of a theory of types would wreak havoc. No longer would all propositions be the result of operation of elementary propositions. The truth functional account of propositions would have been shattered. Moreover, the rules of logical syntax could be stated. In effect, the doctrine of showing would have to be given up. No longer would anything be mirrored in the language. There would only be various levels of discourse. Said discourse could probably be *only descriptive*. The mystical would have been wrung dry. Succinctly, a theory of types and the metaphysics of the *Tractatus* are simply incompatible.

In conclusion, what began as an isolated criticism of Russell's theory of types at 3.33 may now be seen to incorporate the entire *Tractatus*. I can only point to some of the differences for one's theories that lie in a choice between an approach countenancing a theory of types in contrast to one similar in outlook to the *Tractatus*. In one sense, the issue is the contemporary problem of metaphysics—how to establish a linguistic framework to account for experience. One faces the problems of ontology. Adopting a type theory may lead one to commitments to abstract entities, to the use of classes, and to what Goodman might call a platonistic system. On the other hand, theories developed *à la* Wittgenstein might appear as naked nominalism, attending only to the bare facts. Which approach one chooses in part depends on one's purposes. In these last remarks, I have sought only to indicate briefly one of the issues involved. Now, as Wittgenstein might say, "This disposes of my paper."

NOTES

1. Bertrand Russell, *Logic and Knowledge*, ed. by Robert Marsh, George Allen and Unwin, London (1968), p. 61.
2. *Ibid.*, p. 25.
3. *Ibid.*, p. 63.
4. Gottlob Frege, *The Basic Laws of Arithmetic*, trans. by Montgomery Furth, University of California Press, Los Angeles (1964), p. 3.
5. *Ibid.*, p. 25.
6. *Ibid.*, p. 13.
7. *Ibid.*, p. 13.
8. *Ibid.*, p. 72.
9. *Ibid.*, pp. 123-126.

10. Ludwig Wittgenstein, *Tractatus-Logico-Philosophicus*, trans. by D. F. Pears and B. F. McGuinness, Routledge and Kegan Paul, London (1961), p. 31. Hereafter, all references to the Tractatus will follow the quoted material, using standard notational form.

11. Alfred North Whitehead and Bertrand Russell, *Principia Mathematica*, Vol. I, University Press, Cambridge (1935), p. 38.

12. Gottlob Frege, *Philosophical Writings*, trans. Geach and Black, Basil Blackwell, Oxford (1966), p. 159.

13. Alonzo Church, "A formulation of the simple theory of types," *The Journal of Symbolic Logic*, vol. 5 (1940), pp. 56-68.

14. *Ibid.*, p. 57.

15. *Ibid.*, p. 57.

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