

GENTZEN SYSTEMS FOR MODAL LOGIC

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Nice Gentzen formulations of the normal modal systems T and S4 have been known for some time; see, for example, Kanger [5] or Curry [1]. A similar formulation of S5 has also been given, but it is not so nice as the Elimination Theorem is not provable for it. I shall present here sets of rules for several of the non-normal modal systems which are akin to T and S4. Each of the L-systems to be defined here has an Elimination Theorem which may be proved by the methods of Gentzen [3]. These systems are useful, in that each of them has a decision procedure, following, for example, Kleene [6], §80.

1 Epistemic Systems In order to provide a decision procedure for Lewis' system S2, Ohnishi and Matsumoto [11] defined a system, Q2, which had the property that a formula, A , was provable in S2 if and only if the consecution $N(p \supset p) \Vdash A$ was provable in Q2. Q2 was formed by adjoining to any appropriate formulation of the classical propositional calculus, such as Gentzen's LK [3], the rules:

$$(N \Vdash) \frac{\Gamma, A \Vdash \Delta}{\Gamma, NA \Vdash \Delta} \quad \text{and} \quad (\Vdash N) \frac{\Gamma \Vdash A}{N\Gamma \Vdash NA}$$

where for $(\Vdash N)\Gamma$ must be non-empty.

Here, and elsewhere, A, B, C , etc. are well formed formulas formed from atomic formulas by means of propositional connectives, including N for necessity; Γ, Δ , etc. are any finite sequences of (zero or more) constituent formulas, A, B, C , etc. Consecutions α, β , etc. are expressions $\Gamma \Vdash \Delta$. $N\Gamma$ is the result of applying the operator N to each member of Γ .

Q2 is equivalent to the Hilbert style system E2 introduced by Lemmon (see, for example, [8]), in the sense that A is provable in E2 if and only if $\Vdash A$ is provable in Q2.

Besides the axioms and rules for the classical propositional calculus, E2 has only the axioms

A.1 $NA \supset A$

A.2 $N(A \supset B) \supset .NA \supset NB$

and the rule

$$R.2 \frac{\vdash A \supset B}{\vdash NA \supset NB}$$

The proofs of the counterparts of A.1 and A.2 in Q2 are nearly trivial. That *modus ponens* is admissible is guaranteed by the Elimination Theorem, i.e., that the rule

$$\frac{\Gamma \Vdash A, \Delta \quad \Gamma', A \Vdash \Delta'}{\Gamma, \Gamma' \Vdash \Delta, \Delta'}$$

is admissible in Q2. R2 is clearly admissible in Q2 given (\Vdash -N). Hence E2 is contained in Q2.

To show that Q2 is likewise contained in E2, we let the E2 counterpart of the consecution $\alpha = A_1, \dots, A_n \Vdash B_1, \dots, B_m$ be the formula $\alpha^* = A_1 \supset \dots \supset A_n \supset \sim B_1 \supset \dots \supset \sim B_{m-1} \supset B_m$, or, in case $m=0$, $\alpha^* = A_1 \supset \dots \supset A_{n-1} \supset \sim A_n$. (No counterpart for the case in which $n = m = 0$ need be given since the consecution \Vdash is not provable in Q2.) If α is provable in Q2, then α^* is provable in E2. This is clear in case α is prime or if α is the conclusion of an inference from β (β') by one of the non-modal rules, supposing, for the induction, that if β is provable in Q2 in fewer steps than α , then β^* is provable in E2. If α is from β by (\Vdash -N), then

$$\frac{\beta = A_1, \dots, A_n \Vdash B}{\alpha = NA_1, \dots, NA_n \Vdash NB}$$

By the inductive hypothesis, $\Vdash_{E2} A_1 \supset \dots \supset A_n \supset B$; so $\Vdash_{E2} NA_1 \supset N(A_2 \supset \dots \supset A_n \supset B)$ by (R.2), from which it follows that $\Vdash_{E2} NA_1 \supset NA_2 \supset \dots \supset NA_n \supset NB$, i.e., $\Vdash_{E2} \alpha^*$, by repeated applications of A.2 and transitivity.

In case α is from β by the rule (N \Vdash -), we have

$$\frac{\beta = C_1, \dots, C_n, A \Vdash B_1, \dots, B_m}{\alpha = C_1, \dots, C_n, NA \Vdash B_1, \dots, B_m}$$

So, by the inductive hypothesis $\Vdash_{E2} C_1 \supset \dots \supset C_n \supset A \supset \sim B_1 \supset \dots \supset \sim B_{m-1} \supset B_m$ (or $\Vdash_{E2} C_1 \supset \dots \supset C_n \supset \sim A$). Since $\Vdash_{E2} NA \supset A$ (A.1), it follows that $\Vdash_{E2} C_1 \supset \dots \supset C_n \supset NA \supset \sim B_1 \supset \dots \supset \sim B_{m-1} \supset B_m$ (or $\Vdash_{E2} C_1 \supset \dots \supset C_n \supset \sim NA$), which is to say, $\Vdash_{E2} \alpha^*$.

