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COMPLETENESS OF RELEVANT QUANTIFICATION THEORIES

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In [20], Meyer and Dunn answered affirmatively for the relevant sentential logics E and R the question, "Is the rule γ, 'From $\vdash A$ and $\vdash \overline{A} \vee B$, to infer B,' admissible?" This result, which confirmed an old conjecture of Anderson and Belnap, establishes the weak completeness of these and a number of related logics. In the present paper, some of whose principal results were announced without proof in [21], we shall extend the methods of past papers to prove both the admissibility of γ and, in a reasonable sense, weak completeness for the first-order extension RQ of R. In doing so, we replace the intuitively uninformative R-matrices of [20] with the theory of DeMorgan monoids, which furnishes a surprisingly smooth and natural algebraic semantics for R and, by extension, for RQ.

1. Furnishing RQ with a viable algebraic semantics and a proof of γ is no unimportant task. In the first place, the Anderson-Belnap system R of relevant implication is at the sentential level the most stable and interesting of the relevant logics. R contains in exact and well-motivated ways both the intuitionistic and the classical sentential calculi.[1] R_I, the implicational fragment of R,[2] is the oldest of the relevant logics, having been independently investigated twenty years ago by Moh-Shaw-Kwei and by Church in important papers, which provide interesting deductive-methodological motivation (A relevantly implies B only if A is used in some deduction of B).[3]

1. An exact translation of the Curry system HD into R, and hence of the intuitionistic sentential calculus, is presented in [18]; cf. [4] and [17]. In &, v, -, R contains all classical tautologies; cf. [6] and also [1].

2. In unpublished work Meyer has proved that R is a conservative extension of R_I when the latter is axiomatized as by Church in [10]. This settles an open question for R of the sort raised by Anderson for E and E_I in [2]. Cf. also Prawitz's [24].

3. Cf. [10] and [22].

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Though the analogues of truth-functional connectives that appear in Church's paper were highly artificial,[4] it turns out that the Anderson-Belnap addition of strong and natural axioms governing &, v, and - does not upset these deductive-methodological insights into the logic of a relevant →. Anderson and Belnap provide furthermore for the full system R motivation of a semantic kind, of which the crudest but most memorable result is that it is not provable that A relevantly implies B when A and B fail to share a variable.[5]

Not only does R have what it needs to be a *prima facie* candidate for the analysis of the notion of *logical* relevance, it does not have what it does not need. In particular, it is free of the cumbersome Lewis-style theory of modality of its sister system E of entailment, a slightly weaker system; it is free also of the fallacies of relevance of the slightly stronger Dunn system R-mingle.[6] Among the relevant logics, it is accordingly R that entices us to take the Goldilocks view—it appears to be just right.

For all of its motivation, however, R is but a sentential logic and is hence inadequate to many of the purposes for which we want formalized logics. Its extension to the quantificational system RQ is straightforward, however, if one follows Anderson and Belnap in adding quantificational axioms which directly generalize the principles they have laid down for truth-functions. It turns out, as Anderson and Belnap have established in a series of papers and reported again in [3], that there is no loss of motivation either on the deduction-theoretic or on the semantic side in the passage to RQ.

Anderson-Belnap style intuitive motivation is one thing, however, and semantical viability and practical utility another. The latter, for reasons given in [2], [3], and [20], is dependent upon the admissibility of γ. For if γ is not admissible in RQ, one can reproduce the argument for (1) of [20] to show that there is a sentence of RQ which is (a) unprovable, but (b) whose negation, if added to RQ as a new non-logical axiom, leads to inconsistency.[7] More simply—if γ fails some sentence which has no counter-example is nonetheless logically invalid according to RQ. Fortunately, as

4. Something close to Church's analogue of & will be introduced by D2 below; that connective o, which may be understood alternatively as a consistency operator or as an intensional analogue of conjunction, plays as interesting role *a propos* of our algebraic semantics to come. o is not, however, to be confused with the genuinely truth-functional connective & of RQ.

5. See [7].

6. For an algebraic study of R-mingle, see [11]; the system results from R by adding $A \rightarrow (A \rightarrow A)$ as a new axiom scheme. While sharing many features with R, R-mingle permits proof of $A \rightarrow B$ whenever *both* $\overline{A}$ and B are theorems of the system. See also [19].

7. The converse does not hold; $\neg(p \vee \neg p)$ is intuitionistically inconsistent though $p \vee \neg p$ is (notoriously) unprovable; yet γ holds. The reader is not to infer, either from present or past remarks, that we do not dig intuition or its ism. How could we fail to, since R captures the one and contains the other?

we show, this does not happen. On the side of practical utility, the possible pay-off on our proof of γ for RQ is a subject we defer to the end of this paper.

2. The syntax of RQ is ordinary. We use 'a', 'b', and 'c' for real variables, 'x', 'y', and 'z' for apparent variables (only the latter being accessible to quantifiers), and assume denumerable stocks of each. We use 'F^n', 'G^n', 'H^n' for n-ary predicate symbols, and assume denumerable stocks for every n from 0 on. We proceed to recursive definitions of *terms* and *formulas*.

(i) A variable, real or apparent, is a term.
(ii) The sentential constant f is a formula.
(iii) If $t_1, \ldots, t_n$ are terms and F^n is an n-ary predicate symbol, $F^n t_1 \ldots t_n$ is an atomic formula.
(iv) If x is an apparent variable and A and B are formulas, $(x)A$, $(A \,\&\, B)$, $(A \vee B)$, and $(A \rightarrow B)$ are formulas.
(v) Nothing is a term or a formula except in accordance with (i)-(iv).

An occurrence of a term t in a formula A is a free occurrence of t in A if it is not part of a subformula of A of the form $(t)B$, and a term t occurs free in a formula A if there is a free occurrence of t in A. We use '$A(t/u)$' for the result of substituting the term t at each free occurrence of the term u in the formula A, provided that there are no free occurrences of u in A in a context $(t)B$; otherwise the notation is undefined.

Let $x_1, \ldots, x_n$ be all the free apparent variables of the formula A; then $(x_1) \ldots (x_n)A$ is a *closure* of A. A *sentence* of RQ is a formula in which no apparent variables occur free.

Introduced by definition are the following:[8]

D0. $\overline{A} =_{df} A \rightarrow \mathrm{f}$
D1. $A \leftrightarrow B =_{df} \underline{(A \rightarrow B)} \,\&\, (B \rightarrow A)$
D2. $A \circ B =_{df} \overline{A \rightarrow \overline{B}}$
D3. $\exists x A =_{df} \overline{(x)\overline{A}}$
D4. $\mathrm{t} =_{df} \overline{\mathrm{f}}$

As axioms of RQ, take all closures of formulas of the following kinds:

A1. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
A2. $A \rightarrow ((A \rightarrow B) \rightarrow B)$
A3. $A \,\&\, B \rightarrow A$
A4. $A \,\&\, B \rightarrow B$

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8. We drop henceforth outermost parentheses, rank the binary connectives &, o, v, →, ↔ in order of increasing scope, and in the case of associative connectives (&, o, v) leave the reader free to put in parentheses where he thinks they should go. Our formulation of RQ differs from that of [8] chiefly on points of elegance; Belnap presented RQ as an extension of the first-order version EQ of E, and so he carries along into RQ axioms reflecting the E-style modal distinctions. Naturally the formulations are equivalent.

A5. $(A \rightarrow B) \,\&\, (A \rightarrow C) \rightarrow (A \rightarrow B \,\&\, C)$
A6. $A \rightarrow A \vee B$
A7. $B \rightarrow A \vee B$
A8. $(A \rightarrow C) \,\&\, (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$
A9. $A \,\&\, (B \vee C) \rightarrow A \,\&\, B \vee A \,\&\, C$
A10. $\overline{\overline{A}} \rightarrow A$
A11. $(A \rightarrow \overline{A}) \rightarrow \overline{A}$
A12. $(x)A \rightarrow A(t/x)$, *where t is a term*
A13. $(x)(A \rightarrow B) \rightarrow ((x)A \rightarrow (x)B)$
A14. $A \rightarrow (x)A$, *where x is not free in A*
A15. $(x)(A \vee B) \rightarrow (A \vee (x)B)$, *where x is not free in A*
A16. $(x)A \,\&\, (x)B \rightarrow (x)(A \,\&\, B)$

The rules of RQ are *modus ponens* (for $\rightarrow$) and adjunction (for &).

In proving the weak completeness of RQ, we shall at one point adapt the argument of Henkin's [14] by adding more real variables. (This can be avoided by using the techniques of Leblanc's [15], but since our adaptation involves new tricks we opt for familiarity *vis à vis* the old ones.) So we call the result of adding at most denumerably many fresh real variables (and accordingly inflating the supply of terms, formulas, sentences, and logical axioms) a *linguistic extension* of RQ.

Our next subject will be theories. Since we wish to develop our algebraic semantics independently for the sentential part R of RQ, we shall for certain purposes ignore the quantificational axioms A12-A16. An R-theory (RQ-theory) T is any set of sentences of RQ (or a linguistic extension thereof) which contains all instances of A1-A11 (all closures of instances of A1-A16) and which is closed under adjunction and *modus ponens*. Where T is an R-theory (RQ-theory) and $A \in T$, we write $\vdash_T A$; we say that A is *derivable* from the set S of sentences in the theory T, in symbols $S \vdash_T A$, if A is a member of every R-theory (RQ-theory) which contains $S \cup T$.

