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AXIOMATIZATION OF FRAGMENTS OF S5

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The principal result of this paper* is that the well-formed formulas of the system S5 of modal logic can be expressed in a kind of normal form which is restricted in a certain way. That normal forms exist for S5 is well known, but the results given here indicate that the normal forms can be expressed largely—in a sense to be made clear later—using only implication and necessity. From this result, it then follows that there is a uniform pattern for axiomatizing those functionally incomplete parts of the system S5 which contain at least implication and necessity. These functionally incomplete parts are the 'fragments' of the title.

The principal result, in turn, follows from the discovery of a sequence of expressions in S5, containing only implication and necessity, having certain welcome properties. That such functors might exist was suggested by the results of Canty and Scharle, and Massey, mentioned below. In order to find the expressions, the next step was to use a digital computer to search, in effect, all possibilities in hope of finding, in a reasonable amount of time, a successful match. Fortunately, many matches were found. This is one of the few published cases in which a digital computer has been used to find a solution to a non-trivial problem in logic.

Having found a solution for the two-variable case this way (the case for one variable is trivial), one sees that the next step is either to use the computer again, or to proceed by more conventional methods. To use the computer again, it turns out, would involve extremely long computation times, so this was not attempted. But, as will be shown in this paper, an analytical approach shows at least the existence of solutions in general.

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And the methods used are always constructive (in the mathematical sense of the term), although it may be practically impossible to follow through the proofs to produce the expressions needed.

The axiomatization result parallels the well-known method of Henkin for axiomatizing fragments of two-valued propositional calculus, but because of the nature of S5 we do not have one general format for all cases, but must construct a format depending upon the number of variables involved, also, the expressions quickly become unmanageably long. And, finally, we have the inelegance that we are restricted to those fragments containing necessity as well as implication (Henkin managed, of course, with implication only).

1 In this paper, we will use the following symbolism for propositional functors: $\supset$ for material implication, $\prec$ for strict implication, $\equiv$ for material equivalence, $=$ for strict equivalence, & for conjunction, $\vee$ for (inclusive) alternation, $\sim$ for negation, $\Box$ for necessity, and $\Diamond$ for possibility.

1.1 Axiomatizations of S5 The axiomatization of S5 given below is due, essentially, to Gödel.[1] As listed here, axiom *G1* is a single axiom for the two-valued propositional calculus (due to Meredith),[2] but any complete axiom set for two-valued logic serves in its place. Axioms:

G1 $((((p \supset q) \supset (\sim r \supset \sim s)) \supset r) \supset t) \supset ((t \supset p) \supset (s \supset p))$
G2 $\Box p \supset p$
G3 $\Box (p \supset q) \supset (\Box p \supset \Box q)$
G4 $\sim\Box\sim\Box p \supset \Box p$

The rules are substitution, material detachment and the Gödel rule:

if α is a theorem, then $\Box\alpha$ is also.

For later purposes, we will need these theorems of S5[3]:

G5 $\Diamond\Diamond p = \Diamond p$
G6 $\Diamond\sim\Diamond p = \sim\Diamond p$
G7 $\Diamond\Box p = \Box p$
G8 $\Diamond\sim\Box p = \sim\Box p$
G9 $\Diamond(p \,\&\, \Diamond q) = (\Diamond p \,\&\, \Diamond q)$
G10 $\Diamond(p \,\&\, \sim\Diamond q) = (\Diamond p \,\&\, \sim\Diamond q)$
G11 $\Diamond(p \vee q) = (\Diamond p \vee \Diamond q)$

1.2 Normal Forms for S5 The system S5 has the fortunate property of having only a finite number of non-equivalent expressions containing a fixed number of variables.[4] This was apparently first established by Carnap, as

1. Gödel [7].

2. Meredith [12].

3. See Feys [6].

4. See Feys [6].

noted below. A peculiarity of the form of the expressions is that they are simply related to two-valued propositional expressions for a larger number of variables—this leads immediately to the kind of normal form for S5 in which S5 expressions are standard substitutions in two-valued expressions, as noted by Canty and Scharle. By way of the normal forms for two-valued expressions, we have normal forms in a conjunctive-disjunctive form for S5 expressions. As it turns out the simplest way of describing S5 functions is by way of a sort of truth-table technique, first explicitly stated by Massey, which we also will use below to aid in exploring these normal forms for S5.

In the next section of this paper, these normal forms, and the relationship with two-valued logic, will be investigated further to allow us to axiomatize certain subsystems of the system S5.

Note that the strict equivalences *G5* through *G11* above allow us to simplify the iterated modalities in S5. In any part of an expression which is a sequence of negations, possibilities and necessities, replace that sequence with one of the following expressions:

a null expression, in case that this sequence consists of an even number of negations;

'$\sim$', in case that this sequence consists of an odd number of negations;

'$\Diamond$', in case that the first modality from the right in the sequence is a '$\Diamond$', and there are an even number of negations to the right of the possibility, and an even number to the left;

'$\Diamond$', in case that the first modality in the sequence is a '$\Diamond$', and there are an odd number of negations to both the right and left of the necessity;

'$\Diamond\sim$', in case that the first modality from the right is a possibility, and there are an even number of negations to the left of this possibility and an odd number of negations to the right of it, or in case that the first modality is a necessity and there are an odd number of negations to the left of the necessity and an even number of negations to the right of the necessity;

'$\sim\Diamond$', in case that the first modality from the right is a possibility, and there are an odd number of negations to the left of the possibility and an even number of negations to the right of the possibility, or in case that the first modality is a necessity and there are an even number of negations to the left of the necessity and an odd number of negations to the right of the necessity;

'$\sim\Diamond\sim$', in case that the first modality from the right is a possibility and there are an odd number of negations both to the right and the left of the possibility, or in case that the first modality is a necessity and there are an even number of negations both to the right and to the left of the necessity.

