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A NOTE ON FINITE INTERMEDIATE LOGICS

J. G. ANDERSON

By an *intermediate propositional calculus* (**IPC**) we mean a propositional calculus whose formulae, or synonymously, sentences are constructed in the standard way from a denumerable set of propositional variables and connectives $\rightarrow$, $\wedge$, $\vee$, $\neg$, and whose theorems are determined by the single rule modus ponens and the axiom schemes for Heyting's (intuitionistic) propositional calculus together with a single extra axiom scheme which is a two-valued tautology. We denote Heyting's propositional calculus by **HC** and the **IPC** with 'extra' axiom scheme X by **HC** $+ X$. Two **IPC** are said to be *equivalent* if they have identical sets of theorems. (Up to equivalence we lose no generality in specifying that an **IPC** has just a single 'extra' axiom scheme from the case where an arbitrary finite number are allowed since a propositional calculus with axiom schemes those of **HC** together with, say, $X_1, \ldots, X_n$ is equivalent to **HC** $+ X_1 \wedge X_2 \wedge \ldots \wedge X_n$.)

We use the term *model* (often called matrix) of a propositional calculus in an entirely conventional sense, and since every model of an **IPC** is a fortiori a model of **HC** the study of models of **IPC**'s can be interpreted as the study of Heyting algebras, also called pseudo-Boolean algebras, which are pseudo-complemented lattices with smallest element. There is an account in [4]. A model is said to be *characteristic* for an **IPC** if a sentence is a theorem of the **IPC** if and only if it is valid in the model. An **IPC** which has a finite characteristic model will be called *finite*.

The purpose of this paper is to describe an effective test to determine of an arbitrary **IPC** whether it is equivalent to a given finite **IPC**. The test will be of a particular sort: associated with a finite **IPC** **HC** $+ U$, with characteristic model $\mathfrak{M}$, will be a finite set of finite models, and an arbitrary **IPC** **HC** $+ X$, will be equivalent to **HC** $+ U$ if and only if (i) X is valid in $\mathfrak{M}$, and (ii) X is invalid in each of the associated finite models. As an example of such a result in a particular case we have that **HC** $+ X$ is equivalent to **HC** $+ P \vee \neg P$, classical propositional calculus, if and only if X is a two-valued tautology and is invalid in the model which, considered as a lattice, has three linearly ordered elements: a result of Jankov.

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The main tool we use will be Kripke-models. There are several related objects with this name in the literature: we specify the somewhat unsophisticated definition introduced in [1], both because of the authors familiarity with it and because it seems to suit the combinatorial nature of the arguments to be used. To follow the rest of this paper it will be necessary to have some acquaintance with the definitions and results contained in §'s 3, 4, 5, 6 of [1] from where all unexplained notation will come.

Presupposing this terminology we can outline the strategy we adopt. Referring to the conditions (i) and (ii) above it is easy to see that the significance of (i) is that the set of theorems of the finite **HC** + U will contain the set of theorems of **HC** + X if and only if (i) holds. The difficulty is to find the 'associated' models and show that the containment is reversed if and only if (ii) holds. How we deal with this difficulty is as follows. Given a finite model $\mathfrak{M}$ (which we can take to be a Heyting algebra) we find a sentence M which has the property that $\mathfrak{M}$ is a characteristic model of **HC** + M and M has only a finite number of minimal rejecting CKr-models. We prove a lemma to show that this gives us what we want.

Lemma 1 *Suppose* **HC** + X *is an* **IPC** *and* X *has a finite number of minimal rejecting* CKr-*models* $\phi_1, \ldots, \phi_n$. *Then the set of theorems of an* **IPC HC** + Y *include the theorems of* **HC** + X (*i.e.*, X *is a theorem of* **IC** + Y) *if and only if* Y *is invalid in each of* $\langle\phi_1\rangle, \ldots, \langle\phi_n\rangle$.

Remark: In the statement of the lemma, as elsewhere in the paper, we use the same symbol to denote a formula and the formula scheme associated with it. There seems to be no serious danger of ambiguity in doing this.