Let T be an R-theory (RQ-theory). T is *consistent* if not $\vdash_T$ f. T is *prime* if and only if whenever $\vdash_T A \vee B$, $\vdash_T A$ or $\vdash_T B$. T is *rich* if and only if whenever $\vdash_T A(a/x)$ for every real variable a of T, $\vdash_T (x)A$. T is *normal* if and only if T is consistent and prime (and T is *completely normal* if T is consistent, prime, and rich). This extends the terminology of [20].

3. In this section we present our semantics for R, which is the propositional part of our semantics for RQ. Essential is the notion of a DeMorgan monoid, which plays for the relevant logics the part played classically by Boolean algebras and intuitionistically by pseudo-Boolean algebras.[9]

Let $\mathfrak{D}$ be a quadruple $\langle D, \cdot, -, \vee\rangle$, where D is a non-empty set and - is a

9. Much of this section comes from Dunn's dissertation (University of Pittsburgh, 1966), which may be consulted for points of detail. Related material appears in Meyer's dissertation (University of Pittsburgh, 1966). The central Theorem 2, however, is new here; its proof is by adaptation of the techniques of [20].

unary and $\cdot$ and $\vee$ are binary operations on D. We sometimes omit '$\cdot$' in favor of simple juxtaposition and enter the following definitions for all a and b in D.

d0. $a \wedge b =_{df} -(-a \vee -b)$
d1. $a : b =_{df} -(-a \cdot b)$
d2. $a^2 = a \, . \, a$
d3. $a \leq b$ *iff* $a \vee b = b$

Then $\mathfrak{D}$ is a DeMorgan monoid provided that the following postulates hold, for all elements $a, b, c \in D$ and for an element $1 \in D$.

p0. $(ab)c = a(bc)$
p1. $ab = ba$
p2. $1a = a$
p3. $a \vee b = b \vee a$
p4. $(a \vee b) \vee c = a \vee (b \vee c)$
p5. $a = a \vee (a \wedge b)$
p6. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
p7. $--a = a$
p8. $a(b \vee c) = ab \vee ac$
p9. $(b : a)a \leq b$
p10. $a \leq a^2$

A word about these postulates is in order. p0-p2 are the postulates for a commutative monoid. p3-p5 are lattice postulates; their duals are delivered using p7 and d0, making D a lattice with respect to $\wedge$ and $\vee$, which is distributive by p6. (Structures satisfying d0, d3, and p3-p7 are called DeMorgan lattices in [23], suggesting the name for the present structures.) p8 makes D *lattice-ordered*, implying in particular (cf. [9]) that when $b \leq c$, $ab \leq ac$. p9 suffices with previous postulates to make : an operation of residuation in the sense of [9] and [13]; that is, $b : a$ turns out to be the least upper bound of all elements c such that $ac \leq b$ in a DeMorgan monoid. Finally, the square-increasing postulate p10 implies with p8 that D shall be exponentially ordered, in the sense $p^k \leq p^n$ for $0 < k \leq n$, on obvious definitions.

We enter two definitions and a lemma to show that an algebraist speaking DeMorganese is soul brother to the logician talking R. In the DeMorgan monoid $\mathfrak{D}$, for all $a, b \in D$, let

d4. $a \Longrightarrow b =_{df} b : a$
d5. $a \Longleftrightarrow b =_{df} (a \Longrightarrow b) \wedge (b \Longleftrightarrow a)$[10]

Lemma 1. *In every DeMorgan monoid* $\mathfrak{D} = \langle D, \cdot, -, \vee \rangle$, *the following hold for all* $a, b, c \in D$.

10. $a \Leftrightarrow b$ might be equivalently defined as $(a \Rightarrow b) \cdot (b \Rightarrow a)$. Similarly, D2 might be equivalently recast in RQ with o for &.

t1. $a \leq b$ *iff* $-b \leq -a$
t2. $ac \leq b$ *iff* $c \leq a \Rightarrow b$
t3. $1 \leq a \Rightarrow b$ *iff* $a \leq b$
t4. $1 \leq a \Leftrightarrow b$ *iff* $a = b$
t5. $a = 1 \Rightarrow a$
t6. $-a = a \Rightarrow -1$
t7. $ab = -(a \Rightarrow -b)$

Proof. Ad t1. $a \vee b = b$ iff $-(a \vee b) = -b$ iff $-(--a \vee --b) = -b$ (by p7) iff $-a \wedge -b = -b$ (by d0) iff $-b \leq -a$ (by lattice properties). Ad t2. Suppose $c \leq a \Rightarrow b$. By the lattice-ordering postulate p8, $ac \leq a(a \Rightarrow b)$, whence by p9 (and d4), $ac \leq b$. Suppose $ac \leq b$. By t1 $-b \leq -(ac)$ and by p8, p9, etc., $a - b \leq a - (ac) \leq -c$, whence by t1 again, double negation, d1 and d4 $c \leq a \Rightarrow b$. t3 is immediate from t2, since 1 is the identity. t4-t7 are trivial and are left to the reader.

A *filter* in a DeMorgan monoid $\mathfrak{D} = \langle D, \cdot, -, \vee \rangle$ is a filter in the lattice-theoretic sense—i.e., F is a filter in $\mathfrak{D}$ if and only if $F \subseteq D$, $F \neq \phi$ and $a \wedge b \in F$ iff $a \in F$ and $b \in F$. If the identity 1 is a member of the filter F in $\mathfrak{D}$, F is a *1-filter.*

In the algebraic development of classical and intuitionist logics, filters in general correspond to theories; for the relevant logics, 1-filters play that role. (Filters in general, not uninterestingly, correspond to sets of sentences closed under logical consequence; these need not be R-theories, however, since relevance restrictions do not permit logical axioms to be consequences of *arbitrary* sentences. Only 1-filters are of further interest for present purposes.) We introduce for 1-filters terminology paralleling that introduced in [2] for theories. Let F be a 1-filter in the DeMorgan monoid $\mathfrak{D}$. F is *consistent* if and only if $-1 \notin F$. F is *prime* if and only if whenever $a \vee b \in F$, $a \in F$ or $b \in F$. F is *normal* if and only if F is both consistent and prime.

In each DeMorgan monoid, there is a smallest 1-filter—namely the *principal filter determined by 1,* consisting of just those monoid elements a such that $1 \leq a$; we call this, for short, the P-filter, and extend our terminology further by calling the DeMorgan monoid $\mathfrak{D}$ itself *consistent, prime,* or *normal* just in case its P-filter is respectively consistent, prime, or normal.

At this point, we can make contact with some previously developed ideas in the algebraic semantics of the relevant logics. Belnap, in [8], provided such a semantics for the first-degree fragment of R and of E, and by extension for RQ and the quantificational version EQ of E. The key notion was that of an intensionally complemented distributive lattice with truth-filter, which is a pair $\langle \mathfrak{M}, F \rangle$, where $\mathfrak{M}$ is a DeMorgan *lattice* (cf. p. 101) and F is a filter in $\mathfrak{M}$ which contains exactly one of a, $-a$ for each lattice element a. (F is called a truth-filter in [8] because the idea is that lattice elements correspond to propositions and that exactly one of each proposition and its negation is true.) Belnap also introduced the notion of an *implicative extension* of one of his lattices, a triple $\langle \mathfrak{M}, F, \Rightarrow \rangle$, near enough, where $\langle \mathfrak{M}, F \rangle$ is an intensionally complemented distributive lattice with truth-filter and $\Rightarrow$ is a binary operation on the underlying set M. Let us call such a creature a *Belnap algebra* provided that in the sense of [8] it satisfies the axioms and rules of R. Then when a pair $\langle \mathfrak{D}, F \rangle$ is such that $\mathfrak{D}$ is a DeMorgan *monoid* and F is a 1-filter in $\mathfrak{D}$, a sufficient and necessary condition that $\langle \mathfrak{D}, F \rangle$ be a Belnap algebra is that F be normal (where $\Rightarrow$ is defined by d4). The crucial point is that, first, if F is normal we may use p10 to show $1 \leq a \vee -a$ and hence by primeness $a \in F$ or $-a \in F$, and hence also by t1 $a \wedge -a \leq -1$, so that because F is consistent not both $a \in F$ and $-a \in F$; second, if $\langle \mathfrak{D}, F \rangle$ is a Belnap algebra, then the 1-filter F cannot contain -1 and is hence consistent; moreover F contains one each of the pairs a, $-a$ and b, $-b$ but cannot contain all three of $-a$, $-b$, $a \vee b$ on pain of inconsistency; so F is prime and hence normal. Since Belnap conjectures in [8] that Belnap algebras form the key to the semantics of R, we see that this will be true if normal DeMorgan monoids suffice for the semantics of R, since the P-filter will serve in this case to do the job of Belnap's truth-filters.