These rules change, for example:

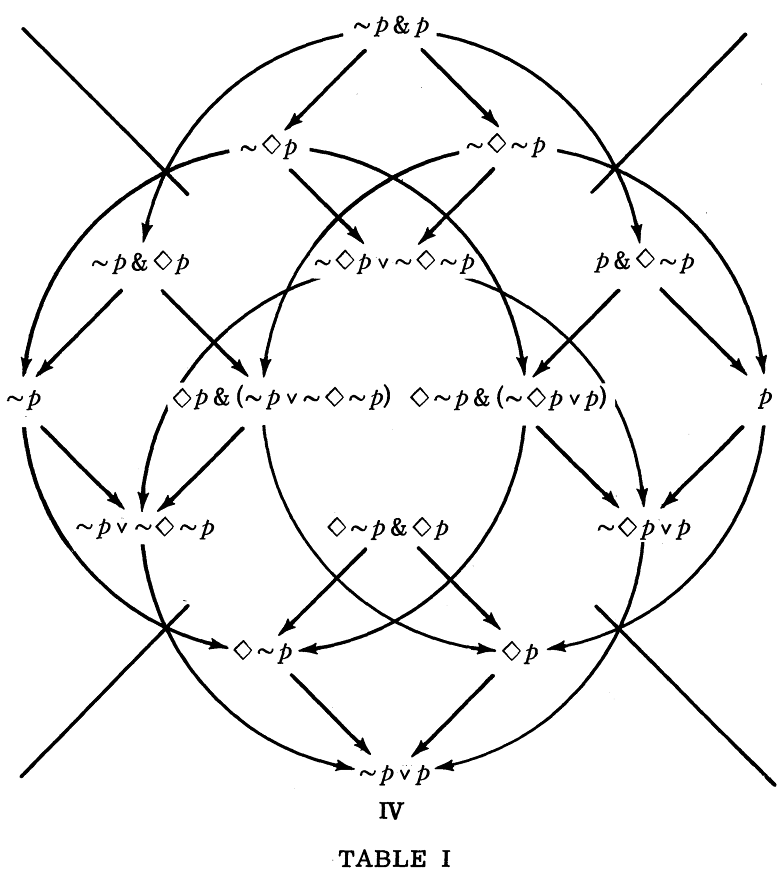
$$\Box\Diamond\sim\Diamond\Box\Diamond p \text{ to } \sim\Diamond p$$
$$\sim\Diamond\sim\Diamond p \text{ to } \Diamond p$$
$$\Box p \text{ to } \sim\Diamond\sim p$$

It is easily seen that there are only six different expressions which can be formed with 'p', '$\sim$', and '$\Diamond$', namely

$$\begin{array}{ccc} p & \Diamond p & \Diamond \sim p \\ \sim p & \sim \Diamond p & \sim \Diamond \sim p \end{array}$$

However, there are ten additional expressions which can be formed using '&' (and '$\vee$') also. These are displayed in Table I.[5]

I



III II

IV

TABLE I

In Table I, arrows are drawn to indicate implications—for example, there is an arrow connecting 'p' and '$\Diamond p$', indicating that '$p \supset \Diamond p$' is valid in S5. The sixteen expressions displayed in the table correspond, as will be shown later, to the sixteen binary functors of two-valued propositional calculus.

5. This table is derivative of the table described in Scharle [13].

That all of these expressions are distinct in S5 can be shown by use of the following matrices:

&	1	2	3	4	$\sim$	$\Diamond$
*1	1	2	3	4	4	1
2	2	2	4	4	3	1
3	3	4	3	4	2	1
4	4	4	4	4	1	4

which assign all valid formulas in S5 the designated value of 1, and which distinguish among all sixteen formulas displayed in the table. To show that there are no other non-equivalent formulas in S5, we use the equivalences *G5* through *G11*. Consider any expression in S5 constructed from one variable, '&', '$\sim$', 'v' and '$\Diamond$'. We will show that the set of sixteen formulas is closed under application of those functors, in the sense that for any α and β in the set, there are γ, δ, ϵ, and ζ in the set such that

$\sim\alpha$ is strictly equivalent to γ
$\alpha \ \& \ \beta$ is strictly equivalent to δ
$\alpha \vee \beta$ is strictly equivalent to ϵ
$\Diamond\alpha$ is strictly equivalent to ζ

In the case of negation, the negation of one of the sixteen expressions is located in the display 180° away from the expression—see for example, the locations of 'p' and '$\sim p$', '$\Diamond p$' and '$\sim\Diamond p$'. In the case of the conjunction of two formulas, find the expression which is lowest in a chain of arrows pointing to the two expressions. This expression will be equivalent to the conjunction of the two expressions. For example:

the conjunction of $\Diamond\sim p \ \& \ (\sim\Diamond p \vee p)$ and p is equivalent to $p \ \& \ \Diamond\sim p$
the conjunction of $\sim p \vee \sim\Diamond\sim p$ and p is equivalent to $\sim\Diamond\sim p$

In the case of the alternation of two expressions, find the expression in the display which is the highest expression in a chain of arrows away from both of the two expressions. For example:

the alternation of $\Diamond\sim p \ \& \ (\sim\Diamond p \vee p)$ and p is equivalent to $\sim\Diamond p \vee p$
the alternation of $\sim p \vee \sim\Diamond\sim p$ and p is equivalent to $\sim p \vee p$

In the case of possibility, the following considerations work: for an expression in quadrant I of Table I possibility of the expression is equivalent to the expression itself (and likewise for quadrant III). In all other cases, possibility for an expression is found in quadrant III, in the position in that quadrant corresponding to the position the original expression has in its quadrant. For example:

possibility of	$\Box p$	is equivalent to	$\Box p$
possibility of	$\Diamond\sim p \ \& \ (\sim\Diamond p \vee p)$	is equivalent to	$\Diamond\sim p$
possibility of	$\sim p \vee \Box p$	is equivalent to	$\sim p \vee p$
possibility of	$\sim p \ \& \ \Diamond p$	is equivalent to	$\sim p \ \& \ p$

In short, then, application of negation, conjunction, alternation, and possibility cannot produce an expression distinct from all of these sixteen expressions. A similar argument will show that there is a finite number of distinct expressions for a given finite number of variables.

We claim that every expression of S5 is equivalent to an expression in the following normal form[6]: a conjunction of alternations

$$\pi_1 \vee \ldots \vee \pi_k \vee \alpha_1 \vee \ldots \vee \alpha_i \vee \sim\beta_1 \vee \ldots \vee \sim\beta_j$$

where each α and β is of the form

$$\Diamond(\pi_1 \ \& \ \ldots \ \& \ \pi_l)$$

where each π is one of the variables in the original expression or its negation. To substantiate this claim, we first note that each variable, considered as an expression, is in normal form. If we prove closure of the set of normal forms (in the sense that the result of any operation is strictly equivalent to a member of the set), we prove *a fortiori* that every expression of S5 is equivalent to a normal form expression. That the set of normal forms is closed under negation, conjunction, and alternation follows by the laws for distributing conjunction and alternation and by DeMorgan's laws, for two-valued propositional calculus. In other words, this set is closed for the same reason that the set of conjunctive-disjunctive normal forms is closed for two-valued logic.