Proof of Lemma 1: If $\vdash_Y X$ then X is valid in every model of **HC** + Y. But $\phi_1, \ldots, \phi_n$ being X-rejecting Kr-models imply that X is invalid in $\langle\phi_1\rangle, \ldots, \langle\phi_n\rangle$, in turn implying that Y is invalid in $\langle\phi_1\rangle, \ldots, \langle\phi_n\rangle$ (see Lemma 6.2 [1]). To prove sufficiency suppose that Y is invalid in each of $\langle\phi_1\rangle, \ldots, \langle\phi_n\rangle$, and take it that Y has k propositional letters. Then, since each $\langle\phi_i\rangle$ is just the closure of the assignments of ϕ_i, for each i, $1 \leq i \leq n$, there are sentences $Y_{i,1}, Y_{i,2}, \ldots, Y_{i,k}$ such that $Y_{i,1}, \ldots, Y_{i,k} : \phi_i$ rejects Y. Put $\Gamma = \bigcup_{1\leq i\leq n}\{Y_{i,1}, \ldots, Y_{i,k}\}$ and we have, for any minimal X-rejecting model $\phi \in \{\phi_1, \ldots, \phi_n\}$, that there are $Y_1, \ldots, Y_k$, elements of the finite set Γ, such that $Y_1 \ldots Y_k$: ϕ is Y-rejecting to show that $\vdash_Y X$, (see Theorem 5.1 [1]).

Before proceeding with the construction of a sentence M from a given $\mathfrak{M}$, as indicated above, we must mention several technical results. First, by the *length* of a Kr-model ϕ we mean the greatest number of nodes of ϕ which form a chain, beginning with the origin, ending with an endpoint such that each member of the chain, apart from the origin, is an immediate successor of the previous member. We write $\mathrm{lgth}(\phi)$ for the length of ϕ. By induction on the length of the Kr-models it is easy to prove:

Lemma 2 *The set of* n-assgt *contracted Kripke-models of length* k *is finite, for all* $k \geq 1$.

In [1] a crucial result was: *If ϕ and ψ are two n-*assgt CKr*-models such that $\psi \not\subseteq \phi$ and $x_1, \ldots, x_n$ are any n propositional letters then there exists an (effectively obtainable) sentence* W$[\psi, \phi]$ *with propositional letters $x_1, \ldots, x_n$ such that ψ is minimal* W$[\psi, \phi]$*-rejecting and ϕ is* W$[\psi, \phi]$*-accepting.*

We extend this to

Lemma 3 *If ϕ is an n-*assgt CKr*-model and $x_1, \ldots, x_n$ are any n propositional variables then there exists an (effectively obtainable) sentence* W$[\phi]$ *with propositional letters $x_1, \ldots, x_n$ such that* W$[\phi]$ *is accepted by ϕ but rejected by all* CKr*-models ψ such that $\psi \not\subseteq \phi$.*

Proof: First consider the set

$$\Psi = \{\psi \mid \psi \text{ is an } n\text{-assgt CKr-model and } \mathrm{lgth}(\psi) = \mathrm{lgth}(\phi) + 1\}$$

which is finite by Lemma 2. For any $\psi \epsilon \Psi$ we have that $\psi \not\subseteq \phi$ since $\mathrm{lgth}(\psi) > \mathrm{lgth}(\phi)$. Thus, from the above-mentioned result, there is a sentence W$[\psi, \phi]$ with propositional letters $x_1, \ldots, x_n$ which is accepted by ϕ and rejected by ψ. Since Ψ is finite we can form the sentence $\prod_{\psi \epsilon \Psi} \mathrm{W}[\psi, \phi]$ which is accepted by ϕ but rejected by all elements of Ψ. But any n-assgt CKr-model of length $> \mathrm{lgth}(\phi)$ must have a submodel of length $\mathrm{lgth}(\phi) + 1$, i.e., a submodel which is an element of Ψ. Immediately we have that $\prod_{\psi \epsilon \Psi} \mathrm{W}[\psi, \phi]$ is rejected by all CKr-models of length $> \mathrm{lgth}(\phi) + 1$.

Now define another set, Θ, of n-assgt CKr-models, whose finiteness will again follow from Lemma 2, by

$$\Theta = \{\theta \mid \theta \text{ is a } n\text{-assgt CKr-model, } \mathrm{lgth}(\theta) \leq \mathrm{lgth}(\phi) \text{ and } \theta \not\subseteq \phi\}.$$

From the result mentioned above, for each $\theta \epsilon \Theta$ we can find a sentence W$[\theta, \phi]$ in $x_1, \ldots, x_n$ accepted by ϕ but rejected by θ. Thus the sentence $\prod_{\theta \epsilon \Theta} \mathrm{W}[\theta, \phi]$ is accepted by ϕ and rejected by all of Φ.