In relating our DeMorganist and our logician, we play familiar tricks. The crucial one is to interpret each sentence of our formal language as making a statement about DeMorgan monoids. Success (at the sentential level) is measured as follows: R is semantically consistent provided that each sentence of RQ which belongs to all R-theories (and is hence derivable from A1-A11 alone) is, as interpreted, true of all DeMorgan monoids; R is semantically complete if each sentence of RQ true as interpreted of all DeMorgan monoids belongs to all R-theories. We shall prove success on both counts.

Let $\mathfrak{D} = \langle D, ., -, \vee \rangle$ be a DeMorgan monoid. By an R-*interpretation in* $\mathfrak{D}$, we mean a function I defined on the sentences of RQ with values in D, subject to the following conditions:

(i) $I(A \rightarrow B) = IA \Rightarrow IB$
(ii) $I(A \,\&\, B) = IA \wedge IB$
(iii) $I(A \vee B) = IA \vee IB$
(iv) $I(\mathrm{f}) = -1$

Lemma 1 may be applied to show that defined connectives of RQ correspond to similar DeMorgan operations—i.e.,

(v) $I(\mathrm{t}) = 1$
(vi) $I(A \circ B) = IA \,.\, IB$
(vii) $I(A \leftrightarrow B) = IA \Leftrightarrow IB$

A DeMorgan monoid, considered as the range of an interpretation of R, may be viewed as a system of propositions.[11] In such a system, our task is to separate the good ones from the bad ones, in order to carry out our program. A fortuitous choice is to count a monoid element a *true* if and

11. Cf. [8] and the Anderson-Belnap papers cited there.

only if $1 \leq a$, and *false* if and only if $a \leq -1$.[12] So we count A *true* (*false*) on the R-interpretation I in the DeMorgan monoid $\mathfrak{D}$ if and only if $I(A)$ ($I(\overline{A})$) belongs to the P-filter in $\mathfrak{D}$. Similarly, A is R-*valid* if and only if A is true on all R-interpretations I in all DeMorgan monoids $\mathfrak{D}$.

Theorem 1. *Let A be a sentence of* RQ. *A is* R-*valid if and only if, for all* R-*theories T*, $\vdash_T A$.

Proof. Suppose first $\vdash_T A$ for all R-theories T. As noted above, this means that there is a proof of A from instances of A1-A11, adjunction, and *modus ponens*. Show by induction on the length of that proof that A is R-valid, verifying that the axioms are R-valid and that the rules preserve R-validity.

This proves the consistency part of the theorem. To prove the completeness half, define for each R-theory T its *Lindenbaum* algebra $\mathfrak{T}^*$ in the following way. For each sentence A of RQ, let A^* be the set of sentences B of RQ such that $\vdash_T A \leftrightarrow B$. Let T^* be the class of all such A^* and define for all A^*, $B^* \in T^*$ the DeMorgan monoid operations thus: $A^* . B^* = (A \circ B)^*$, $-A^* = \overline{A}^*$, $A^* \vee B^* = (A \vee B)^*$. It is readily verified, applying *mutatis mutandis* the techniques of [25], that $\mathfrak{T}^* = \langle T^*, ., -, \vee \rangle$ is a DeMorgan monoid.

To finish the proof of Theorem 1, suppose now that not $\vdash_T B$ for some R-theory T. For each sentence A of T, let A^* be as above and let $I(A) = A^*$. It is readily verified that I is an R-interpretation in the Lindenbaum algebra $\mathfrak{T}^*$ of T and that $\vdash_T A$ if and only if $1 = t^* \leq A^*$ in $\mathfrak{T}^*$. So in particular B is not true on I in $\mathfrak{T}^*$. Contraposing and generalizing, for all sentences A of RQ, if A belongs to the class of R-valid sentences, A belongs to all R-theories, ending the proof.

Theorem 1 may be viewed as a specialization to our more natural semantics for R of (i)-(ii) of Theorem 2 of [20]. But just as in [20], there is a crucial gap between the kind of completeness guaranteed by the theorem and what is required. For in suggesting that the reader picture a DeMorgan monoid as a system of propositions, we nonetheless characterized truth and falsity in a way that will not do on most intuitions; there are DeMorgan monoids in which certain elements are both true and false, or neither true nor false. Believing, as some of us do, that both inconsistent and incomplete theories have their uses, and that accordingly there may be reasons

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12. It was with this fortuitous choice in mind that we introduced the sentential constant t (intuitively, for an arbitrary RQ-theory T the conjunction of the theorems of T) into the syntax and the corresponding monoid identity 1 into the semantics; dual remarks apply to f and -1. Both syntax and semantics can be handled without these frills; cf. [17], p. 196. (Besides these formal reasons for liking t and its kin, our colleague Nino Cocchiarella supplies another; our t may be correlated with what he calls the *world-proposition* and which in his view no correct ontology can do without. Having come ourselves to like t on shallow formal grounds, we rejoice to have discovered that its utility is no accident but is grounded in the Nature of Things.)

not to bifurcate the class of propositions into just the true ones and the false ones, still R would be a strange system if it *required* us to postulate that some propositions are both, or neither, of true, false.

This is incompatible with the Goldilock's view expressed above; what is the case, happily, is that if we stick to DeMorgan monoids in which exactly one of a, $-a$ is true for all a, we can still reject all R-invalid sentences. In making contact with Belnap's semantics for first-degree R, we have already noted that the DeMorgan monoids meeting this condition are just the normal ones. Hence let us call a sentence A of RQ *normally* R-*valid* if and only if A is true on all interpretations in normal DeMorgan monoids.

Theorem 2. *For all sentences* A *of* RQ, A *is* R-*valid if and only if* A *is normally* R-*valid.*

Observation. We adapt rather than apply directly the technique of [20] in order to render its extension to RQ uncluttered.

Proof. If A is R-valid it is trivially normally R-valid. Suppose then that A is not R-valid. Then there is a DeMorgan monoid $\mathfrak{D} = \langle D, ., -, \vee\rangle$ and an R-interpretation I such that A is not true on I in $\mathfrak{D}$. We show how to transform $\mathfrak{D}$ into a normal DeMorgan monoid $\mathfrak{D}^*$ and I into an interpretation I^* such that A is not true, and hence by normality false, on I^*. There are two stages in the transformation, which we call respectively *priming* and *splitting*.

Stage 1. We trade in $\mathfrak{D}$ for a prime DeMorgan monoid $\mathfrak{D}'$. We use two facts. First, by the Stone prime filter theorem (cf. [29]), if F is a filter in $\mathfrak{D}$ and $a \notin F$, there is a prime filter F' such that $F \subseteq F' \subseteq D$ and $a \notin F'$. In particular, since A is not true on I, $I(A) \notin P$, where $P = \{b\colon 1 \leq b\}$ (i.e., P is the P-filter in $\mathfrak{D}$). So there is a prime filter P' such that $P \subseteq P'$ and $I(A) \notin P'$.

P' is a 1-filter, since P is. This brings us to our second fact, namely that 1-filters determine homomorphisms and hence quotient monoids in the theory of DeMorgan monoids. (This fact is already reflected in the proof of Theorem 1; here we admit it.) Let F be a 1-filter in $\mathfrak{D}$, and for all $a \in D$ let $a' = \{b\colon b \Leftrightarrow a \in F\}$. Let D' be the set of all such a', and define operations $.'$, $-'$, $\vee'$ on D' by setting, for all b', $c' \in D'$, $b' .' c' = (b . c)'$, $-(b') = (-b)'$ $b' \vee' c' = (b \vee c)'$. There is then no difficulty in verifying that (i) these operations are well-defined, (ii) $\mathfrak{D}/F = \langle D', .', -', \vee'\rangle$ is a DeMorgan monoid, and (iii) $'$ is a homomorphism from $\mathfrak{D}$ onto $\mathfrak{D}/F$, in the sense that $'$ preserves monoid operations and that $1'$ is the identity of the *quotient monoid* $\mathfrak{D}/F$.[13]

We conclude stage 1 by setting $\mathfrak{D}' = \mathfrak{D}/P'$, where P' is as above. Define an R-interpretation in $\mathfrak{D}'$ by setting $I'(B) = (I(B))'$ for all sentences B of RQ.

13. Thus among other things a homomorphism from one DeMorgan monoid to another is a *lattice-homomorphism* (cf. [9]) and accordingly in particular preserves lattice order; i.e., if $a \leq b$, $h(a) \leq h(b)$. We note also that of course the identity in a DeMorgan monoid, as in any commutative monoid, is unique.

Noting that $a' \leq b'$ in $\mathfrak{D}'$ if and only if $a \Rightarrow b \in P'$ we conclude in particular that $1' \nleq I'(A)$—i.e., if A is not R-valid, there is an R-interpretation in a prime DeMorgan monoid on which A is not true.

Stage 2. We may at this stage already assume (for the typographer's sake) that $\mathfrak{D} = \langle D, ., -, \vee\rangle$ is a prime DeMorgan monoid in which A is not true on the interpretation I. What we show now is that there is a normal DeMorgan monoid $\mathfrak{D}^*$ of which $\mathfrak{D}$ is a homomorphic image. The construction of $\mathfrak{D}^*$ will enable us to construct the crucial interpretation I^* on which A is not true.