Suppose now that we prefix an expression in normal form with the functor for possibility. By the use of the laws of distribution of conjunction and alternation and of the laws *G10* and *G11*, we can move the possibility sign inside until the scope of all possibility operations is a conjunction of variables and their negations:

$$\begin{array}{lll} \text{transform } \Diamond(\alpha \ \& \ (\beta \vee \gamma)) & \text{to } \Diamond((\alpha \ \& \ \beta) \vee (\alpha \ \& \ \gamma)) \\ \text{transform } \Diamond(\alpha \vee \beta) & \text{to } \Diamond\alpha \vee \Diamond\beta \\ \text{transform } \Diamond(\alpha \ \& \ \Diamond\beta) & \text{to } \Diamond\alpha \ \& \ \Diamond\beta \end{array}$$

Because of the equivalence

$$\Diamond p = \Diamond(p \ \& \ q) \vee \Diamond(p \ \& \ \sim q)$$

we can require that the normal forms have the expressions α, β to be of the form

$$\Diamond(\pi_1 \ \& \ \ldots \ \& \ \pi_n)$$

where for each variable, either it or its negation appears in $\pi_1 \ \& \ \ldots \ \& \ \pi_n$. Thus we have essentially the result that for any formula α in S5 with n variables $p_1 \ldots p_n$, there is a non-modal formula $\beta(p_1, \ldots, p_k)$, where $k = 2^n$, such that

$$\alpha = \beta(p_1, \ldots, p_n), \Diamond(p_1 \ \& \ \ldots \ \& \ p_n), \ldots, \Diamond(\sim p_1 \ \& \ \ldots \ \& \ \sim p_n)$$

6. The following discussion is a presentation of results due to Carnap [5].

As a corollary of this, we can represent each modal expression in S5 with a truth-table, where the following take the values Truth and Falsehood: each of the variables which occur in the expression $\Diamond(\pi_1 \,\&\, \ldots \,\&\, \pi_n)$, where '$\pi_1 \,\&\, \ldots \,\&\, \pi_n$' is a conjunction of the variables which are in the expression, or their negations. For examples, see the truth-tables presented in Table II given on p. 52.

If we examine these truth-tables, it becomes obvious that there is a redundancy. Because $p \prec\!\!3 \Diamond p$ the rows in which 'p' is valued true and '$\Diamond p$' is valued false have no significance, nor the rows in which '$\sim p$' is valued true and '$\Diamond \sim p$' is valued false. A similar argument holds for the case of more than one variable, to discard half of the rows. As there will be in the unedited form $2 + 2^n$ expressions valued true or false, there will be 2^{n+2^n} rows, and the editing will discard half of the rows, giving $2^{2^{n+2^n-1}}$ independent possibilities of distinct assignments of truth values to expressions in S5, which means that there are $2^{2^{n+2^n-1}}$ different expressions, up to strict equivalence, in S5.[7] This truth-tabular method has been used by Massey[8] to produce a truth-tabular decision procedure for S5.

Let us look at some of the truth-tables with one-variable expressions:

p	$\sim p$	$\sim p \,\&\, \Diamond p$	$p \,\&\, (\sim p \vee \sim\Diamond\sim p)$
T	T	F	T
F	F	F	F
T	T	T	F
F	T	F	T

In Table III, see p. 53, are some truth-tables for two-variable expressions. The tables are structured into two levels: first, there are sets of rows (here separated by horizontal lines), and second, within each set are the rows with assignments of T and F to each of the variables. Within a given set of rows, the variables are assigned values from a non-empty subset of all possible assignments of T and F. Non-modal operations are taken by standard truth-table techniques—in a row in which the expression α is valued T, the expression $\sim\alpha$ is valued F, and is valued T otherwise, and if α and β are both valued T in a given row, then $\alpha \,\&\, \beta$ is valued T in that row, and is valued F otherwise. Modal operations on expressions receive their valuations from the values in all of the rows in a given set of rows—if α is valued T in some row of a given set, then $\Diamond\alpha$ is valued T in all of the rows of that set, and otherwise is valued F in all of the rows of the set. If α is valued F in some row of a given set, then $\Box\alpha$ is valued F in all rows of the set, and is valued T otherwise.

The expression $\Diamond p \,\&\, (\sim p \vee \sim\Diamond\sim p)$ is of particular interest as this expression, when taken together with the expression p will uniquely

7. See Carnap [5].

8. Massey [11].

TABLE II

p	$\Diamond p$	$\Diamond \sim p$	$\sim p \,\&\, \Diamond p$	$\sim \Diamond \sim p$	$\sim \Diamond p \vee \sim \Diamond \sim p$	$p \,\&\, (\sim p \vee \sim \Diamond \sim p)$	$\sim p \vee \sim \Diamond \sim p$	$\sim \Diamond p$
T	T	T	F	F	F	F	F	F
T	T	F	F	F	T	T	T	F
T	F	T	F	F	T	F	F	F
T	F	F	F	F	T	F	T	F
F	T	T	T	T	F	T	T	F
F	T	F	T	F	T	T	T	F
F	F	T	F	T	T	F	T	T
F	F	F	F	F	T	F	T	T

p	q	$\Diamond(p \,\&\, q)$	$p \prec q$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	T
T T	T F	T T	F F
T F	T T	T T	T T
T F	T F	T T	T T
T F	F T	F F	F F
T F	F F	F F	F F
F F	T F	F F	T T
T T F	T F T	T T T	F F F
T T F	T F F	T T T	F F F
T F F	T T F	T T T	T T T
T F F	F T F	F F F	F F F
T T F F	T F T F	T T T T	F F F F

TABLE III

characterize each row of a one-variable truth-table, in the sense that any expression of S5 with one variable is strictly equivalent to a non-modal function of p and the above-mentioned expression. Obviously, there are such expressions for any number of variables—we need take only that expression which is valued T in half of the rows in which p is valued T and q is valued T, then another expression which is true in half of the rows in which p, q and the first expression are valued T, and a third expression which is valued T in one half of the rows in which p, q, and the two previous expressions are valued T, and similarly for other possible combinations of truth values. This observation will show the truth of this theorem.[9]

Theorem 1 *For all n there exist expressions in* S5,

$$\alpha_1(p_1, \ldots, p_n), \ldots, \alpha_k(p_1, \ldots, p_n)$$

with $k = 2^n - 1$, *such that for any expression* $\beta(p_1, \ldots, p_n)$, *of n variables in* S5, *there exists precisely one non-modal function*

$$\gamma(p_1, \ldots, p_k)$$

such that

$$\beta(p_1, \ldots, p_n) = \gamma(\alpha_1(p_1, \ldots, p_n), \ldots, \alpha_k(p_1, \ldots, p_n))$$

One final normal form expression for S5 will make use of a convention that the conjunction of zero propositions is the constant true proposition, and the alternation of zero propositions is the constant false proposition. In two-valued propositional calculus, each expression with n variables is equivalent to a (possibly null) conjunction of wffs of the form

$$p_1 \,\&\, \ldots \,\&\, p_i \supset q_1 \vee \ldots \vee q_j$$

where $i, j \geqslant 0$, $i + j = n$, and $p_1 \ldots p_i\, q_1 \ldots q_j$ ranges over permutations of the set of n variables. In particular, the expression above will be a conjunct of the normal form if and only if the original expression if valued F in that row of a truth-table in which $p_1, \ldots, p_i$ are all valued T and $q_1, \ldots, q_j$ are all valued F.