Thus, if $\alpha = \Vdash A$ is provable in Q2, then $\alpha^* = A$ is provable in E2, and conversely. The two systems are equivalent, in the sense defined. Hence I shall now call Q2 LE2.

Matsumoto [9] also presented the system Q3, to provide a decision procedure for S3, by modifying the rule (\Vdash -N) of Q2 to

$$(\Vdash - N) \frac{\Gamma \Vdash \mathbf{t}, A \quad N\Gamma \Vdash NA}{N\Gamma \Vdash NA}$$

where \mathbf{t} is any designated tautology, such as $(p \supset p)$, and Γ is again non-empty.

Q3 is equivalent to Lemmon's E3, which results from adding the axiom schema

A.3 $NA \supset N(NB \supset NA)$

to E2. It is easy to show that E3 is contained in Q3. To establish that Q3 is contained in E3 we use the same representation as given above. The admissibility of the counterpart of (N||-) follows as with E2. That the counterpart of this (||-N) is admissible in E3 is shown as follows: Suppose that $\beta_1^* = C_1 \supset \dots \supset C_n \supset \sim Nt \supset A$ and $\beta_2^* = NC_1 \supset \dots \supset NC_n \supset A$, we show that if $\vdash_{E3} \beta_1^*$ and $\vdash_{E3} \beta_2^*$, then $\vdash_{E2} \alpha^* = NC_1 \supset \dots \supset NC_n \supset NA$.

1. $\vdash NC_1 \supset \dots \supset NC_n \supset A$ given
2. $\vdash C_1 \supset \dots \supset C_n \supset \sim Nt \supset A$ given
3. $\vdash NC_1 \& \dots \& NC_n \supset A$ 1, truth functional logic (TFL)
4. $\vdash N(C_1 \& \dots \& C_n) \supset A$ 3, ($\vdash N(A \& B) \equiv NA \& NB$)
5. $\vdash C_1 \supset \dots \supset C_n \supset (Nt \supset N(C_1 \& \dots \& C_n)) \supset A$ 2, 4, (TFL)
6. $\vdash C_1 \& \dots \& C_n \supset (Nt \supset N(C_1 \& \dots \& C_n)) \supset A$ 5, (TFL)
7. $\vdash N(C_1 \& \dots \& C_n) \supset N(Nt \supset N(C_1 \& \dots \& C_n)) \supset NA$ 6, R.2, A.2
8. $\vdash N(C_1 \& \dots \& C_n) \supset N(Nt \supset N(C_1 \& \dots \& C_n)) \supset N(C_1 \& \dots \& C_n) \supset NA$ 7, (TFL)
9. $\vdash N(C_1 \& \dots \& C_n) \supset N(Nt \supset N(C_1 \& \dots \& C_n))$ A.3
10. $\vdash N(C_1 \& \dots \& C_n) \supset NA$ 8, 9, M, P
11. $\vdash NC_1 \& \dots \& NC_n \supset NA$ 10, ($\vdash N(A \& B) \equiv NA \& NB$)
12. $\vdash NC_1 \supset \dots \supset NC_n \supset NA$ 11, (TFL)

This suffices to show that Q3 is contained in E3 and that the systems are thus equivalent. I shall call Q3 LE3. (The equivalence of Q2 and E2 and of Q3 and E3 was observed by Ohnishi in [10].)

Lemmon's systems ET and E4 result from adding

A.4 $Nt \supset NNt$

and

A.5 $NA \supset NNA$

respectively to E.2. Their L formulations are formed by modifying the rule (||-N) again. LET has the rule

$$\frac{\theta, \Gamma \Vdash A}{\theta, N\Gamma \Vdash NA}$$

where θ is either empty or contains only the constituent Nt , and not both θ and Γ are empty. LE4 has the rule

$$\frac{N\Gamma \Vdash A}{N\Gamma \Vdash NA}$$

provided, again, that Γ is not empty. LET and LE4 may easily be shown to be equivalent to ET and E4 in the same manner as before.

If the condition that Γ (Γ or θ) not be empty is dropped from the rules (||-N) for LE4 and LET, we have the familiar rules for the Gentzen formulations of S4 and T.