Now any n-assgt CKr-model which is not a submodel of ϕ is either an element of Θ or contains a submodel of Ψ so that if we define

$$\mathrm{W}[\phi] = \prod_{\mu \epsilon \Theta \cup \Psi} \mathrm{W}[\mu, \phi]$$

then W$[\phi]$ clearly satisfies the requirements of the lemma.

By using much the same sort of arguments as in the proof of Lemma 3 in a routine manner it can be shown that for any set $x_1, \ldots, x_n$ of propositional letters and any positive integer p there can be effectively found a sentence W_p using propositional letters $x_1, \ldots, x_n$ such that every CKr-model of length p is a minimal W_p-rejecting CKr-model. From this we obtain:

Lemma 4. *If ϕ is an n-*assgt CKr*-model and* $\mathrm{lgth}(\phi) > m$ *then the* Kr*-model $W_2 \ldots W_{m+1} : \phi$ is equivalent to*

$$\lambda_{m+1} = \begin{array}{cccccc} \multicolumn{6}{c}{\overbrace{0 \text{---} 0 \text{---} 0 \ \cdots\cdots\cdots\cdots\cdots \ 0 \text{---} 0}^{m\,+\,1\ \textit{nodes}}} \\ \mathsf{F} & \mathsf{F} & \mathsf{F} & & \mathsf{F} & \mathsf{T} \\ \mathsf{F} & \mathsf{F} & \mathsf{F} & & \mathsf{T} & \mathsf{T} \\ \vdots & \vdots & & & \vdots & \\ \mathsf{F} & \mathsf{F} & \mathsf{T} & & \mathsf{T} & \mathsf{T} \\ \mathsf{F} & \mathsf{T} & \mathsf{T} & & \mathsf{T} & \mathsf{T} \end{array}$$

Theorem 1 *If $\mathfrak{M}$ is a finite Heyting algebra then for some integer $n \geq 1$ there is an n-*assgt *Kripke-model ϕ such that the set of sentences valid in $\mathfrak{M}$ is identical to the set valid in $\langle\phi\rangle$.*

This is a paraphrase of corollaries 1.3(2) and 1.5 of H. Ono, published in [2]. (Ono's definition of Kripke-model differs from ours; but it is not hard to interpret one definition in terms of the other.)

We can now approach the main result. Suppose that $\mathfrak{M}$ is a finite model of HC and, using Theorem 1, the set of theorems valid in $\mathfrak{M}$ is just the set valid in $\langle\phi\rangle$, a n-assgt CKr-model. The elements of $\langle\phi\rangle$ are a closed set of assignments $a_1, \ldots, a_k$ (in fact the closure of the assignments of ϕ) on the model structure of ϕ. As mentioned before we are aiming at a sentence M such that $\mathfrak{M}$, (or $\langle\phi\rangle$) is a characteristic model if IC + M and M has only a finite number of minimal rejecting CKr-models. Since M is to be valid in $\langle\phi\rangle$ it will not be rejected by any Kr-model that can be formed from a subset of the assignments $\{a_1, \ldots, a_k\}$ and the model structure of ϕ. It will turn out to be convenient to construct M from $m = \max\{k + 1, \mathrm{lgth}(\phi)\}$ propositional letters. Consider the set Φ of Kr-models defined to be the set of all m-assgt Kr-models that can be formed from the model structure of ϕ and assignments $(b_1, \ldots, b_m)$ where $b_i \in \{a_1, \ldots, a_k\}$ $1 \leq i \leq m$, i.e.,

$$\Phi = \{b_1, \ldots, b_m : \phi \mid b_i \in \{a_1, \ldots, a_k\}, 1 \leq i \leq m\}.$$

Put

$$\Phi^* = \{\mu \mid \mu \text{ is contracted, and } \exists\phi,\ \phi \in \Phi,\ \mu \sim \phi\}.$$

We will want M to be accepted by all the elements of Φ as remarked above. For any given m propositional letters $x_1, \ldots, x_m$ define a sentence M in $\{x_1, \ldots, x_m\}$ by

$$M = \textstyle\sum_{\phi\in\Phi^*} \mathsf{W}[\phi],$$

where $\mathsf{W}[\phi]$ is as defined in Lemma 3, from which lemma it is apparent that M is accepted by all $\theta \in \Phi^*$ and rejected by all m-assgt Kr-models μ such that $\mu \not\leq \theta$ for any $\theta \in \Phi^*$.