Corresponding to this property, we have for S5 that every expression is equivalent to a conjunction of expressions of the form

$$p_1 \,\&\, \ldots \,\&\, p_i \,\&\, \alpha_1 \,\&\, \ldots \,\&\, \alpha_j \dashv q_1 \vee \ldots \vee q_k \vee \beta_1 \vee \ldots \vee \beta_l$$

where $i, j, k, 1 \geqslant 0$, $i + k = n$, $j + l = 2^n = 1$, the p's and q's are the variables in the original expression, and the α's and β's are the expressions as described above in Theorem 1.

2 Fragments of Propositional Calculi In this section we will survey the results for two-valued propositional calculus, in axiomatizing fragments, and see from the analysis of the method used how this method may also be used for fragments of S5.

For the purposes of exposition, we will consider that a propositional calculus consists of three sets:

a) a set of propositional functors, and the set of propositional variables (which may be some fixed infinite set of letters, for example);

9. Canty and Scharle [4], and Massey [10].

b) the set of well-formed formulas which can be constructed from these sets of functors and variables;
c) the subset of the set of well-formed formulas which consists of all the formulas which are valid under a given interpretation.

Given the interpretation and the set of functors, the other two sets are fixed. For example, using the ordinary truth-table interpretation and the set of all two-valued functors, the set (b) is the set of all expressions in two-valued propositional logic, and the set (c) is the set of all tautologies. This we will call the full two-valued propositional calculus. Or, given a standard interpretation of S5 formulas, and the set of all functors of modal logic, we obtain as (c) the set of all valid formulas in S5. (In the latter case, the set of functors may be determined precisely, but we will not need to express this precisely for our purposes.) This we will call the full S5.

A fragment of a propositional calculus is a propositional calculus whose set of functors is a subset of the set of functors of the given propositional calculus. Some examples of fragments of the full two-valued propositional calculus are given by specifying these sets of functors:

Take the set consisting of $\supset$ and **f**. The fragment determined by these functors is functionally complete, in the sense that we may give definitions of all functors of two-valued logic using $\supset$ and **f** only.

Take as the set of functors $\{\supset\}$. This determines a functionally incomplete fragment of two-valued logic.

We are interested here in axiomatizing fragments of propositional calculi, that is in giving rules and axioms which will yield as theorems all of the formulas in set (c), given the set (a) of functors.

2.1 Henkin's Fragments of Propositional Calculus Henkin has investigated the problem of axiomatizing a large class of fragments of the full two-valued propositional calculus, namely those in which the set of functors (a) contains $\supset$ and one other functor. The problem of such a general method is to find a way of using information from the truth-table for a given functor to determine an axiom set specific to the functor.

We have seen above how a given row of a truth-table for a functor has a correspondence to a normal form—suppose that we have a functor ϕ which takes n arguments. Suppose that $\phi p_1 \ldots p_i q_1 \ldots q_j$ (where $i + j = n$) is false when $p_1 \ldots p_i$ are true and $q_1 \ldots q_j$ are false. This may be characterized by either of the following properties:

In the normal form for $\phi p_1 \ldots p_i q_1 \ldots q_j$, which is a conjunction of implications, the following implication is one of the conjuncts

$$p_1 \,\&\, \ldots \,\&\, p_i \supset q_1 \vee \ldots \vee q_j \tag{1}$$

or

the following is a tautology

$$\phi p_1 \ldots p_i q_1 \ldots q_j \,\&\, p_1 \,\&\, \ldots \,\&\, p_i \supset q_1 \vee \ldots \vee q_j \tag{2}$$

In case that $\phi p_1 \ldots p_i q_1 \ldots q_j$ is true in that row of the truth-table in which $p_1 \ldots p_i$ are all true and $q_1 \ldots q_j$ are false, then we have that (1) is not a conjunct in the normal form for $\phi p_1 \ldots p_i q_1 \ldots q_j$, or, equivalently, that the following is a tautology

$$p_1 \,\&\, \ldots \,\&\, p_i \supset q_1 \vee \ldots \vee q_j \vee \phi p_1 \ldots p_i q_1 \ldots q_j \tag{3}$$

And we have similar statements for arbitrary permutations of the p's and q's (a more precise statement of the property will be deferred, as it is not very enlightening at the present). If we reflect upon the expressions (1), (2), and (3) above (and their permutations), we may note that:

a) The normal form of a given expression is uniquely determined by the rows of the truth-table, that is, given the truth-table for a given functor, we have established here rules for including or excluding (1) as a conjunct of the normal form. The normal form may thus be considered to be a kind of syntactic means of expressing all of the semantic information contained in a truth-table. This is one essential part of the problem of completeness of a formal system. It is strongly suggestive, then, that we can completely axiomatize the functor ϕ by, say a rule of replacement for every occurrence of ϕ by its equivalent normal form.
b) We have a set of 2^n tautologies containing ϕ, which are distinctive for ϕ. That is, each n-ary functor will have a distinct set of formulas (2) and (3), chosen by the above process. This set of formulas is in some sense characteristic of the functor ϕ, and we may be lead to speculate that the expressions (2) and (3) will constitute an axiom set for the functor ϕ (given, of course, an adequate axiomatization of the remainder of the logic).

Unfortunately, in these forms, the results are not too interesting, if they are true. First of all, we are here approaching an axiomatization of a fragment of a propositional calculus which is functionally complete. We obviously are using, in addition to the functor $\supset$, the functors & and $\vee$. But also hidden here are the logical constants **t** and **f**, because of the possibility of an empty conjunction or alternation: take the simple case in which the functor is $\sim$. If we follow the rules to get the normal form for $\sim p$, we get a conjunction of one formula

$$p \supset$$

or, in more conventional notation, using **f** for an empty alternation

$$p \supset \mathbf{f}$$

Or, to get the tautologies (2) and (3) for $\sim p$, we get

$$\sim p \,\&\, p \supset \mathbf{f} \tag{2'}$$
$$\mathbf{t} \supset p \vee \sim p \tag{3'}$$

(where we have introduced the constant **t** for the empty conjunction). These are indeed correct, but uninteresting, as we know that negation can be defined using $\supset$ and **f**. It is an essentially trivial task to give an axiom set for a given functor on the basis of a functionally complete system.

What we will do is to make this into a non-trivial task by eliminating all occurrences of functors in the expressions (2) and (3) except for $\supset$ and ϕ.