2 Deontic Systems Deontic modal systems, DX , result from the system X (= $E2, E3, ET, E4, T, S4$) when its axiom $A.1$ is replaced by

$$A.1' \quad NA \supset \sim N\sim A.$$

This requires modification in the rule $(N\|\vdash)$ in their Gentzen counterparts. Let the new rule be

$$(N\|\vdash)_D \quad \frac{\Gamma, A\|\vdash}{N\Gamma, NA\|\vdash}$$

where it is required that the succedent in both the premiss and conclusion be void and that every constituent in the antecedent of the conclusion be prefixed by an N . The systems LDX are formed by replacing the rule $(N\|\vdash)$ of LX by $(N\|\vdash)_D$. In addition it is necessary to modify the rule $(\|\vdash N)$ of $LDE4$ and $LDS4$ to

$$\frac{\Gamma' \|\vdash A}{N\Gamma \|\vdash NA}$$

where Γ' is a sequence which results by prefixing zero or more members of Γ with an N . A similar change is required for the right premiss of $(\|\vdash N)$ for $LDE3$. (These modifications would also be acceptable for $LE3, LE4,$ and $LS4$.)

Proof of the Elimination Theorem for these systems goes through unimpeded by these changes. Hence, it is easy to show that $LDE2, LDE3, LDET, LDE4, LDT,$ and $LDS4$ are equivalent to the systems $DE2, DE3, DET, DE4, DT,$ and $DS4$, as defined, for example, in [4].

3 Lewis' S-Systems As observed above, Gentzen formulations of the normal systems T and $S4$ are obtained by allowing Γ in $(\|\vdash N)$ to be empty. In [10] Ohnishi defines a Gentzen system for Lewis' system $S2$; it has exactly the same rules as LT but a restriction is placed on its tree proofs to the effect that

(R) *No inference by $(\|\vdash N)$ may occur below an inference by $(\|\vdash N)$ in which Γ is empty.*

Thus $\|\vdash N(p \supset p)$ is provable in $LS2$, since

$$\frac{\frac{p \|\vdash p}{\|\vdash p \supset p} \quad (\|\vdash \supset)}{\|\vdash N(p \supset p)} \quad (\|\vdash N)$$

meets the condition, but $\|\vdash NN(p \supset p)$ is not, since its proof

$$\frac{\frac{\frac{p \|\vdash p}{\|\vdash p \supset p} \quad (\|\vdash \supset)}{\|\vdash N(p \supset p)} \quad (\|\vdash N)}{\|\vdash NN(p \supset p)} \quad (\|\vdash N) \text{—invalid}$$

violates the restriction (R).

If Γ is allowed to be empty in $(\|\vdash N)$ for $LE3$ but a similar restriction

is placed on proofs, the result is a system, LS3, equivalent to Lewis' S3. (Cf. Ohnishi [10].)

The restriction (R) may be generalized to read, for $n \geq 1$

(R)ⁿ No more than $n - 1$ inferences by (\parallel -N) may occur below an inference by (\parallel -N) in which Γ is empty.

If, for $n \geq 1$, this is applied to proofs by the LT rules for which Γ in (\parallel -N) is allowed to be void, systems LE2ⁿ are defined (where the n corresponds to the n of the restriction) which are equivalent to the systems E2ⁿ described by Lemmon [8] and Kripke [7]. These are formed by the addition of the axiom Nⁿt (i.e., the result of prefixing t with n N's) to E2 so formulated that *modus ponens* is its sole rule of inference. Thus E2⁰ = E2; E2¹ = E2 + Nt = S2; E2² = E2 + NNt; etc.

The condition (R)ⁿ presents no impediment to the proof of the Elimination Theorem for any of these systems. Thus LE2ⁿ may be shown to be equivalent to E2ⁿ. That E2ⁿ is contained in LE2ⁿ is clear, for all its axioms are either provable in E2, and so provable in LE2 which is contained in LE2ⁿ, or else the formula Nⁿt whose proof in LE2ⁿ is trivial. The admissibility of *modus ponens*, the sole rule of E2ⁿ, is guaranteed by the Elimination Theorem. To show that LE2ⁿ is contained in E2ⁿ we use the same representation α^* as before. The result then follows from the fact that

for every $n \geq 1$, if β occurs in a proof of α which contains no more than $n - 1$ inferences by (\parallel -N) which are below an inference by (\parallel -N) in which Γ is empty, then $\vdash_{E2^n} \beta^*$

which may be proved by induction on n , making use of the fact that if $\vdash_{E2^{n-1}} B$, then $\vdash_{E2^n} NB$ ($n \geq 1$). Hence, E2ⁿ is equivalent to LE2ⁿ.

