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A NEW POSTULATE–SYSTEM FOR MODULAR LATTICES

BOLESŁAW SOBOCIŃSKI

In [1], p. 384, Theorem 2, Milan Kolibiar has proved that the following two formulas

$A1 \quad [abcd]: a, b, c, d \epsilon A . \supset . ((a \cap b) \cap c) \cup (a \cap d) = ((d \cap a) \cup (c \cap b)) \cap a$

and

$A2 \quad [ab]: a, b \epsilon A . \supset . (a \cup (b \cap b)) \cap b = b$

constitute an adequate postulate-system for modular lattices. It remains an open problem, *cf.*, [1], p. 386, Remark, whether Kolibiar's postulate-system can be sustituted by a shorter one.*

In this note a positive answer will be given for this problem. Namely, it will be proved:

(A) *Any algebraic system*

$$\mathfrak{A} = \langle A, \cup, \cap \rangle$$

where $\cup$ *and* $\cap$ *are two binary operations defined on the carrier set A, is a modular lattice, if it satisfies the following two mutually independent postulates:*

$B1 \quad [abc]: a, b, c \epsilon A . \supset . (a \cap b) \cup (a \cap c) = ((c \cap a) \cup (b \cup b)) \cap a$
$B2 \quad [abc]: a, b, c \epsilon A . \supset . a = (c \cup (b \cup a)) \cap a$

Proof:

1 It is self-evident that *B1* and *B2* are provable formulas in the field of any modular lattice. In order to prove the converse let us assume *B1* and *B2*. Then:

$B3 \quad [ac]: a, c \epsilon A . \supset . a = (a \cap a) \cup (a \cap c) \qquad [B2, b/a, c/c \cap a; B1, b/a]$
$B4 \quad [abc]: a, b, c \epsilon A . \supset . (a \cap c) \cap b = a \cap ((a \cap c) \cap b)$
PR $[abc]: \mathrm{Hp}(1) . \supset .$
$(a \cap c) \cap b = ((a \cap a) \cup (((a \cap c) \cap (a \cap c)) \cup ((a \cap c) \cap b))) \cap ((a \cap c) \cap b)$
$[1; B2, a/(a \cap c) \cap b, b/(a \cap c) \cap (a \cap c), c/a \cap a]$

*See final remark, **NB**, at the end of this paper.

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$= ((a \cap a) \cup (a \cap c)) \cap ((a \cap c) \cap b)$ $[B3, a/a \cap c, c/b]$

$= a \cap ((a \cap c) \cap b)$ $[B3]$

B5 $[abc]: a, b, c \epsilon A . \supset . (a \cap (b \cap b)) \cup (a \cap c) = ((c \cap a) \cup b) \cap a$

PR $[abc]: \mathrm{Hp}(1) . \supset .$

$(a \cap (b \cap b)) \cup (a \cap c) = ((c \cap a) \cup ((b \cap b) \cup (b \cap b))) \cap a$ $[1; B1, b/b \cap b]$

$= ((c \cap a) \cup b) \cap a$ $[B3, a/b, c/b]$

B6 $[ab]: a, b \epsilon A . \supset . a \cap a = ((a \cap b)) \cap (a \cap b)) \cup (a \cap a)$

PR $[ab]: \mathrm{Hp}(1) . \supset .$

$a \cap a = ((a \cap a) \cup (a \cap b)) \cap a = (a \cap ((a \cap b) \cap (a \cap b))) \cup (a \cap a)$ $[1; B3, c/b; B5, b/a \cap b, c/a]$

$= ((a \cap b) \cap (a \cap b)) \cup (a \cap a)$ $[B4, c/b, b/a \cap b]$

B7 $[a]: a \epsilon A . \supset . a \cap (a \cap a) = a \cap a$

PR $[a]: \mathrm{Hp}(1) . \supset .$

$a \cap (a \cap a) = ((a \cap a) \cup (a \cap a)) \cap (a \cap a)$ $[1; B3, c/a]$