Let N be the set of elements of D which are both true and false, i.e., $N = \{a\colon 1 \leq a \;\&\; a \leq -1\}$. (If N is empty, $\mathfrak{D}$ is already normal.) Let $-N$ be a set disjoint from D and in 1-1 correspondence with N; let g be a bijection from N onto $-N$. Define

(i) $D^* = D \cup -N$.

(ii) $h: D^* \rightarrow D$ is a function defined by cases thus:
- (a) if $a \in D$, $h(a) = a$;
- (b) if $a \in N$, $h(g(a)) = a$.

(iii) $-^*$ is a unary operation on D^* defined thus:
- (a) if $a \in N$, $-^*a = g(-a)$;
- (b) if $a \in D^* - N$, $-^*a = -h(a)$.

(iv) $.^*$ is a binary operation on D^* defined thus:
- (a) if $h(a)\,.\,h(b) \in D - N$, $a \,.^*\, b = h(a)\,.\,h(b)$;
- (b) if $h(a)\,.\,h(b) \in N$, then
  - (1) if both $a \in N$ and $b \in N$, $a \,.^*\, b = a\,.\,b$;
  - (2) if $a \notin N$ or $b \notin N$, $a \,.^*\, b = g(h(a)\,.\,h(b))$.

(v) $\vee^*$ is a binary operation on D^* defined thus:
- (a) if $h(a) \vee h(b) \in D - N$, $a \vee^* b = h(a) \vee h(b)$;
- (b) if $h(a) \vee h(b) \in N$, then
  - (1) if $a \in -N$ and $b \in -N$, $a \vee^* b = g(h(a) \vee h(b))$;
  - (2) if $a \in D$ and $b \in -N$, $1 \nleq a$ in $\mathfrak{D}$, $a \vee^* b = b \vee^* a = g(a \vee h(b))$;
  - (3) otherwise $a \vee^* b = b \vee^* a = h(a) \vee h(b)$.

Let $\mathfrak{D}^* = \langle D^*, .^*, -^*, \vee^*\rangle$ be defined by (i)-(v). Straightforward verification establishes the following points:[14]

(vi) $\mathfrak{D}^*$ is a DeMorgan monoid.

(vii) The identity 1 of $\mathfrak{D}$ is the identity of $\mathfrak{D}^*$, and the P-filter P^* of elements greater than or equal to 1 in $\mathfrak{D}^*$ is normal; hence by definition $\mathfrak{D}^*$ is normal.

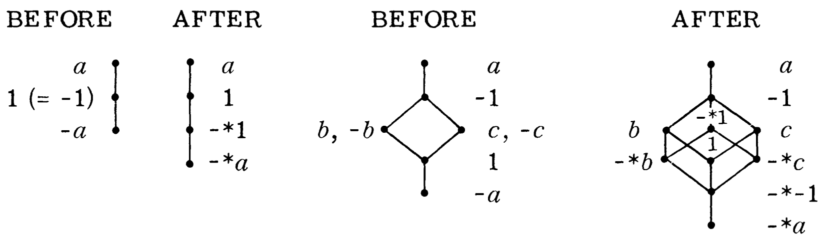
(viii) The function h from D^* to D defined in (ii) is a homomorphism from $\mathfrak{D}^*$ onto $\mathfrak{D}$.

We finish the proof by rejecting the R-invalid formula A in $\mathfrak{D}^*$. Where I is the R-interpretation on which A is not true in $\mathfrak{D}$, let I^* be a function defined

14. The reader interested in further details may consult corresponding sections of [25] and of Dunn, *op. cit.*

recursively on all sentences of RQ thus: if B is atomic or of the form $(x)C$, $I(B) = I^*(B)$; otherwise $I^*(C \rightarrow D) = -^*(I^*(C) \,.^*\, -^*I^*(D))$, $I^*(C \vee D) = I^*(C) \vee^* I^*(D)$, $I^*(\mathrm{f}) = -^*1$, and $I^*(C \,\&\, D) = I^*(C) \wedge^* I^*(D)$, $\wedge^*$ defined on D^* by d0 from $\vee^*$ and $-^*$. Straightforward induction shows for all sentences B of RQ that $h(I^*(A)) = I(A)$. But $1 \not\leq I(A)$ in $\mathfrak{D}$; since h by (viii) is a homomorphism, $1 \not\leq I^*(A)$ in $\mathfrak{D}^*$, since lattice homomorphisms preserve order. This completes the proof of Theorem 2.

We call the technique of stage 2 *splitting* because its effect is to provide for each element of $\mathfrak{D}$ which is both true and false one counterpart in $\mathfrak{D}^*$ which is definitely true and another which is definitely false. We enclose a couple of snapshots of DeMorgan monoids (order from bottom to top, value of . not indicated) to show its effect.



That the rule γ is admissible for R-valid sentences (i.e., at the sentential level) is an easy consequence of Theorem 2. Let A be R-valid. If B is invalid, B is by Theorem 2 false in some normal DeMorgan monoid on an R-interpretation I; by normality $\bar{A} \vee B$ is false on I and hence also is R-invalid. Contraposing and generalizing, if A and $\bar{A} \vee B$ are both R-valid, so also is B. This concludes our remarks about the theory of DeMorgan monoids and the role they play in developing at the sentential level an algebraic semantics for relevant implication.

4. In this section we extend the ideas developed above to full-blooded relevant quantification theory. There are at least two ways to do this, of which we opt for the one which makes it easiest to deliver our promised results. The way we do not choose is to complicate the algebra by providing an operation corresponding to the universal quantifier as the DeMorgan operations correspond to sentential connectives. What we do rather is to leave the algebra as is and constrict instead the notion of what counts as an interpretation.

Let L be the set of sentences of RQ. Let $\mathfrak{D}$ be a DeMorgan monoid and let P be its P-filter (i.e., the set of its true elements). By an RQ-interpretation in $\mathfrak{D}$, we mean a function I defined on L with values in D, subject to conditions (i)-(iv) of the definition of an R-interpretation and (1)-(2) below.

(1) $I((x)A) \in P$ if and only if $I(A(a/x)) \in P$ for all real variables a of L.
(2) If the sentence A of RQ is of one of the kinds A12-A16, $I(A) \in P$.

The effect of (1) is that we opt for a *substitution interpretation* of the quantifiers in the sense discussed in [16], [12], [15], [27] and elsewhere.

That is, we do not provide in our semantics for domains of objects over which our quantifiers are to range, but count instead a general statement true provided that all its instances are true. This is rather more a matter of convenience than of substance, since the adaptation to RQ of the more customary model-theoretic semantics would proceed along similar lines and lead to similar results.[15] (2) reflects the decision announced at the outset; we avoid the technicalities of algebraic analysis of quantification by requiring of an RQ-interpretation simply that it make all logical truths true and that it respect the rules of *modus ponens* and adjunction. This is accomplished as follows. That the rules are respected by an RQ-interpretation I in the sense that the conclusion of a rule is true on I whenever its premisses are is secured because $\mathfrak{D}$ is a DeMorgan monoid; so is the truth on I of *sentences* of the kind A1-A11; (2) takes care of *sentences* A12-A16. What remains to be shown is that closures of arbitrary *formulas* A1-A16 are true on I, and for this we use (1) to argue inductively; the induction is on the number of prefixed universal quantifiers.

Thus an RQ-interpretation I is simply an R-interpretation which shows quantifiers the respect they deserve. So truth on I is as above. And A is RQ-*valid* (*normally* RQ-*valid*) if and only if A is true on all RQ-interpretations in all DeMorgan monoids (normal DeMorgan monoids). We turn now to the principal task of this section, which is to extend to RQ the main results of **3**.

Writing $\vdash_{RQ} A$ if and only if A is a theorem of RQ itself and noting the equivalence of $\vdash_{RQ} A$ and $\vdash_T A$ for all RQ-theories T (i.e., that RQ is the smallest RQ-theory), we observe that semantic consistency remains trivial.

Lemma 2. *Suppose* $\vdash_{RQ} A$. *Then* A *is* RQ-*valid.*

Proof. It suffices to observe that an arbitrary RQ-interpretation I in a DeMorgan monoid $\mathfrak{D}$ makes the axioms of RQ true and that truth on I is preserved under *modus ponens* and adjunction.

Lemma 3. *Suppose* $\vdash_{RQ} A$. *Then* A *is normally* RQ-*valid.*

Proof. Immediate from Lemma 2.

To prove semantic completeness for RQ is more difficult. Our strategy is as follows. Assume that A is a non-theorem of RQ. In the manner of Henkin's completeness proof we construct a rich and prime RQ-theory such that not $\vdash_T A$. Forming the Lindenbaum algebra for T as

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15. Were we interested in strong rather than weak completeness theorems for RQ, some complication of the present semantics would be in order. Cf. [15], p. 233, for an appropriate kind of complication. But whereas what counts classically as weak completeness can be rendered in the relevant logics either as validity or normal validity of just the provable sentences, since the notions coincide in extension, strong completeness either holds or fails relevantly depending upon which among equivalent classical notions one takes it to be. The choice is tedious without being illuminating; see the last sentence of the paper.

before, we obtain a prime DeMorgan monoid $\mathfrak{D}$ and an interpretation I such that A is not true on I; this shows that A is not RQ-valid. We then employ the splitting technique of the proof of Theorem 2 to turn $\mathfrak{D}$ into a normal DeMorgan monoid $\mathfrak{D}^*$ and I into an interpretation I^* on which A is false. So as explained above A remains falsifiable on normal semantic intuitions, from which we derive as a corollary a proof of γ for RQ.