Theorem 2 *The* IPC HC + M *has finite characteristic model* $\mathfrak{M}$.

Proof: First note that, by construction, M is valid in $\langle\phi\rangle$ and hence $\mathfrak{M}$ is a model of HC + M. Now we must show that, if X is a formula valid in $\mathfrak{M}$, then X is a theorem of HC + M. We accomplish this by exhibiting a finite set of formulae Γ such that if ψ is a minimal X-rejecting CKr-model then we can substitute formulae from Γ into M to obtain an instance rejected on ψ, (see Theorem 5.1 of [1]). Suppose that X has propositional letters $x_1, \ldots, x_n$. We recall that M has $m = \max\{k + 1, \mathrm{lgth}(\phi)\}$ variables. Let ψ be a minimal X-rejecting CKr-model. We treat two possibilities.

$\mathrm{lgth}(\psi) \leq m$. Suppose that ψ has $p \leq n$ distinct assignments; call them $b_1, \ldots, b_p$. We consider in turn the possibilities that $p \leq m$ and $p > m$.

If $p \leq m$ then the m-assgt Kr-model

$$b_1 b_2 \ldots b_p \overbrace{b_1 \ldots b_1}^{m - p} : \psi \tag{1}$$

cannot be equivalent to any submodel of a member of Φ^*, for otherwise, by appropriate substitution of assignments from that element of Φ^* we could reject X in $\langle\phi\rangle$. Thus the Kr-model (1) is M-rejecting. If $p > m$ then the Kr-model $b_1, \ldots, b_m : \psi$ is not equivalent to any submodel of an element of Φ^* since, as is easy to see, two equivalent CKr-models have the same number of distinct assignments and no element of Φ^* can have more than k distinct assignments. Thus in both these cases we have that there exist $y_1, \ldots, y_m \in \{x_1, \ldots, x_n\}$ such that $y_1, \ldots, y_m : \psi$ is M-rejecting.

$\mathrm{lgth}(\psi) > m$. In this case we have sentences $W_2, \ldots, W_{m+1}$ in $\{x_1, \ldots, x_n\}$, as in Lemma 4, such that $W_2, \ldots, W_{m+1} : \psi \sim \lambda_{m+1}$. But no member of Φ^* has length greater than $\mathrm{lgth}(\phi) < m + 1$ from which we deduce that $W_2, \ldots, W_{m+1} : \psi$ is equivalent to no submodel of Φ^* and hence rejects M, to finish consideration of this case.

Altogether, for any X-rejecting CKr-model ψ we have that there exist formulae $S_1, \ldots, S_m$, elements of the finite set of formulae $\{x_1, \ldots, x_n\} \cup \{W_2, \ldots, W_{m+1}\}$ such that $S_1, \ldots, S_m : \psi$ is M rejecting, and hence X is a theorem of HC + M, to conclude the proof.

It is easy to see that M has a finite number of minimal rejecting CKr-models. Consider the set

$$\Phi(M) = \{\theta \mid \theta \text{ is a minimal } X\text{-rejecting CKr-model and } \mathrm{lgth}(\theta) \leq \mathrm{lgth}(\phi) + 1\}.$$

This is finite, from Lemma 2, and effectively obtainable; though by the awkward algorithm of listing all the m-assgt CKr-models of length $\leq \mathrm{lgth}(\phi) + 1$ and simply checking whether they reject M or not. Moreover it consists of all the minimal X-rejecting CKr-models since every m-assgt CKr-model of length $> \mathrm{lgth}(\phi) + 1$ contains an M-rejecting submodel of $\mathrm{lgth}(\phi) + 1$ and consequently cannot be minimal.