A relaxation of (R) similar to the above immediately produces LS4 from LS3. I.e., if one or more inferences by (\parallel -N) may be below an inference by (\parallel -N) in which Γ is void, then the rule (\parallel -N) for LS4 is derivable:

$$\begin{array}{c} \text{(\parallel-K)} \quad \frac{\parallel-t}{Nt\parallel-t} \quad \parallel-t \\ \text{(\parallel-N)} \quad \frac{\parallel-t}{\Gamma\parallel-Nt, A} \quad N\Gamma^*\parallel-A \\ \text{(K \parallel-K)} \quad \frac{\Gamma\parallel-Nt, A \quad N\Gamma^*\parallel-A}{N\Gamma\parallel-NA} \quad \text{(\parallel-N)} \end{array}$$

This is as it should be, for, it is known, E3² = S4.

Devices similar to that used in the rule (\parallel -N) for ET may be employed to give L systems for the systems S4ⁿ which result from the extension of T by the axiom: NⁿA \supset Nⁿ⁺¹A (cf. e.g., Feys [2], p. 127). For LS4ⁿ we modify the T rule (\parallel -N) to

$$\frac{\theta, \Gamma\parallel-A}{\theta, N\Gamma\parallel-NA}$$

where θ is either empty or every constituent in it has the form NⁿB.

Obviously, $LS4^1 = LS4$. If we required that either θ or Γ not be empty, we would obtain rules for a family of systems $LE4^n$ which would be equivalent to the result of adding $N^n A \supset N^{n+1} A$ to ET . Thus, $E4^n$ stands to ET as $S4^n$ stands to T .

4 The \mathbb{L} Modal System A version of Łukasiewicz' modal system (without variable functors), here called \mathbb{L} , may be obtained by adjoining to $E2$ either the schema:

$$A \supset B \supset .NA \supset NB$$

or the schema

$$NB \supset .A \supset NA$$

(cf. Lemmon [8]; also Kripke [7]). A Gentzen system, $L\mathbb{L}$, equivalent to \mathbb{L} results by adding to rules for the classical propositional calculus the rule $(N\|\vdash)$ as for $E2$ and the rule

$$(\|\vdash N) \frac{\Gamma, NB \|\vdash A}{\Gamma, NB \|\vdash NA}$$

where now Γ may be any sequence, empty or not.

5 Intuitionistic Modal Systems If the modal postulates for $E2$ - $S4$, etc. were added to the intuitionistic propositional calculus for a base rather than to the classical propositional calculus, intuitionistic variants on those systems would be obtained. Equivalent Gentzen systems result simply by adding the appropriate $(N\|\vdash)$ and $(\|\vdash N)$ rules to L -rules for the intuitionistic logic.

These intuitionistic modal systems are not, however, the same as those defined in [4] and for which semantics were provided, for in those systems the formula $NA \vee \sim NA$ was provable. Gentzen formulations of these systems IX ($X = E2$ - $S4$, etc.) having this formula result by adding the modal rules to an L -system for intuitionistic logic which allows multiple constituents in the succedent and modifying the rule $(\|\vdash \sim)$ to read

$$\frac{\Gamma, A \|\vdash \Delta}{\Gamma \|\vdash \sim A, \Delta}$$

where Δ is either empty or every member of it has the form NB . (Ordinarily for intuitionistic negation one would require Δ to be empty.)

The proof that LIX is equivalent to IX is essentially the same as before. The only trick is to show that the counterpart of the new rule $(\|\vdash \sim)$ is admissible in IX . We argue that if $\beta = C_1, \dots, C_n, A \|\vdash NB_1, \dots, NB_m$ is provable in LIX , then by the inductive hypothesis $\beta^* = C_1 \supset \dots \supset C_n \supset A \supset \sim NB_1 \supset \dots \supset \sim NB_{m-1} \supset NB_m$ is provable in IX ; hence, by permutation, so is $C_1 \supset \dots \supset C_n \supset \sim NB_1 \supset \dots \supset A \supset NB_m$. From this we have $\vdash C_1 \supset \dots \supset C_n \supset \sim NB_1 \supset \dots \supset \sim \sim A \supset \sim \sim NB_m$ by contraposition twice, and so $\vdash C_1 \supset \dots \supset C_n \supset \sim \sim A \supset \sim NB_1 \supset \dots \supset \sim NB_{m-1} \supset \sim \sim NB_m$ again by permutation. Given that $NB_m \vee \sim NB_m, \sim \sim NB_m \supset NB_m$ is provable in IX , so by transitivity it follows that $\alpha^* = C_1 \supset \dots \supset C_n \supset \sim \sim A \supset \sim NB_1 \supset \dots \supset \sim NB_{m-1} \supset NB$ is also provable in IX .

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