Since the first part of our proof will be deduction-theoretic, we introduce a definition and develop a pair of syntactic facts that will be needed.

D5. $A \supset B =_{df} A \,\&\, \mathrm{t} \rightarrow B$.[16]

Lemma 4. (Deduction theorem for RQ.) *Let T be an* RQ*-theory, S a set of sentences and A a sentence of* RQ *or a linguistic extension thereof. Then*

$$S \cup \{A\} \vdash_T B \text{ if and only if } S \vdash_T A \supset B.$$

Proof. The usual proof of the deduction theorem suffices (as e.g. in [26]), noting that essential formulas involving $\supset$ as defined by D5 are theorems of RQ.

Lemma 5. *Let T be an* RQ*-theory. Let S be a set of sentences and $A(a/x)$ and $B_1, \ldots, B_n$ be sentences of* RQ *or a linguistic extension thereof. Let the real variable a be foreign to S, T, and $B_1, \ldots, B_n$. Then*

(i) $S \vdash_T A(a/x)$ *if and only if* $S \vdash_T (x)A$;
(ii) $S \vdash_T B_1 \vee \ldots \vee B_n \vee A(a/x)$ *if and only if* $S \vdash_T B_1 \vee \ldots \vee B_n \vee (x)A$.

Proof. Ad (i). If $A(a/x)$ belongs to every RQ-theory which contains $S \cup T$, there is a finite sequence $C_1, \ldots, C_p$ of sentences of a linguistic extension of RQ such that each C_i is a logical axiom, a member of $S \cup T$, or a consequence of predecessors by *modus ponens* or adjunction. Rewrite bound variables so that x does not occur in $C_1, \ldots, C_p$ and define a sequence $C_1', \ldots, C_p'$ by letting $C_i' = C_i$ if a does not occur in C_i and letting $C_i' = (x)B$ if C_i is of the form $B(a/x)$, where x occurs free in B. Using the fact that closures of formulas of the form of logical axioms are logical axioms, show by induction that $S \vdash_T C_i'$ for each i from 1 to p; in particular, since $C_p' = (x)A$, $S \vdash_T (x)A$, which was to be shown. The converse is trivial.

Ad (ii). Use (i) and the confinement axiom A15. We remark that (ii) of Lemma 5, and hence the confinement axiom, is an essential ingredient in our proof that RQ has a normal semantics and hence that γ holds for it. This is not surprising in view of the importance of the distribution axiom A9, of which A15 is a kind of infinite analogue, for the proof of γ in the sentential case, on which we remarked in [20]. Since the presence of the intuitionistically invalid A15 causes the translation of [18] to become inexact at the level of predicate logic, however, it would be interesting to

16. $\supset$ is an enthymematic implication in the sense of [18].

consider dropping A15 in order to preserve that translation. Question—does γ hold or fail if A15 is dropped?

Continuing with A15 (which is by the way used in the Anderson-Belnap proof that the arrow-free fragment of RQ is the classical first-order functional calculus, *cf.* [5], [28]), we are ready for a theorem.

Theorem 3. *Let T be an* RQ-*theory, and let A be a sentence in the vocabulary of T such that not* $\vdash_T A$. *Then there is an* RQ-*theory T' such that* (i) $T \subseteq T'$, (ii) *T' is prime,* (iii) *T' is rich, and* (iv) *not* $\vdash_{T'} A$.

Observation. Theorem 3 is a generalization of the classical theorem which permits us to extend any consistent theory to a maximally consistent theory. But whereas classically this result implies Theorem 3 directly, the absence in RQ of implicational paradoxes blocks the classical argument. Accordingly we argue directly for Theorem 3, noting that our eventual theory T' is not maximally consistent but maximally A-free; to get a theory which is both consistent and maximally A-free requires not only Theorem 3 but also application of the splitting technique; cf. the end of the paper.

Proof. We are going to build a prime and rich extension T' of T which fails to contain A. We shall do so by building an infinite sequence of infinite sequences of ever larger theories; the union of which will be the desired T'. Again, generalization of simple classical techniques for assuring richness is blocked; what this involves in practice is that at each stage i of the construction we shall define a set R_i of *rejected sentences* no *disjunction* of which will be permitted in any of the theories we build.

We accordingly define, for each natural number i from 0 on, a set V_i of *real variables,* the set L_i of *sentences* of a linguistic extension of RQ built up from V_i by the formation rules of 2, an RQ-*theory* T_i, and a *rejection set* R_i. Since L_i will be in each case denumerable we assume it ordered by the positive integers and call its ordering *alphabetical.* Using the alphabetical order of L_i we define for fixed i and for every natural number j an *instantial variable* a_{ij}, a *critical sentence* A_{ij} and an *auxiliary* RQ-*theory* T_{ij}.

Let N be the set of natural numbers. We begin by setting $V_0 = V$, where V is the set of real variables which occur in sentences of T, $T_0 = T$, and $R_0 = \{A\}$. The instantial variables a_{ij} we require to be distinct from each other and from all members of V_0, and we define V_{i+1} to be $V_i \cup \bigcup_{j \in N} \{a_{ij}\}$. This determines V_i, and hence L_i, for each natural number i.

T_{i+1} is defined using the auxiliary RQ-theories T_{ij}—specifically, $T_{i+1} = \bigcup_{j \in N} T_{ij}$. The critical sentence A_{ij} is the $(j+1)$th sentence of L_i in alphabetical order. For each L_i and subset S of L_i, let $[S, L_i]$ be the smallest RQ-theory H such that $S \subseteq H \subseteq L_i$. We then define the auxiliary RQ-theories T_{ij} inductively as follows. For fixed i, $T_{i0} = [T_i, L_i]$. Thus except for the case $i = 0$, T_{i0} results from T_i by increasing the vocabulary and closing T_i under logical consequence in the expanded vocabulary; clearly T_{i0} is an *inessential extension* of T_i in the sense $T_{i0} \cap L_{i-1} = T_i$. Supposing T_{ij} defined, we define $T_{i,j+1}$ by cases as follows:

(i) If for some disjunction B of members of R_i, $\{A_{ij}\} \vdash B$ in T_{ij}, then $T_{i,j+1} = T_{ij}$;
(ii) Otherwise $T_{i,j+1} = T_{ij} \cup \{A_{ij}\}$.

We turn to the important rejection sets R_i. The central idea is that whenever at any stage in our theory-building we must leave out a sentence of the form $(x)B$, we want to make sure that at some point we also leave out an instance $B(a/x)$. Accordingly define the *i-instance*, for fixed i, of a critical sentence A_{ij} of the form $(x)B$ as $B(a_{ij}/x)$, where x is free in B. Call a sentence of L_{i-1} *generally i-untenable* provided that it is of the form $(x)B$ and that it does not belong to T_i. For each i from 1 on, let Q_i be the set of *i-1-instances* of generally i-untenable sentences. Finally, let $R_{i+1} = R_i \cup Q_{i+1}$, for each i from 0 on.

Let $L' = \bigcup_{i\in N} L_i$, $T' = \bigcup_{i\in N} T_i$, $R' = \bigcup_{i\in N} R_i$. We finish the proof of Theorem 3 by showing that T' is indeed an RQ-theory and that conditions (i)-(iv) in the statement of the theorem hold.

T_0 is an RQ-theory, and for each i from 1 on T_i is the union of a chain of RQ-theories; it is easy then to prove that each T_i is an RQ-theory, and, repeating the argument, that T' is an RQ-theory.

We observe next that for each sentence $B \in R'$, $B \notin T'$. Suppose for *reductio* that there exists a $B \in R' \cap T'$. Then there are least i, j such that $B \in R_i \cap T_j$; clearly $R_i \cap T_i = \emptyset$, so $i < j$. So there is a least integer k such that $B \in R_i \cap T_{j-1,k}$, $i \leq j - 1$. We distinguish two cases.

Case 1. $k = 0$.

1.1. $i = j - 1$. $i > 0$, since $T_0 = T_{00}$. Since B first shows up in R_i, by the leastness of i, B is the instance $C(a/x)$ of a generally i-untenable sentence $(x)C$, where a is foreign to L_{i-1}. Since $\vdash C(a/x)$ in T_{i0}, apply Lemma 5 to show $\vdash (x)C$ in T_{i0}. But as remarked T_{i0} is an inessential extension of T_i; so $(x)C \in T_i$, contradicting its general i-untenability.

1.2. $i < j - 1$. But then B is in the vocabulary of T_{j-1}; since $T_{j-1,0}$ is an inessential extension of T_{j-1}, $B \in T_{j-1}$, contradicting the leastness of j.