We can now state the main result:

Theorem 3 *Let* HC + U *be a finite* IPC *with characteristic model* $\mathfrak{M}$. *Then there exists a finite set of finite models,* $\mathfrak{M}^*$, *effectively obtainable, such that, if* HC + Y *is an* IPC, HC + Y *is equivalent to* HC + U *if and only if*

(i) *Y is valid in* $\mathfrak{M}$,
(ii) *Y is invalid in each member of* $\mathfrak{M}^*$.

Proof: Given $\mathfrak{M}$ and using Theorem 1 we can define M as above so that, by Theorem 2, $\mathfrak{M}$ is a characteristic model for $\mathbf{HC} + M$ and so $\mathbf{HC} + M$ is equivalent to $\mathbf{HC} + U$. We then put $\mathfrak{M}^* = \{\langle\theta\rangle \mid \theta \in \Phi(M)\}$ and necessity is immediate. Sufficiency follows from Lemma 1.

It should be remarked that not every sentence has but a finite number of minimal rejecting CKr-models. We will shortly show that $p \vee (p \rightarrow q) \vee \neg q$ has an infinite number of them. However the **IPC** with extra axiom scheme $P \vee (P \rightarrow Q) \vee \neg Q$ is a finite **IPC** (see, e.g. [1]) and consequently is equivalent to an **IPC** whose extra axiom scheme does have a finite number of minimal rejecting CKr-models. It is not known whether such a result is true in general, i.e., it is not known whether, associated with any sentence X is another sentence Y, with a finite number of minimal rejecting CKr-models, such that the **IPC** $\mathbf{HC} + X$ is equivalent to $\mathbf{HC} + Y$. Such a result would be very strong if the association of Y with X was effective: it would, for instance, imply the decidability of all the **IPC**. It also seems to imply that operations can be defined in a natural sort of way on **IPC** so as to make the set of **IPC** a Heyting algebra which is a characteristic model of **HC**.

We finish by describing an infinite set of $p \vee (p \rightarrow q) \vee \neg q$ - rejecting CKr-models. Define a sequence $\{\rho_i\}$ of 2-assgt CKr-models by

$$\rho_0 = \begin{matrix}0\\ \mathbf{T}\\ \mathbf{T}\end{matrix} \qquad \rho_1 = \begin{matrix}0\\ \mathbf{F}\\ \mathbf{F}\end{matrix} \qquad \rho_2 = \begin{matrix}0 & \text{---} & 0\\ \mathbf{F} & & \mathbf{T}\\ \mathbf{F} & & \mathbf{T}\end{matrix}$$

$$\rho_n = \begin{matrix} & & \rho_{n-3}\\ 0 & \begin{matrix}\nearrow\\ \searrow\end{matrix} & \\ \mathbf{F} & & \rho_{n-2}\\ \mathbf{F} & & \end{matrix} \qquad n > 2$$

For each $n \geq 0$ ρ_n is contracted, by induction on n. Also for all $n \geq 0$ ρ_n affirms $p \rightarrow q$, and thus affirms $p \vee p \rightarrow q \vee \neg q$. Now consider the sequence $\{\rho_i{}^*\}$ of Kr-models defined

$$\rho_n{}^* = \begin{matrix} & \rho_n & & \\ 0 & \text{------} & & 0\\ \mathbf{F} & \searrow & & \mathbf{T}\\ \mathbf{F} & & 0 & \mathbf{F}\\ & & \mathbf{F} & \\ & & \mathbf{T} & \end{matrix}$$

Since $\begin{matrix}0\\ \mathbf{F},\\ \mathbf{T}\end{matrix}\begin{matrix}0\\ \mathbf{T}\\ \mathbf{F}\end{matrix} \not\subseteq \rho_n$ for any $n \geq 0$ ρ_n^* is contracted. Also ρ_n rejects $p \vee (p \rightarrow q) \vee \neg q$ for all $n \geq 0$, and since $\rho_i{}^* \not\subseteq \rho_j{}^*$, $i \neq j$ each $\rho_n{}^*$ is a minimal $p \vee (p \rightarrow q) \vee \neg q$ - rejecting CKr-model.

Remark: The main result was first obtained at the 1969 Summer Research Institute of the Canadian Mathematical Congress and was announced in [3]. The present proof is modified in the light of the result of H. Ono (Theorem 1) and also uses results from my thesis under Prof. R. Harrop.

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University of Waterloo
Waterloo, Ontario, Canada