First, each occurrence of the functor 'v' can be eliminated by use of the identity

$$(p \vee q) \equiv ((p \supset q) \supset q)$$

Then, as each occurrence of '&' is on the left of an implication, it may be removed by the equivalence

$$(p \;\&\; q \supset r) \equiv (p \supset (q \supset r))$$

We are able to remove the occurrences of null conjunctions on the left of the implication by simply removing the implication in virtue of the equivalence

$$\mathbf{t} \supset p \equiv p$$

Finally, we are able to remove the occurrence of null alternations on the right of the implication by the device of introduction of a new variable. Instead of having '$p \supset \mathbf{f}$', we have '$p \supset r$'. In the form in which Henkin states the axioms, we have for a given n-ary functor ϕ, supposing that $\phi x_1 \ldots x_n$ is false when $p_1 \ldots p_i$ are all true and $q_1 \ldots q_j$ are all false, and the x's are a permutation of the p's and q's, the axiom:

$$p_1 \supset \ldots \supset p_i \supset (q_1 \supset r) \supset \ldots \supset (q_j \supset r) \supset (\phi x_1 \ldots x_n \supset r) \qquad (2'')$$

Suppose that $\phi x_1 \ldots x_n$ is true under the same conditions, then we take as an axiom the following:

$$p_1 \supset \ldots \supset p_i \supset (q_1 \supset r) \supset \ldots \supset (q_j \supset r) \supset (\phi x_1 \ldots x_n \supset r) \supset r \qquad (3'')$$

As additional axioms, we take a complete axiom set for the pure implicational fragment of two-valued logic (i.e., axioms and rules which yield all tautologies expressible with $\supset$ only). Once these axioms have been discovered it becomes a straightforward task to show the completeness and consistency.

As a simple example of the system involved, consider the functor to be $\supset$. From the truth-table for negation: $\sim p$ is false when p is true. Hence, take the following axiom:

$$p \supset (\sim p \supset r) \qquad (2''')$$

Also, $\sim p$ is true when p is false, hence take the following axiom:

$$(p \supset r) \supset (\sim p \supset r) \supset r \qquad (3''')$$

For the details of the proof and related results, refer to the papers of Henkin [8] and Thomas [14].

We will now attempt to use a similar heuristic argument with S5.

2.2 Normal Forms for S5 That is, we will attempt to find axiom sets for fragments of S5, by exploiting a correspondence between rows of a truth-table and conjuncts of a normal form.

Recall our Theorem 1 above. It is strongly suggestive of a correspondence between, for example, all two-place functions of S5 and all five-place functions of two-valued logic. If we can find the proper expressions, X, Y, and Z, say, such that for any binary functor of S5 there is a five-place non-modal functor, W, such that

$$\phi pq \text{ is strictly equivalent to } WpqXpqYpqZpq$$

then the above argument could be followed out for, at least, two-place functors.

What was a major portion of our argument above was the elimination of all functors other than implication. In the case of a modal logic, it seems a little too severe a restriction, but we are able to find X, Y, and Z as needed, which are definable with only implication and necessity. These will be described shortly. But the importance of the existence of such a triple of expressions in S5 is that we are then able to say that

1) For every two-place functor in S5, there is a conjunctive-disjunctive normal form involving only implication and necessity;
2) We are able to construct from the truth-tables for any two-place functor of S5 an axiom set of 2^5 axioms.

Let us first examine the properties that we need, for the elementary case of one-place function is S5. We need that there is an expression in S5, which is constructed using only the functors $\supset$, $\Box$, such that each row of a truth-table is uniquely characterized by the values of 'p' and this expression. In particular, the expression $\Box(p \supset \Box p)$ fulfills the need, as we can see by examining the truth table for it.

p	$\Box(p \supset \Box p)$
T	T
F	T
T	F
F	F

With the case of two-variable expressions, there is still the possibility of finding appropriate expressions. The task may be characterized as a constructive one, as there are a finite number of rows in a truth-table, and a finite number of non-equivalent containing two variables. But the number of cases to be checked is very large. Note that the task involves finding three expressions constructed from implication, necessity, and two variables such that each row in a truth-table is characterized by the values of the three expressions and the two variables.

Fortunately, it is possible to have the task done by a computer program, and then verify the results by conventional methods. The program which does this was written in the programming language PL/1, and in the assembler language for the IBM System/360.[10] In the System/360, the standard amount of information which can easily be handled in operations—which in technical language is called the "word"—is 32 bits. This size corresponds well with the size of two-variable truth-tables in S5—32 cases. So we can represent the variables 'p' and 'q' as single words in the computer, which is to say that the column of truth values under 'p' in the truth-table is represented by a word having '1' bits where 'p' is valued 'T', and '0' bits where 'p' is valued 'F', and similarly for 'q'. Then we take all possible combinations of these computer words by applying sequences of machine operations which correspond to the functors of implication and necessity. In case that the resulting word has an equal number of '1' and '0' bits, the sequence of operations (that is, equivalently, the well-formed formula which has such a truth-table) and the word itself (that is, the truth-table) are stored for later reference, for these are possible X, Y, and Z functors. Below, labelled (A) through (H), are some of the resulting expressions. Using one algorithm for generating these expressions, in the first 3000 expressions considered, 143 satisfied this first criterion.

After a large number of these words are generated, we cross-check the words by triples for this property:

there are three expressions α, β, and γ such that as n ranges between 0 and 31, the five bit pattern taken from the n'th position in the words for p, q, α, β, and γ assumes all possible combinations.

That is the computer checks (in the long run) all possible functions definable by implication and necessity. There are, of course, a large number of such functions, and without extensive computer use, all possibilities cannot be checked. As a rough upper limit, using the Lemmas 1 and 2 below, we may estimate that there may be as many as 2^{31} non-equivalent expressions (that is, of the order of magnitude of one billion) definable with implication and necessity. Thus, it is fortunate that we are able in a fairly short time, to find triples which satisfy all the conditions.

First of all, let us give some of the expressions from which we may select triples. The number of combinations of triples selected from these is fairly large, and we will not describe all of them. Likewise, the expressions, if written with implication and necessity alone, become quite long, so we will first use these auxiliary definitions:

$p \# q$ is defined to be $p \prec (q \supset \Box p)$
$p \$ q$ is defined to be $(p \supset q) \prec (p \prec q)$
$p \vee q$ is defined to be $(p \supset q) \supset q$

The following eight expressions are all definable by $\supset$ and $\Box$, and are valued **T** in half of the rows of the truth-table, and **F** in half of the rows of

10. See [1] and [2].

the truth-table. In Table IV, see p. 61, is a complete truth-table worked out for the first three of these expressions.