$= ((a \cap a) \cup (((a \cap a) \cap (a \cap a)) \cup (a \cap a))) \cap (a \cap a)$ $[B6, b/a]$

$= a \cap a$ $[B2, a/a \cap a, b/(a \cap a) \cap (a \cap a), c/a \cap a]$

B8 $[a]: a \epsilon A . \supset . (a \cap a) \cap (a \cap a) = ((a \cap a) \cap (a \cap a)) \cup ((a \cap a) \cap (a \cap a))$

PR $[a]: \mathrm{Hp}(1) . \supset .$

$(a \cap a) \cap (a \cap a) = (((a \cap a) \cap (a \cap a)) \cup (a \cap a)) \cap (a \cap a)$ $[1; B6, b/a]$

$= ((a \cap a) \cap ((a \cap a) \cap (a \cap a))) \cup ((a \cap a) \cap (a \cap a))$ $[B5, a/a \cap a, b/a \cap a, c/a \cap a]$

$= ((a \cap a) \cap (a \cap a)) \cup ((a \cap a) \cap (a \cap a))$ $[B7, a/a \cap a]$

B9 $[a]: a \epsilon A . \supset . (a \cap a) \cap (a \cap a) = a \cap a$

PR $[a]: \mathrm{Hp}(1) . \supset .$

$(a \cap a) \cap (a \cap a) = ((a \cap a) \cap (a \cap a)) \cup ((a \cap a) \cap (a \cap a))$ $[1; B8]$

$= (((a \cap a) \cap (a \cap a)) \cup ((a \cap a) \cup (a \cap a))) \cap (a \cap a)$ $[B1, a/a \cap a, b/a \cap a, c/a \cap a]$

$= a \cap a$ $[B2, a/a \cap a, b/a \cap a, c/(a \cap a) \cap (a \cap a)]$

B10 $[a]: a \epsilon A . \supset . a = a \cap a$

PR $[a]: \mathrm{Hp}(1) . \supset .$

$a = (a \cap a) \cup (a \cap a) = ((a \cap a) \cap (a \cap a)) \cup ((a \cap a) \cap (a \cap a))$ $[1; B3, c/a; B9; B9]$

$= (a \cap a) \cap (a \cap a) = a \cap a$ $[B8; B9]$

B11 $[a]: a \epsilon A . \supset . a = a \cup a$ $[B3, c/a; B10; B10]$

B12 $[abc]: a, b, c \epsilon A . \supset . (a \cap b) \cup (a \cap c) = ((c \cap a) \cup b) \cap a$ $[B1; B11, a/b]$

B13 $[ab]: a, b \epsilon A . \supset . a = (b \cup a) \cap a$ $[B2, b/a, c/b; B11]$

B14 $[ab]: a, b \epsilon A . \supset . a \cap b = a \cap (a \cap b)$

PR $[ab]: \mathrm{Hp}(1) . \supset .$

$a \cap b = ((a \cap a) \cup (a \cap b)) \cap (a \cap b)$ $[1; B13, a/a \cap b, b/a \cap a]$

$= (((b \cap a) \cup a) \cap a) \cap (a \cap b)$ $[B12, b/a, c/b]$

$= a \cap (a \cap b)$ $[B13, b/b \cap a]$

B15 $[ab]: a, b \epsilon A . \supset . a = a \cup (a \cap b)$ $[B3, c/b; B10]$

B16 $[ab]: a, b \epsilon A . \supset . a = (a \cap b) \cup a$

PR $[ab]: \mathrm{Hp}(1) . \supset .$

$a = a \cap a = ((a \cap a) \cup (a \cap b)) \cap a$ $[1; B10; B3, c/b]$

$= (a \cap (a \cap b)) \cup (a \cap a) = (a \cap b) \cup a$ $[B12, b/a \cap b, c/a; B14; B10]$

B17 $[ab]:a,b\,\epsilon\,A\,.\supset.\,a\cap b=b\cap a$

PR $[ab]:\mathrm{Hp}(1)\,.\supset.$

$a\cap b=(a\cap b)\cup(a\cap b)=((b\cap a)\cup b)\cap a$ [1; *B11*, $a/a\cap b$; *B12*, c/b]