Case 2. $k > 0$. By (i) under the recursive definition of the T_{il}, $T_{j-1,k} = T_{j-1,k-1}$ if B (as a rejected sentence of R_{j-1}) is derivable from the critical sentence at this stage in the construction, which contradicts the leastness of k. The cases being exhaustive and leading uniformly to contradiction, we conclude that $R' \cap T' = \emptyset$.

In particular, this shows that since $A \in R_0 \subseteq R'$, $A \notin T'$, proving (iv) of the theorem; moreover $T = T_0 \subseteq T'$, proving (i). We next prove (ii) by showing T' prime.

Since T' is the union of the T_i, clearly it will be prime if each of the T_i are prime, for each i from 1 on. (Since $T_0 \subseteq T_1$, we ignore it.) Let i be the least positive integer such that T_i is not prime. Then $\vdash B \vee C$ but neither $\vdash B$ nor $\vdash C$ in T_i. Since both B and C are in L_{i-1}, each is a critical sentence in the construction of T_i; by definition of the $T_{i-1,j}$, we conclude that there disjunctions D and E of members of R_{i-1} such that D is derivable from B and E is derivable from C in T_i. One sees easily, using the disjunction and distribution axioms and the deduction theorem, that $B \vee C \vdash D \vee E$ in T_i. But then since $B \vee C \epsilon T_i, D \vee E \epsilon T_i$.

$D \vee E$ is itself a disjunction of rejected sentences in R_{i-1}, since both D and E are. Let $B_1, \ldots, B_j$ be the members of this disjunction which belong to R_{i-2}, if any, and let $C_1, \ldots, C_k$ be the rest. T_i is itself a union of the $T_{i-1,j}$, and by the instructions for building the latter the bad disjunction must belong to $T_{i-1,0}$, since otherwise $T_{i-1,j}$ would in violation of orders *newly* contain a bad disjunction.

$i \neq 1$, since only A belongs to R_0; by idempotence disjunction of A's reduce to A; but we have proved $A \epsilon T'$. Accordingly each of $C_1, \ldots, C_k$ is the instance of an i-1-untenable formula, say $(x)D_1, \ldots, (x)D_k$ respectively. So since

$$\vdash B_1 \vee \ldots \vee B_j \vee C_1 \vee \ldots \vee C_k \textit{ in } T_{i-1,0},$$

where each C_i contains an instantial variable distinct from the others and from all variables of L_{i-1}, we conclude by repeated application of Lemma 5 and elementary properties of disjunction,

$$\vdash B_1 \vee \ldots \vee B_j \vee (x)D_1 \vee \ldots \vee (x)D_k \textit{ in } T_{i-1}.$$

Since $i > 1$ was chosen as the least natural number such that T_i is not prime, T_{i-1} is prime. Accordingly one of $B_1, \ldots, B_j, (x)D_1, \ldots, (x)D_k$ belongs to T_{i-1}. But this is not the case: the B_p all belong to R' and thus as we have observed do not belong to T' and *a fortiori* do not belong to T_{i-1}; neither are the $(x)D_q$, being generally i-1-untenable, in T_{i-1}. This refutes the assumption that there is a T_i, $i > 0$, which is not prime, and establishes (ii)—T' is prime.

We finish the proof of Theorem 3 by showing (iii) T' is rich. In view of A12 and the fact that T' is an RQ-theory, it is plain that for every real variable a, $\vdash_{T'} B(a/x)$ whenever $\vdash_{T'} (x)B$. On the other hand, suppose it is not the case that $\vdash_{T'} (x)B$. Then for some i, $(x)B$ is generally i-untenable; for that i, there is for some real variable a an instance $B(a/x) \epsilon R_i \subseteq R'$; since $R' \cap T' = \emptyset$, not $\vdash_{T'} B(a/x)$. So $\vdash_{T'} (x)B$ if and only if $\vdash_{T'} B(a/x)$ for every real variable a of L'—i.e., T' is rich, which completes the proof.

Here is the next step of the program set out in the observation made above.

Lemma 6. *Suppose A is* RQ-*valid. Then* $\vdash_{\mathrm{RQ}} A$.

Proof. Suppose it is not the case that $\vdash_{\mathrm{RQ}} A$. By Theorem 3, for some RQ-theory T', not $\vdash_{T'} A$, where T' is prime and rich. Let $\mathfrak{L}'^*$ be the Lindenbaum algebra of T', defined as for Theorem 1. Since the number of real variables of T' is denumerable and only a finite number of them occur in A, clearly we may assume without loss of generality that the variables and hence the sentences of the language of T' are just those of RQ. Define a function I from the set of sentences of RQ to $\mathfrak{L}'^*$ by setting $I(B) = B^*$ for each sentence B, where $B^* = \{C: \vdash_{T'} B \leftrightarrow C\}$. We now show that I is an RQ-interpretation on which A is false in the prime DeMorgan monoid $\mathfrak{L}'^*$, proving that A is not RQ-valid.

Since T' is an RQ-theory and *a fortiori* an R-theory, by the proof of Theorem 1, $\mathfrak{T}'^*$ is a DeMorgan monoid in which A is not true on I; moreover the primeness of $\mathfrak{T}'^*$ follows readily from that of T'. Trivially I is an R-interpretation; what remains to be shown is that I satisfies (1) and (2) definition making it an RQ-interpretation.

Let P' be the P-filter of $\mathfrak{T}'^*$. We observe that $\vdash_{T'} B$ if and only if $1 = t^* \leq B^*$. Since T' is rich, $I((x)B)$ accordingly belongs to P' if and only if $I(B(a/x)) \in P'$ for all real variables a of RQ; so I satisfies (1); furthermore since all RQ-axioms are in T', their values under I are in P', satisfying (2). So if not $\vdash_{RQ} A$, A is not RQ-valid. Contraposing we have the lemma, ending its proof. We now apply the splitting technique to get normality.

Lemma 7. *Suppose A is normally* RQ-*valid. Then* $\vdash_{RQ} A$.

Proof. Suppose it is not the case that $\vdash_{RQ} A$. By the proof of Lemma 6, there is a prime DeMorgan monoid $\mathfrak{D}$ and an RQ-interpretation I such that A is not true on I. Split $\mathfrak{D}$ as in stage 2 of the proof of Theorem 2 to obtain a normal DeMorgan monoid $\mathfrak{D}^* = \langle D^*, .^*, -^*, \vee^* \rangle$, where $\mathfrak{D} = \langle D, ., -, \vee \rangle$. Define a function from the set of sentences of RQ to D^* by setting $I^*(B) = I(B)$ for each atomic sentence B; $I^*(C \rightarrow D) = I^*(C) \Rightarrow^* I^*(D)$, $I^*(C \mathbin{\&} D) = I^*(C) \wedge^* I^*(D)$, $I^*(C \vee D) = I^*(C) \vee^* I^*(D)$, and $I^*(\mathrm{f}) = -^* 1$, as before. $I^*(B(a/x))$ having been defined for all real variables a of RQ, define $I^*((x)B)$ as follows:

(i) N, $-N$, g, and h being as before if $I((x)B) \in N$ and $I^*(B(a/x)) \in -N$ for some real variable a of RQ, $I^*((x)B) = g(I((x)B))$.
(ii) Otherwise $I^*((x)B) = I((x)B)$.

I^* having been defined on all sentences of RQ, we must now show it an RQ-interpretation. Since by the proof of Theorem 2, $\mathfrak{D}^*$ is a normal DeMorgan monoid and since I^* is clearly an R-interpretation in $\mathfrak{D}^*$, again all that remains to be shown is that (1) and (2) of the definition of RQ-interpretation hold.

To prove (2), we observe first that each of A12-A16 is of the form $B \rightarrow C$, that $I^*(B \rightarrow C) = I^*(B) \Rightarrow^* I^*(C)$, and that $I^*(B) \Rightarrow^* I^*(C)$ is in the P-filter of $\mathfrak{D}^*$ if and only if $I^*(B) \leq^* I^*(C)$, by t3 of Lemma 1.

Let h be as in (i). By (viii) of Theorem 2, stage 2, h is a homomorphism from D^* onto D, and it is again evident that $h(I^*(B)) = I(B)$ for all sentences B of RQ. So by definition of h, $I^*(B) = I(B)$, or $I^*(B) \in -N$ and $I^*(B) = g(I(B))$. Further checking of definitions establishes that the order relation on $\mathfrak{D}^*$ is an extension of that on $\mathfrak{D}$ and that (cf. [20], p. 468)

(a) If $a \in N$ and $b \in -N$, $a \not\leq^* b$;
(b) Otherwise $a \leq^* b$ if and only if $h(a) \leq h(b)$.

(The authors are themselves committed to the principle, "You check your definitions and I'll check mine." The reader with similar prejudices is invited to look again at the pictures above which will at least make it plausible to him that we have done our part of the job.)