(A) $(p \supset (p \# q) \dashv p)) \supset (p \$ q)$
(B) $(p \vee (q \supset (p \# q))) \supset (q \dashv p)$
(C) $(p \vee ((p \# q) \supset q)) \supset \Box(p \vee q)$
(D) $(p \supset (q \vee (p \# q))) \supset (p \$ q)$
(E) $(p \vee q \vee (p \# q)) \supset \Box(p \vee q)$
(F) $(p \vee (q \supset (q \# p))) \supset (q \$ p)$
(G) $(p \supset (q \vee (q \# p))) \supset (p \dashv q)$
(H) $(q \supset ((p \# q) \supset p)) \supset (q \dashv p)$

Now, if we examine the truth-tables for the expressions (A), (B), and (C), and compare them with the truth-tables for p and q, it is easy to see that each row of the truth-table is characterized by the values which (A), (B), and (C) and p and q take. That is to say that for a given row of the truth-table, we have assigned values to these five expressions—for example, TTTTT in the first row of the truth-table, TTTTF in the ninth row of the table—and this assignment occurs only in one row of the truth-table, hence that there is a one-to-one correspondence between rows of the truth-table and the values given to p, q, (A), (B), and (C). As there are five such expressions, the total number of possible assignments of T and F to them is 32—the number of rows of the truth-table, so that a corollary of this is that given any assignment of truth-values to the expressions p, q, (A), (B), and (C), there corresponds one (and only one) row of a truth-table, or, equivalently, any non-modal combination of these expressions corresponds to a unique row of the truth-table, or, equivalently, that the expressions (A), (B) and (C) are an appropriate choice for the two-variable case of Theorem 1. And they are all expressed using implication and necessity.

Now, in order for formulas to satisfy these properties in triples, they must have one property, namely that they take on the value T in precisely one-half of the rows of the truth-table. Were this not the case with each of the expressions it would obviously be impossible for them to assume all possible truth assignments once and only once. That even expressions exist which have this property, and are constructed with implication and necessity, is not obvious, and is tedious to work out. The expressions (A) through (H), as mentioned above, have this property, and are thus among them possible candidates for the same joint property that (A), (B), and (C) share. As a matter of fact, all of these expressions do enter into triples (according to the results of the computer program) with this property. It is a time-consuming task to verify that they do—and one quickly appreciates the ability of the computer to take over this task when one goes through the verification. But we have found one set of expressions with the desired property, so there is no need to discover the other sets among (A) through (H).

Although by using a computer we are able to find sets of expressions, definable by implication and necessity, satisfying the conditions of Theorem 1, for the two-variable case, the task for more than two variables becomes

p	q	$p\$q$	$p\#q$	A	B	C
T	T	T	T	T	T	T
T	F	T	T	T	T	T
F	T	T	T	F	T	T
F	F	T	T	T	T	T
T T	T F	F F	T T	T T	F F	T T
T F	T T	T T	F F	F T	T T	T T
T F	T F	T T	T F	T T	T T	F F
T F	F T	F F	T T	F F	T F	T T
T F	F F	F F	T T	T T	T F	F T
F F	T F	T T	T T	F F	T T	F T
T T F	T F T	F F F	F F F	F F T	F F F	T T T
T T F	T F F	F F F	F F F	T T T	F F F	F F F
T F F	T T F	T T T	F F F	F T F	T T T	F F F
T F F	F T F	F F F	T T T	F F F	T F F	F F T
T T F F	T F T F	F F F F	F F F F	F F T F	F F F F	F F F F

TABLE IV

impossible practically. For example, with three variables, the truth-tables will have $2^7 = 128$ rows, and we will now have to search among 5-tuples of the extremely large number of possible functions. At this point, we must proceed theoretically, which will mean that we will not give explicit expressions for the formulas—although the method we will give is constructive, in the sense that in principle we can write out the expressions.

Therefore, at this point it is appropriate to make our language more precise:

$\vdash \alpha$ means that α is valid in S5;

$\vdash \alpha = \beta$ means that the strict equivalence of α and β is valid in S5 (i.e., that α and β have the same truth-values);

the n-ary functor ϕ is definable by the set of functors Γ, means that there is an expression α which is constructed only from the propositional variables $p_1, \ldots p_n$ and functors from the set Γ, such that

$$\vdash \alpha = \phi p_1 \ldots p_n$$

We will now prove, in two steps (namely, Lemma 1 and Lemma 2) that for any n, the desired expressions for Theorem 1 are definable by $\{\supset, \Box\}$.

Lemma 1 *If* $\vdash \Box(p_1 \& \ldots \& p_n) \supset \phi p_1 \ldots p_n$, *then* ϕ *is definable by* $\{\supset, \Box\}$.

Proof: Assume that

$$\vdash \Box p_i \supset \beta$$

We have the equivalence of $p \supset q$ and $q \equiv (\sim p \supset q)$ in two-valued logic, so the validity of the above entails that there exists a wff γ of S5 such that

$$\vdash \beta = (\gamma \supset \Box p_i) \tag{1}$$

We may assume that γ contains only the functors $\supset$, $\&$, $\sim$, and $\Box$, and the variables $p_1, \ldots, p_n$. We will now eliminate all occurrences of negation on the right side of (1), by moving '$\sim$' outside of the scope of functors by the rules:

replace $\delta \supset \sim\epsilon$ by $\sim(\delta \& \epsilon)$
replace $\sim\delta \supset \epsilon$ by $(\delta \supset \epsilon) \supset \epsilon$
replace $\delta \& \sim\epsilon$ by $\sim(\delta \supset \epsilon)$
replace $\sim\delta \& \epsilon$ by $\sim(\epsilon \supset \delta)$
replace $\Box\sim\delta$ by $\sim((\delta \supset \Box(\delta \supset \Box\delta)) \supset \Box(\delta \supset \Box\delta))$

It is easy to verify that in each case the replacement may be done with truth-values preserved, and note in particular that the last expression above is equivalent to $\sim\Diamond\delta$.

Now we have an expression equivalent to β which contains only $\supset$, $\&$, and $\Box$ (in particular, the last occurrence of negation, if any, would occur in the right side of (1) only on the left of the implication, and could thus be eliminated by the second replacement rule). We may now further transform the right side of (1) by moving all occurrences of '&' outside the scope of other functors by the following replacement rules:

replace $(\delta \& \epsilon) \supset \zeta$ by $\delta \supset (\epsilon \supset \zeta)$
replace $\delta \supset (\epsilon \& \zeta)$ by $(\delta \supset \epsilon) \& (\delta \supset \zeta)$
replace $\Box(\delta \& \epsilon)$ by $\Box\delta \& \Box\epsilon$

Again, it is straightforward to verify that these replacements may be done preserving truth-values in S5.

If we follow the replacement rules as indicated above, then the expression on the right side of (1) will be transformed into an expression which is strictly equivalent to it, and which contains only implication and necessity. This proves the lemma.

Lemma 2 *For every n: there are k* ($= 2^{2^n-1}$) *n-ary functors* θ_i *of* S5, *definable by* $\{\Box, \supset\}$, *such that for every expression* α *in* S5 *containing the variables* $p_1 \ldots p_n$ *there is a unique* $(k+n)$*-ary non-modal functor* ψ *such that*

$$\vdash \alpha = \psi(p_1, \ldots, p_n, \theta_1(p_1, \ldots, p_n), \ldots, \theta_k(p_1, \ldots, p_n))$$

Proof: This is essentially Theorem 1 with the restriction that the functors be definable by $\{\Box, \supset\}$. The proof will be by induction on n. For $n = 1$, the functor $*$ defined by

$$\vdash *(p) = \Box(p \supset \Box p)$$

serves as such a functor, as was shown above.