$=b\cap a$ [*B16*, a/b, b/a]

B18 $[ab]:a,b\,\epsilon\,A\,.\supset.\,a=(a\cup b)\cap a$

PR $[ab]:\mathrm{Hp}(1)\,.\supset.$

$a=(a\cap b)\cup a=(a\cap b)\cup(a\cap a)$ [1; *B16*; *B10*]

$=((a\cap a)\cup b)\cap a=(a\cup b)\cap a$ [*B12*, c/a; *B10*]

B19 $[ab]:a,b\,\epsilon\,A\,.\supset.\,a=a\cap(a\cup b)$ [*B18*; *B17*, $a/a\cup b$, b/a]

B20 $[ab]:a,b\,\epsilon\,A\,.\supset.\,a\cup b=b\cup a$

PR $[ab]:\mathrm{Hp}(1)\,.\supset.$

$a\cup b=((b\cup a)\cap a)\cup((b\cup a)\cap b)$ [1; *B13*; *B18*, a/b, b/a]

$=(((b\cap(b\cup a))\cup a)\cap(b\cup a)$ [*B12*, $a/b\cup a$, b/a, c/b]

$=(b\cup a)\cap(b\cup a)=b\cup a$ [*B19*, a/b, b/a; *B10*, $a/b\cup a$]

B21 $[ab]:a,b\,\epsilon\,A\,.\,a\cup b=b\,.\supset.\,a\cap b=a$ [*B19*]

B22 $[ab]:a,b\,\epsilon\,A\,.\,a\cap b=a\,.\supset.\,a\cup b=b$ [*B17*; *B16*, a/b, b/a]

D1 $[ab]\therefore a,b\,\epsilon\,A\,.\supset:a\leqslant b\,.\equiv.\,a\cap b=a$

B23 $[ab]\therefore a,b\,\epsilon\,A\,.\supset:a\leqslant b\,.\equiv.\,a\cup b=b$ [*D1*; *B21*; *B22*]

B24 $[ab]:a,b\,\epsilon\,A\,.\,a\leqslant b\,.\,b\leqslant a\,.\supset.\,a=b$ [*D1*; *B10*]

B25 $[abc]:a,b,c\,\epsilon\,A\,.\,a\leqslant b\,.\,a\leqslant c\,.\supset.\,a\leqslant b\cap c$

PR $[abc]:\mathrm{Hp}(3)\,.\supset.$

4. $a\cup b=b\,.$ [1; 2; *B23*]

5. $a\cap c=a\,.$ [1; 3; *D1*, b/c]

6. $a\cup(b\cap c)=(a\cap c)\cup(b\cap c)=(c\cap a)\cup(c\cap b)$

[1; 5; *B17*, b/c; *B17*, a/b, b/c]

$=(c\cap b)\cup(c\cap a)=((a\cap c)\cup b)\cap c$

[*B20*, $a/c\cap a$, $b/c\cap b$; *B11*, a/c, c/a]

$=(a\cup b)\cap c=b\cap c\,.$ [5; 4]

$a\leqslant b\cap c$ [1; *B23*, $b/b\cap c$; 6]

B26 $[abc]:a,b,c\,\epsilon\,A\,.\,a\leqslant c\,.\,b\leqslant c\,.\supset.\,a\cup b\leqslant c$

PR $[abc]:\mathrm{Hp}(3)\,.\supset.$

4. $a\cap c=a\,.$ [1; 2; *D1*, b/c]

5. $b\cap c=b\,.$ [1; 3; *D1*, a/b, b/c]

6. $(a\cup b)\cap c=(b\cup a)\cap c=((b\cap c)\cup a)\cap c$ [1; *B20*; 5]

$=(c\cap a)\cup(c\cup b)$ [*B12*, a/c, b/a, c/b]

$=(a\cap c)\cup(b\cup c)=a\cup b\,.$ [*B17*, b/c; *B17*, a/c; 4; 5]

$a\cup b\leqslant c$ [1; *D1*, $a/a\cup b$, b/c; 6]