Where B is a formula of RQ with at most one free apparent variable x, an *instance of* B is any sentence $B(a/x)$, where a is a real variable of RQ. (Note that our definitions yield, where B is itself a sentence, B itself as sole instance.) We are now going to construct for $I((x)B)$ an analogue to the *truth-partition tables* of [20]; we shall then use these tables to give us essential information in verifying A12-A16 on I^*, given the remarks of the last paragraph. As in [20], we partition D into sets T, N, and F, where an element a of D belongs (a) to T if and only if $1 \leq a$ and $1 \not\leq -a$, (b) to N if and only if $1 \leq a$ and $1 \leq -a$, and (c) to F if and only if $1 \not\leq a$ and $1 \leq -a$. ((a)-(c) are exhaustive by the primeness of $\mathfrak{D}$ and excluded middle, since because $1 \leq a \vee -a$, $1 \leq a$ or $1 \leq -a$. We note that $P = T \cup N$, where P is the P-filter of $\mathfrak{D}$, and we note for future reference that P is also the P-filter of $\mathfrak{D}^*$.)

The value under I of a sentence $(x)B$ of RQ is partially determined by the value under I as indicated in the following table:

Distribution of instances of B under I	Value of $(x)B$ under I is in
(1) All instances are in T	$T \cup N$
(2) Some instance is in N, none are in F	N
(3) Some instance is in F	F

Justification of (1)-(3) is as follows: since I is an RQ-interpretation, $I((x)B) \in P$ if and only if I carries each instance of B into a member of P; This justifies (1), (3), and partially justifies (2). To complete the justification of (2), suppose for *reductio* that B' is an instance of B which belongs to N under I but that $I((x)B) \in T$. But then, since $B' \in N$, $1 \leq -I(B') \leq -I((x)B)$ (since A12 is valid on I and applying Lemma 1), which contradicts $I((x)B) \in T$.

We note that after splitting the resultant monoid $\mathfrak{D}^*$ is partitioned into T, N, $-N$, and F, where T, N, and F are as *defined for* D and $-N$ is as above. (The reader may test these concepts by reference once more to the sample monoids pictured above; in both cases $a \in T$, $-a = -^*a \in F$, $-1 \in N$, and $-^*1 \in -N$.)

We can now construct a corresponding table for I^*.

Distribution of instances of B under I^*	Value of $(x)B$ under I^* is in
(1) All instances are in T	$T \cup N$
(2) Some instance is in N, none are in $F \cup -N$	N
(2′) Some instance is in $-N$, none are in F	$-N$
(3) Some instance is in F	F

Justification of the new table is immediate from the old table and the definition of $I^*((x)B)$; note in particular that if all instances $B(a/x)$ of B are in T under I^* they are also all in T under I, on our observation that $I(B(a/x)) = h(I^*(B(a/x)))$.

We return to our examination of the fate of A12-A16 under I^*. The observations made at the beginning of this proof reduce this to the problem of showing, for each axiom $B \rightarrow C$, that the case $I^*(B) \in N$, $I^*(C) \in -N$ does not arise; for otherwise, since I is on assumption an RQ-interpretation and hence makes all RQ-axioms turn out true, by (b) above, since $I(B) = h(I^*(B)) \leq h(I^*(C)) = I(C)$, $I^*(B) \leq^* I^*(C)$ and so $1 \leq^* I^*(B \rightarrow C)$. Accordingly we may finish the proof that each sentence A12-A16 is true on I^* in $\mathfrak{D}^*$ by assuming for *reductio* that its antecedent is in N and its consequent in $-N$ under I^* and deriving in each case a contradiction. (Define $\Rightarrow^*$ on $\mathfrak{D}^*$ by $a \Rightarrow^* b = -^*(a \;.^*\; -^*b)$.)

Ad A12. Assume $I^*((x)A) \in N$ but $I^*(A') \in -N$ for an instance of A. This contradicts (2′) of the table if no instance A'' of A is in F; otherwise it contradicts (3).

Ad A13. Suppose $I^*((x)(A \rightarrow B)) \in N$, $I^*(((x)A \rightarrow (x)B)) \in -N$. Then $I^*((x)A) \Rightarrow^* I^*((x)B) \in -N$, whence by properties of h and Lemma 1, $I((x)A) \leq I((x)B)$ but $I^*((x)A) \not\leq^* I^*((x)B)$; in view of (a) and (b) above, $I^*((x)A) \in N$ but $I^*((x)B) \in -N$. This yields the conclusion that all instances of A and $A \rightarrow B$ respectively are true on I^* but that some instance of B is not true on I^*, consulting the table; since the R-interpretation I^* respects *modus ponens*, this is a contradiction.

Ad A14. If x is not free in A, $1 \leq I(A) \Leftrightarrow I((x)A)$, whence by Lemma 1, $I(A) = I((x)A)$; since A is by definition the sole instance of A when x is not free in A, we see that $I^*(A) = I^*((x)A)$ by (i)-(ii) above, which makes untenable the hypothesis that $I^*(A) \in N$ and $I^*((x)A) \in -N$.

Ad A15. Suppose $I^*((x)(A \vee B)) \in N$, $I^*(A \vee (x)B) \in -N$, where x is not free in A. By the table, all instances of $A \vee B$ are true on I^*; this means that either the sentence A is true on I^* or else for each instance B' of B, B' is true on I^*; the reason is that since $\mathfrak{D}$ is a prime DeMorgan monoid its P-filter, which coincides with the P-filter of $\mathfrak{D}^*$, is prime. If A is true on I^*, by A6 and the respect which I^* shows for *modus ponens*, so is $A \vee (x)B$, a contradiction; accordingly we assume instead that each instance B' of B is true on I^*. But then by the table $(x)B$ is true on I^* also, whence so again is $A \vee (x)B$ true on I^*, which is still a contradiction. This exhausts the cases and establishes the truth on I^* of A15.

Ad A16. Suppose $I^*((x)A \;\&\; (x)B) \in N$ but $I^*((x)(A \;\&\; B)) \in -N$. By the table, there is an instance $A' \;\&\; B'$ of $A \;\&\; B$ which is not true on I^*. It follows by the filterhood of P that at least one of A', B' is not true on I^*, whence by the table it follows that at least one of $(x)A$, $(x)B$ is not true on I^*, whence by the filterhood of P once more it follows that $(x)A \;\&\; (x)B$ is not true on I^*, which is the desired contradiction.

We have now proved that if a sentence of RQ is of the form A12-A16, it is true on I^*. In constructing our tables we have by the way proved (1) of the definition of RQ-interpretion as well, since inspection shows that if each instance of a formula B of RQ with one free variable is true on I^*, so also is $(x)B$ true on I^*, and conversely. Given this information, and knowing that all sentences A1-A16 are true on I^*, it is clear that all closures of *formulas* of the form A1-A16 are true on I^*, and, since I^* respects the rules of *modus ponens* and adjunction, that all theorems of RQ are true on the RQ-interpretation I^*.

We conclude the proof of the lemma by observing, for our selected non-theorem A of RQ, that A is not true, and hence false, on I^* in the normal DeMorgan monoid $\mathfrak{D}^*$. For suppose otherwise. Then $1 \leq^* I^*(A)$. But then since h is a homomorphism from $\mathfrak{D}^*$ onto $\mathfrak{D}$, $1 = h(1) \leq h(I^*(A)) = I(A)$, contradicting the fact that A is not true on I. Accordingly A is not true on I^*. Contraposing and generalizing, if A is true on all RQ-interpretations in normal DeMorgan monoids, A is a theorem of RQ, which was to be proved.

We gather up the lemmas in our principal theorem.

Theorem 4. *The following conditions are equivalent, for every sentence A of* RQ.
(1) $\vdash_{RQ} A$;
(2) *A is* RQ-*valid;*
(3) *A is normally* RQ-*valid.*

Proof. The equivalence of (1) and (2) is the content of Lemmas 2 and 6, and the equivalence of (1) and (3) is the content of Lemmas 3 and 7.

We have hinted that Theorem 3 may be viewed for relevant quantification theory as a generalization of the classical theorem that every consistent theory may be extended to a theory rich and maximally consistent. For the relevant logics this has an interesting and non-trivial converse.

Theorem 5. *Let T be a rich, prime* RQ-*theory. Then T has a completely normal sub-theory T'.*

Proof. Form the Lindenbaum algebra of T as in the proof of Lemma 6, and let I assign to each sentence A the corresponding equivalence class in the algebra. Split the algebra and define I^* in the manner of Lemma 7, observing that the set of sentences of RQ that are true on I^* form a completely normal RQ-theory and that each of them is also true on I (though not conversely), completing the proof.

Theorem 5 has a certain philosophical interest, not unrelated to the *raison d'être* of the relevant logics. For we are often told that scientists, when they run into theories inconsistent either in themselves or in the wider context of external fact, seek to reshape those theories in order to preserve as much as possible. We agree here, as we always do, with Professor Quine—minimum mutilation is the best policy. (Well, almost always.) Classically, however, minimum mutilation makes no sense—there is only one inconsistent theory and it asserts everything whatsoever, at least to the limits of its vocabulary; the best one can do is to cut such a theory down to one both consistent and complete, and any exhaustive choice between atomic sentences and their negations will constitute a minimum mutilation. Obviously any purported repair of an inconsistent arithmetic which lets us have our choice between $0 = 4$ and $0 \neq 4$ leaves something to be desired, though it is difficult on classical grounds to tell exactly what.