Suppose that for given n, the specified θ_i exist. Then for all rows of the truth-tables in which $\Box p_{n+1}$ is valued T, there are wffs $\theta_i(p_1, \ldots, p_n)$ which characterize the rows uniquely. That is to say, for each valuation of $\theta_i(p_1, \ldots, p_n)$, $i = 1, \ldots, 2^{2^n-1}$ and of $p_1 \ldots p_n$, this is the correct valuation of one and only one row in the truth-table in which $\Box p_{n+1}$ is valued T. Now we may choose expressions β_j which are all valued T when $\Box p_{n+1}$ is valued T, such that the valuations of β_j, p_m, $\theta_i(p_1, \ldots, p_n)$ uniquely characterize all rows of the truth-table for the variables $p_1, \ldots, p_{n+1}$. By Lemma 1, this means that the functors $\theta_j(p_1, \ldots, p_{n+1})$ which are defined by β_j, are definable by the set $\{\Box, \supset\}$.

2.3 Axiomatization of the Fragments of S5 We are now prepared to use the method of Henkin, applied to the normal form theorem represented in Lemma 2, the generalized form of Theorem 1. First, we must look at the minimal fragment to be considered, that is, the fragment with implication and necessity only.

The following axiomatization of the $\{\supset, \Box\}$ fragment of S5 is due to Beth and Nieland.[11]

A1 $p \supset (q \supset p)$
A2 $(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$
A3 $((p \supset q) \supset p) \supset p$
A4 $\Box p \supset p$
A5 $\Box(\Box p \supset q) \supset (\Box p \supset \Box q)$
A6 $(\Box p \supset \Box q) \supset \Box(\Box p \supset q)$

Rules for substitution and detachment for material implication, and the

11. Beth and Nieland [3].

Gödel-Aristotle rule that if α is a theorem, then $\Box\alpha$ is a theorem. The axioms *A1*, *A2*, and *A3* constitute a complete axiomatization of the implicational fragment of the two-valued propositional calculus,[12] so we are justified in using any of the usual theorems and rules of the two-valued propositional logic which contain only implication. A5 immediately entails

A7 $\Box p \supset \Box\Box p$

The following may also be proved easily from the axioms *A1*-*A5*:

$$\Box(p \supset q) \supset (\Box p \supset q) \qquad \text{(by } A4\text{)}$$
$$\Box(\Box(p \supset q) \supset (\Box p \supset q)) \qquad \text{(Gödel rule)}$$
$$\Box(p \supset q) \supset \Box(\Box p \supset q) \qquad \text{(by } A5\text{)}$$

A8 $\Box(p \supset q) \supset (\Box p \supset \Box q)$ (by *A5*)

Conversely, the set *A1, A2, A3, A4, A7, A8* entail *A1-A5*, as may be seen by this simple deduction:

$$\Box(p \supset q) \supset (\Box p \supset \Box q) \qquad (A8)$$
$$\Box(\Box p \supset q) \supset (\Box\Box p \supset \Box q) \qquad \text{(by } A8\text{)}$$
$$\Box\Box p \supset \Box q \supset (\Box p \supset \Box q) \qquad \text{(by } A7\text{)}$$
$$\Box(\Box p \supset q) \supset (\Box p \supset \Box q) \qquad \text{(by } A1\text{-}A3\text{)}$$

As this latter set is the axiom set of Gödel for S4, less axioms for negation, it is plausible that this constitutes a complete axiom set for the $\{\supset, \Box\}$ fragment of S4 (as is proved by Beth and Nieland)—at least it contains at most S4. The axiom *A6* may be seen to be equivalent to the necessary addition for S5 provided we take the following definition by possibility.

$$\Diamond p = \Box(p \supset \Box p) \supset p$$

Then from *A1*-*A6* follow:

$$\Diamond\Box p \supset \Box(\Box(\Box p \supset \Box\Box p) \supset \Box p) \qquad \text{(by } A6\text{)}$$
$$\Box(\Box(\Box p \supset \Box\Box p) \supset \Box p) \supset \Box\Box(\Box p \supset \Box\Box p) \supset \Box\Box p \qquad \text{(by } A8\text{)}$$
$$(\Box(\Box(\Box p \supset \Box\Box p) \supset \Box p) \supset \Box\Box p) \qquad \text{(by } A7\text{)}$$

A9 $\Diamond\Box p \supset \Box p$ (by *A5*)

We will now sketch a completeness proof of this fragment of S5. The notion of validity has above been characterized by truth-tables. Here, we will use the semantic tableau method of Beth, which verifies the same wffs as the truth-tables.

Suppose that we are given a well-formed formula. We may test its validity with truth-tables either by exhausting all rows of the truth-table, or by attempting to find at least one row in which the formula is valued false. This latter procedure may be described systematically by the

12. Due to Tarski and Bernays. See [9].

semantic tableau method. We begin with one main tableau—which may be visualized thus:

True	False

The formulas which we will put on the right side of the vertical line are those which are determined to be false, those on the left side, those which are determined to be true. We begin with the assumption that the given formula is false, and attempt to show that this is impossible or to find at least one case in which the formula is false. In case that we have the situation

T	F
	$\alpha \supset \beta$

because we know that $\alpha \supset \beta$ can be false only if α is true and β is false, we may extend this tableau thus:

T	F
	$\alpha \supset \beta$
α	β

In case that we have

$\alpha \supset \beta$	

Then we know that $\alpha \supset \beta$ is true, hence that either α is false or that β is true. In this case, we construct two sub-tableaux, representing each of the possibilities:

	$\alpha \supset \beta$		
	β	α	

In case that we have

	$\Box\alpha$

$\Box\alpha$ is false provided that there is some portion of the truth-table in which α is false, which is represented here by beginning a new alternative tableau:

		$\Box\alpha$	
		α	

Finally, in case that we have

$\Box\alpha$	

This case, the case that $\Box\alpha$ is true, entails that in all portions of the truth-tables, α is true—or here, that in every alternative tableau created by the above case, that α is true:

$\Box\alpha$	
α	

The formula in question may thus be mechanically reduced to atomic formulas, where we may readily test the possibilities: A tableau is closed (that is, not possible) provided that the same variable appears both in the True column and the False column. If a tableau has sub-tableaux, then the tableau is closed if and only if all of its sub-tableaux are closed. If a tableau has alternative tableaux, then the tableau is closed if and only if one of the alternative tableaux is closed. The closure of the tableau obviously corresponds to the impossibility of finding a falsifying instance of the formula, hence of the validity of the formula. On the other hand, if the tableau is not closed, then we are guaranteed by the exhaustive nature of the search that the formula has a falsifying instance.