B27 $[ab]:a,b\,\epsilon\,A\,.\supset.\,a\cap b\leqslant a$ [*B16*; *B23*, $a/a\cap b$, b/a]

B28 $[ab]:a,b\,\epsilon\,A\,.\supset.\,a\cap b\leqslant b$ [*B27*, a/b, b/a; *B17*]

B29 $[ab]:a,b\,\epsilon\,A\,.\supset.\,a\leqslant a\cup b$ [*B19*; *D1*, $b/a\cup b$]

B30 $[ab]:a,b\,\epsilon\,A\,.\supset.\,a\leqslant b\cup a$ [*B29*; *B20*]

B31 $[abc]:a,b,c\,\epsilon\,A\,.\supset.\,c\leqslant a\cup(b\cup c)$

PR $[abc]:\mathrm{Hp}(1)\,.\supset.$

2. $c\cap(a\cup(b\cup c))=(a\cup(b\cup c))\cap c=c\,.$

[1; *B17*, a/c, $b/a\cup(b\cup c)$; *B2*, a/c, c/a]

$c\leqslant a\cup(b\cup c)$ [1; *D1*, a/c, $b/a\cup(b\cup c)$; 2]

B32 $[abc]:a,b,c\,\epsilon\,A\,.\supset.\,b\leqslant a\cup(b\cup c)$ [*B31*, b/c, c/b; *B20*, a/c, c/b]

B33 $[abc]:a, b, c \epsilon A . \supset . a \leqslant (a \cup b) \cup c$

[*B32*, a/c, b/a, c/b; *B20*, a/c, $b/a \cup b$]

B34 $[abc]:a, b, c \epsilon A . \supset . b \leqslant (a \cup b) \cup c$ [*B33*, a/b, b/a; *B20*]

B35 $[abc]:a, b, c \epsilon A . \supset . (a \cup b) \cup c = a \cup (b \cup c)$

PR $[abc]:\mathrm{Hp}(1) . \supset .$

2. $a \cup b \leqslant a \cup (b \cup c)$. [1; *B26*, $c/a \cup (b \cup c)$; *B29*, $b/b \cup c$; *B32*]

3. $(a \cup b) \cup c \leqslant a \cup (b \cup c)$. [1; *B26*, $a/a \cup b$, b/c, $c/a \cup (b \cup c)$; 2; *B31*]

4. $b \cup c \leqslant (a \cup b) \cup c$.

[1; *B26*, a/b, b/c, $c/(a \cup b) \cup c$; *B34*; *B30*, a/c, $b/a \cup b$]

5. $a \cup (b \cup c) \leqslant (a \cup b) \cup c$. [1; *B26*, $b/b \cup c$, $c/(a \cup b) \cup c$; *B33*; 4]

$(a \cup b) \cup c = a \cup (b \cup c)$ [1; *B24*, $a/(a \cup b) \cup c$, $b/a \cup (b \cup c)$; 3; 4]

B36 $[abc]:a, b, c \epsilon A . a \leqslant b . b \leqslant c . \supset . a \leqslant c$

PR $[abc]:\mathrm{Hp}(3) . \supset .$

4. $a \cup b = b$. [1; *B23*; 2]

5. $b \cup c = c$. [1; *B23*, a/b, b/c; 3]

6. $a \cup c = a \cup (b \cup c) = (a \cup b) \cup c = b \cup c = c$. [1; 5; *B35*; 4; 5]

$a \leqslant c$ [1; *B23*, b/c; 6]

B37 $[abc]:a, b, c \epsilon A . \supset . (a \cap b) \cap c = a \cap (b \cap c)$

PR $[abc]:\mathrm{Hp}(1) . \supset .$

2. $(a \cap b) \cap c \leqslant a$.

[1; *B36*, $a/(a \cap b) \cap c$, $b/a \cap b$, c/a; *B27*, $a/a \cap b$, b/c; *B27*]

3. $(a \cap b) \cap c \leqslant b$.