On the other hand, for logics like RQ which admit non-trivial inconsistent theories, minimum mutilation does make sense; if a theory has carelessly asserted both a sentence and its negation, take one out; on the other hand, where that theory has distinguished between a sentence and its negation by asserting one but not the other, to mutilate minimally is surely to preserve that distinction. The limiting case, corresponding to the classical inconsistent theory, is the theory which is both rich and prime; the content of Theorem 5 is that we can always preserve the distinction between truths and falsehoods which such a theory makes while improving it to distinguish always between truths and falsehoods. Thus, γ holds for RQ.

Theorem 6. *Suppose* $\vdash_{RQ} A$ *and* $\vdash_{RQ} \overline{A} \vee B$. *Then* $\vdash_{RQ} B$.

Proof. Assume the hypothesis but not $\vdash_{RQ} B$. There is then by Theorem 4 a normal DeMorgan monoid $\mathfrak{D}$ and an RQ-interpretation I which makes each of $\overline{B}, A, \overline{A} \vee B$ true; but the conjunction of these three sentences, being the negation of a purely truth-functional tautology, relevantly implies the sentential constant f, so that in $\mathfrak{D}$, $1 \leq I(\mathrm{f}) = -1$, contradicting the consistency and hence the normality of $\mathfrak{D}$, and ending the proof of γ.

5. Back in the beginning of the paper we hinted that its results might prove useful; in conclusion we broaden the hint. For though we have found it a virtue in the relevant logics that they admit non-trivial inconsistent theories, we see in them also a vehicle for consistency proofs. The essential idea hinges on γ.

Hilbert, it will be recalled, sought to prove the consistency of a formalized arithmetic by establishing the existence of an underivable sentence; because of the classical looseness about what counts as a logical consequence of a contradiction, any old sentence, say 527 = 619, would have done; indeed the sentential variable F^0, if present in the language, would have done.

Hilbert's program had its nose tweaked by Gödel, as we all admiringly recall, from which it might be concluded that it is difficult for theories of minimal complexity, like arithmetic, to establish that there are underivable sentences. So it is, so long as our theory of entailment is classical. But it is not hard, on as small a modification of that notion as RQ represents (and, despite the fuss pro and con sometimes heard among the brotherhood, RQ is the product of minimum mutilation—as neutrals in the contest between Anderson and Belnap and their critics, we can testify that the only classical theorems missing from RQ are those which Whitehead and Russell threw into *Principia Mathematica* to provide a welcome comic relief), to show that any RQ-theory may be trivially extended so that F^0 is underivable.

In fact, let **3** be the chain pictured at the end of section **3** whose members are a, 1, and $-a$, letting 1 be the identity and completing the definition of . by setting $(-a)^2 = a.-a = -a$ and $a^2 = a$.[17] Let T be any

17. This is the 3-point Sugihara matrix of [11].

RQ-theory, as inconsistent as you please, such that F^0 does not occur in the vocabulary of T. Expand the vocabulary to include F^0, and let T' be the smallest RQ-theory in the expanded vocabulary which contains T.

Observation. T' is absolutely consistent; i.e., there is a sentence in the vocabulary of T', namely F^0, such that $F^0 \notin T'$.

Proof. Let I be that RQ-interpretation in **3** whose value for each atomic sentence B in the vocabulary of T is 1 and such that $I(F^0) = -a$. Clearly the set of sentences true on I is an RQ-theory T'' which fails to contain F^0 but which contains every sentence in the vocabulary of T and hence in particular every sentence in T. By definition $T \subset T' \subseteq T''$, ending the proof.

Our observation is, among other things, a quick proof (not that one is needed) that the classical predicate calculus is consistent. For as noted above, RQ exactly contains the predicate calculus in &, v, -, and the quantifiers. Entering a definition of the material conditional $\supseteq$, not to be confused with the genuine though flabby conditional $\supset$ of D5, we have

D6. $A \supseteq B =_{df} \overline{A} \vee B$.

The definition yields the following amusing theorem scheme.[18]

T1. $\vdash_{RQ} \mathrm{f} \supseteq A$.

By T1 and the admissibility in RQ of *modus ponens* for $\supseteq$, if RQ were inconsistent, F^0 would be a theorem; furthermore, since f is classical, and since RQ contains the predicate calculus, only if f is a theorem of RQ is f a classical theorem. But by our observation F^0 is not a theorem of RQ; accordingly classical logic is consistent.

The above argument was only for practice; the appealing thing about it is that it looks general. Take any axiomatic theory S formulated in first-order classical logic; we may assume *modus ponens* for $\supseteq$ as sole rule. If the consistency of S is in doubt, try the following recipe.

(1) Reformulate the axioms of S in RQ, putting $\rightarrow$ (or perhaps $\supset$) for those occurrences of $\supseteq$ which are to be taken as genuine conditionals.

(2) Close the reformulated set of axioms under the rules of RQ, getting an RQ-theory T.

(3) Prove in T the original axioms of S.

(4) Find a sentence in the vocabulary of T underivable in T. If necessary, add a new sentential variable F^0 to the vocabulary of T. By our observation, F^0 will be underivable in the inessential extension T' of T.

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18. For Anderson, who in [1] uses 'f' for '$(p)p$', even f $\rightarrow A$ is not amusing. In view of our own past usage, we wish he had used 'F' for his notion, which stands for the falsest proposition around—the one which entails everything; building on earlier remarks, we note that our 'f' stands intuitively for the *disjunction* of all false propositions—i.e., the one entailed by but not necessarily entailing an arbitrary falsehood—the falsehood most nearly true, if you please, and thus deserving to be uttered in a softer voice and written in a lower case.

(5) Prove that *modus ponens* for $\supseteq$ is an admissible rule of T (or of T' if we did not find an underivable sentence in T under (4)).

If we can carry out each of steps (1)-(5), we shall have our proof that the classical first-order theory S is consistent. Our sample proof of the consistency of the pure predicate calculus illuminates the suggested strategy. The step (1) of reformulation is essentially encompassed in the logical axioms of RQ themselves in this case; since *modus ponens* for $\supseteq$ holds not as a primitive rule of RQ but, as Anderson puts it, as a kind of lucky accident, note that it would not have done to have weakened arrows to underlined horseshoes indiscriminately in A1-A16, despite the fact that with γ and D6 these form a more than sufficient set for the classical predicate calculus; the point is that to prove γ admissible in RQ the stronger forms of the logical axioms are required.

The T of step (2) is of course in our sample case just RQ itself. We sidestepped (3) in the sample, but it would have sufficed to have proved in RQ any set of classical axioms in the quantifier and truth-functional connectives which, given γ, are classically complete. We used the method of inessential extension for illustrative purposes under (4); obviously the method of our observation is directly applicable to RQ, so that in this case the inessential extension is inessential indeed.

(5) is of course γ, and the main point. Once it is shown admissible, all the theorems of S are, given (3), immediately theorems on direct translation of T. Furthermore the consistency of T, and hence of S, is immediately assured by the presence under (4) of a single underivable formula, just as Hilbert taught us.

Since γ is the main point, the question of proving its admissibility for an arbitrary consistent RQ-theory T naturally arises. Unfortunately as we noted in [20] it has a disappointing answer—there are consistent RQ-theories, including reasonably natural ones, for which γ is nonetheless inadmissible; take, e.g., the RQ-theory with $f \vee F^0$ as sole non-logical axiom.

The fact that γ is not admissible for arbitrary RQ-theories is not, however, to be read as a counsel of despair. For first-order RQ-theories, at least, Theorem 3 has already accomplished half the job; we can extend arbitrary RQ-theories to theories both rich and prime, preserving non-theoremhood for some selected sentence. This yields automatically a prime DeMorgan monoid $\mathfrak{D}$ and an RQ-interpretation I such that A is false on I; what remains to apply the technique of the present paper is to split $\mathfrak{D}$, which we know how to do, and to use I to define a new RQ-interpretation I^* in the split monoid $\mathfrak{D}^*$, which is where the difficulty lies. Even here, however, the central task can be located; all the present proof requires to validate logical axioms is that $I(B) = h(I^*(B))$ for all atomic sentences B; putting that condition on I^* reduces the proof of γ for an arbitrary RQ-theory T to that of defining I^* in such a way that *non-logical* axioms of T true on I in $\mathfrak{D}$ remain true on I^* in $\mathfrak{D}^*$.

Since no such technique will work in general, we leave it to the interested reader, and to ourselves, so fortuitously to apply it that it works in particular cases. We conclude simply by noting the affinity between this suggested technique and that actually applied by Gentzen in proving the consistency of arithmetic; cf. [3]. For Gentzen so recasts formalized number theory that its absolute consistency is trivial, inasmuch as the empty sequent is trivially non-derivable in a Gentzen consecution calculus. The work then comes in proving the cut theorem, which, if the reader thinks about it, is for classical theories simply γ in peculiar notation. Though the techniques involved here would seem to apply essentially no principles stronger than those which deliver the weak completeness of the classical first-order functional calculus, and amount in Theorem 4 accordingly to a weak completeness result for RQ, Gödel's theorem encourages us to think that the proof of the consistency of real life theories will require yet more sophisticated twists in the proof of γ. But for the moment, dear reader, we have been through enough together; sufficient unto the day is the evil thereof.[19]

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