Now we are prepared to apply the semantic tableau method to the $\{\supset, \Box\}$ fragment of S5. For a fragment of S5 containing $\supset$, $\Box$ and any n-ary functor ϕ, we can establish rules for the semantic tableau by means of Lemma 2. Suppose that we have in a tableau an occurrence of ϕ in the truth column, for example

$\phi\alpha_1 \ldots \alpha_n$	

By Lemma 2, there are expressions $\theta_1, \ldots, \theta_k$ such that the truth of $\phi\alpha_1 \ldots \alpha_n$ is equivalent to the alternation of conjunctions of θ_i and the negations of θ_i. That is, to speak in terms of truth-tables, $\phi\alpha_1 \ldots \alpha_n$ is true in a certain number of rows of the truth-table, for example in the row in which $\theta_i, \theta_j, \ldots$ are all true and $\theta_k, \theta_l, \ldots$ are all false. Therefore, we have this principle for the semantic tableau:

$\phi\alpha_1 \ldots \alpha_n$					
θ_i	θ_j	....	θ_k	θ_l	
.	.	....	.	.	....
.	.		.	.	
.	.		.	.	

That is, we generate one sub-tableau for each row of the truth-table in which $\phi\alpha_1 \ldots \alpha_n$ is true, and in a given sub-tableau place θ_i as true or false according as it is true or false in the given row of the truth-table.

Likewise, in case that $\phi\alpha_1 \ldots \alpha_n$ is false, we have a similar alternation of rows of a truth-table. Hence, there is a principle for the semantic tableau of this form:

$\phi\alpha_1 \ldots \alpha_n$					
θ_i	θ_j		θ_k	θ_l	
.	.	. . . .	.	.	. . . .
.	.		.	.	
.					

For axioms for this fragment of S5, we will take the axiom set for the $\{\supset, \Box\}$ fragment of S5, its rules, and axioms of the following forms:

$$q_1 \supset \ldots \supset q_j \supset \alpha_1 \supset \ldots \supset \alpha_l \supset (r_1 \supset s) \supset \ldots \supset (r_i \supset s) \supset (\alpha_{l+1} \supset s) \supset \ldots \supset (\alpha_n \supset s) \supset ((\phi p_1 \ldots p_n \supset s) \supset s) \tag{1}$$

$$q_1 \supset \ldots \supset q_j \supset \alpha_1 \supset \ldots \supset \alpha_l \supset (r_1 \supset s) \supset \ldots \supset (r_i \supset s) \supset (\alpha_{l+1} \supset s) \supset \ldots \supset (\alpha_k \supset s) \supset (\phi p_1 \ldots p_n \supset s) \tag{2}$$

where:

i) α's are expressions of the form $\theta(p_1 \ldots p_n)$, for θ chosen as in Lemma 2;
ii) the variables $q_1 \ldots q_j\ r_1 \ldots r_i$ are a permutation of $p_1 \ldots p_n$;
iii) form (1) is chosen when the row of the truth-table in which $q_1 \ldots q_j$ $\alpha_1 \ldots \alpha_l$ are all true and $r_1 \ldots r_i\ \alpha_{l+1} \ldots \alpha_k$ are all false is a row in which $\phi p_1 \ldots p_n$ is true;
iv) form (2) is chosen when the row of the truth-table in which $q_1 \ldots q_j$ $\alpha_1 \ldots \alpha_l$ are all true and $r_1 \ldots r_i\ \alpha_{l+1} \ldots \alpha_k$ are all false is a row in which $\phi p_1 \ldots p_n$ is false.

Theorem 2 *The above axiom set is complete for the* $\supset$, $\Box$, ϕ *fragment of* S5.

Proof: We will establish the truth of this in three stages:

It is an inductive proof, to show that every formula which has a closed semantic tableau is provable from the axioms. The induction is on the number of lines in a closed semantic tableau. By considering the rows generated for elimination of the three functors independently, we have the three stages, corresponding to the completeness of *A1-A3* for two-valued propositional calculus, of *A1-A5* for the implication-necessity fragment of S5, and finally for the completeness of the $\{\supset, \Box, \phi\}$ axiom set.

Corresponding to a construction of a semantic tableau which is closed, there is a deduction of the formula through a series of steps, each step being related to a given application of a reduction principle for semantic tableaux, with the steps proceeding in the reverse order as the application of the reduction principles.

Thus, at the bottom of the semantic tableau we have the closure property because the same formula appears in both the True and the False column. This situation corresponds to the provable formulas $p \supset p$, $p \supset (q \supset p)$, $q \supset (p \supset p)$, . . . For in general, each row of a tableau corresponds to an expression in which the formulas on the True column imply the alternation of the formulas in the False column.

We may call stage one of the process the elimination of the rules for implication. Corresponding to the rule

$\alpha \supset \beta$	
β	α

We have this rule of two-valued propositional calculus

$$\frac{\begin{array}{c}\beta \supset \gamma \\ \delta \supset \alpha\end{array}}{\delta \supset ((\alpha \supset \beta) \supset \gamma)}$$

As the axioms *A1-A3* are in fact a complete axiom set for the implicational fragment of two-valued logic, this rule may be used. For the other rule for implication, i.e.,

	$\alpha \supset \beta$
α	β

this is trivially eliminable.

Stage two is the corresponding elimination of the rules for necessity. For necessity on the True column, this may be done in virtue of the axiom *A5,* on the False column, in virtue of the axiom *A4* and the rule: if α then $\Box\alpha$. See Beth and Nieland [3].

Stage three is the corresponding elimination for the functor ϕ. In this case, the rules for the semantic tableau and the axioms for ϕ are clearly designed to make this stage simple. Suppose that we have this tableau:

$\phi\alpha_1 \ldots \alpha_n$			
θ_i	θ_j	θ_k	θ_l

In such a case, our principles for introduction of axioms for ϕ state that there is an axiom which eliminates this rule. That is, that there is an axiom, which has a substitution instance of the form

$$((\theta_i \supset r) \supset (\theta_j \supset r) \supset \theta_k \supset \theta_l \supset (\phi\alpha_1 \ldots \alpha_n \supset r))$$

whereby the rule is eliminable. Similarly for occasions in which ϕ occurs in the False column.

Hence, by induction, corresponding to every closed semantic tableau there is an inference from our axiom set which results in the formula at the top of the tableau. Therefore, the axiom set is complete.

Now, as mentioned, this axiomatization of the fragments of S5 is constructive, although it is not easy to display the axioms in any but fairly simple cases. The method is also restricted more than, say, Henkin's axiomatization in that Henkin assumed that the fragment contained the functor for implication, while we need also that the fragment contains necessity. The method used here in fact does not seem to admit of generalization in that direction. Of course, it is more general in the sense in which the system S5 can be considered as a generalization of two-valued logic.

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