[1; *B36*, $a/(a \cap b) \cap c$, $b/a \cap b$, c/b; *B27*, $a/a \cap b$, b/c; *B28*]

4. $(a \cap b) \cap c \leqslant b \cap c$. [1; *B25*, $a/(a \cap b) \cap c$; 3; *B28*, $a/a \cap b$, b/c]

5. $(a \cap b) \cap c \leqslant a \cap (b \cap c)$. [1; *B25*, $a/(a \cap b) \cap c$, b/a, $c/b \cap c$; 2; 4]

6. $a \cap (b \cap c) \leqslant b$.

[1; *B36*, $a/a \cap (b \cap c)$, $b/b \cap c$, c/b; *B28*, $b/b \cap c$; *B27*, a/b, b/c]

7. $a \cap (b \cap c) \leqslant c$.

[1; *B36*, $a/a \cap (b \cap c)$, $b/b \cap c$; *B28*, $b/b \cap c$; *B28*, a/b, b/c]

8. $a \cap (b \cap c) \leqslant a \cap b$. [1; *B25*, $a/a \cap (b \cap c)$, b/a, c/b; *B27*, $b/b \cap c$; 6]

9. $a \cap (b \cap c) \leqslant (a \cap b) \cap c$. [1; *B25*, $a/a \cap (b \cap c)$, $b/a \cap b$; 8; 7]

$(a \cap b) \cap c = a \cap (b \cap c)$ [1; *B24*, $a/(a \cap b) \cap c$, $b/a \cap (b \cap c)$; 5; 9]

B38 $[abc]:a, b, c \epsilon A . a \leqslant c . \supset . a \cup (b \cap c) = (a \cup b) \cap c$

PR $[abc]:\mathrm{Hp}(2) . \supset .$

3. $a \cap c = a$. [1; *D1*, b/c; 2]

$a \cup (b \cap c) = (a \cap c) \cup (b \cap c) = (c \cap a) \cup (c \cap b)$

[1; 3; *B17*, b/c; *B17*, a/b, b/c]

$= (c \cap b) \cup (c \cap a)$ [*B20*, $a/c \cap a$, $b/c \cap b$]

$= ((a \cap c) \cup b) \cap c = (a \cup b) \cap c$ [*B12*, a/c, c/a; 3]

Since formulas *B11*, *B10*, *B20*, *B17*, *B35*, *B37*, *B15*, *B19*, and *B38* are the consequences of *B1* and *B2*, it is proved that axioms *B1* and *B2* can be accepted as a postulate-system for modular lattices.

2 The mutual independence of axioms *B1* and *B2* is established by using the following algebraic tables:

	$\cap$	α	β	$\cup$	α	β			$\cap$	α	β	$\cup$	α	β
$\mathfrak{M}1$	α	α	β	α	α	α	;	$\mathfrak{M}2$	α	α	α	α	α	α
	β	α	β	β	α	α			β	α	β	β	α	α

which are given by Kolibiar in [1], pp. 385-386.

Namely:

(a) $\mathfrak{M}1$ verifies *B2*, but falsifies *B1* for a/β, b/β, and c/β: (i) $(\beta \cap \beta) \cup (\beta \cap \beta) = \beta \cup \beta = \alpha$, (ii) $((\beta \cap \beta) \cup (\beta \cup \beta)) \cap \beta = (\beta \cup \alpha) \cap \beta = \alpha \cap \beta = \beta$.

(b) $\mathfrak{M}2$ verifies *B1*, but falsifies *B2* for a/β, b/β, and c/β: (i) $\beta = \beta$, (ii) $(\beta \cup (\beta \cup \beta)) \cap \beta = (\beta \cup \alpha) \cap \beta = \alpha \cap \beta = \alpha$.

Thus, the proof of (**A**) is complete.

NB After this paper was composed I learned that Kolibiar's problem had already been solved by J. Riečan, *cf.*, [2], who proved that formulas *B12* and *B2*, given above, constitute an adequate postulate-system for modular lattices. It should be noted that Riečan's system is shorter than mine.

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University of Notre Dame
Notre Dame, Indiana