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THE GEOMETRY OF SOLIDS IN HILBERT SPACES

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In [6] A. Tarski presents a definition of concentric solids in Leśniewski's deductive system of Mereology [1, 2, 5]. If solid is interpreted as sphere in Euclidean geometry, Tarski showed that the set of equivalence classes of spheres determined by the definition of concentric corresponds to the set of points of the Euclidean space. Thus Mereology could be used as a foundation for Euclidean geometry. In this paper* we shall show that the above correspondence still holds in the case where the space is taken to be a real separable Hilbert space of dimension ≥ 2.

The definition of concentric presented below is based on the primitive relation of Mereology, "A is part of B," where A and B are restricted to be instances of the name solid. In a Hilbert space $\mathfrak{H}$ this relation will be interpreted as the sphere A is a subset of the sphere B where a sphere is a set of the form $\{\mathbf{x}/\|\mathbf{x} - \mathbf{a}\| < r\}$ with $\mathbf{a} \in \mathfrak{H}$, $r > 0$, and $\|\ \|$ denoting the norm of $\mathfrak{H}$.

We begin with the development of Tarski's definition of concentric. We note that the definitions below will be formulated using only the primitive relation of Mereology and definitions already given.

Definition 1 *A is disjoint from B* iff A and B are solids and whenever X is part of A then X is not part of B. (We note that for open balls in a Banach space this definition is equivalent to the statement that A and B are disjoint sets.)

Definition 2 *A is externally tangent to B* iff (i) A is disjoint from B and (ii) if A is part of X and is part of Y with B disjoint from X and disjoint from Y then X is part of Y or Y is part of X.

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Definition 3 *A is internally tangent to B* iff (i) A is part of B and (ii) if A is part of X and A is part of Y and X is part of B and Y is part of B then X is part of Y or Y is part of X.

Definition 4 *A and B are externally diametrical to C* iff (i) A and B are externally tangent to C and (ii) if A and B are part of X and Y respectively and X is disjoint from C and Y is disjoint from C then X is disjoint from Y.

Definition 5 *A and B are internally diametrical to C* iff (i) A and B are internally tangent to C and (ii) if X and Y are disjoint from C and A is externally tangent to X and B is externally tangent to Y then X is disjoint from Y.

Definition 6 *A and B are concentric* iff (i) A is identical to B or (ii) A is part of B and whenever X and Y are such that both are externally diametrical to A and each is internally tangent to B then X and Y are internally diametrical to B or (iii) interchange A and B in (ii).

Theorem *If $\mathfrak{H}$ is a real separable Hilbert space of dimension ≥ 2 then the set of points of $\mathfrak{H}$ can be mapped bijectively to the set of equivalence classes of concentric spheres determined by* Definition 6.

The proof of this theorem is contained in the following lemmas in which we shall characterize the definitions above in terms of statements about points in $\mathfrak{H}$. We note that the norm of $\mathfrak{H}$ is denoted by $\|\ \|$ and the inner product is denoted by $(\ ,\ )$. Further we shall use the following convention: A, B, C, X, Y will denote spheres with centers $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{x}$, and $\mathbf{y}$ and with radii r_1, r_2, r_3, r, and s respectively.

Lemma 1 *Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{p}$ be points of $\mathfrak{H}$. Then* $\|\mathbf{v}-\mathbf{u}\| = \|\mathbf{u}-\mathbf{p}\| + \|\mathbf{p}-\mathbf{v}\|$ *iff there exists* $0 < t < 1$ *such that* $\mathbf{p} = \mathbf{u} + t(\mathbf{v}-\mathbf{u})$.

Proof: This lemma is a consequence of the Cauchy-Schwarz inequality for Hilbert Spaces [3].

Lemma 2 *If A and B are externally tangent then there is a unique $\mathbf{p} \in \mathfrak{H}$ with* $\|\mathbf{p}-\mathbf{a}\| = r_1$ *and* $\|\mathbf{p}-\mathbf{b}\| = r_2$ ($\mathbf{p}$ *is the unique point of tangency between A and B*).

Proof: A externally tangent to B implies A and B are disjoint which implies $\|\mathbf{b}-\mathbf{a}\| > r_1$. So let $t = r_1/\|\mathbf{b}-\mathbf{a}\|$ and $\mathbf{p} = \mathbf{a} + t(\mathbf{b}-\mathbf{a})$. It is clear that $0 < t < 1$. If $\|\mathbf{p}-\mathbf{b}\| < r_2$ then we choose t' such that $t'\|\mathbf{a}-\mathbf{b}\| < r_2 - \|\mathbf{p}-\mathbf{b}\|$ and $0 < t' < t$. Then if we let $\mathbf{q} = \mathbf{a} + (t-t')(\mathbf{b}-\mathbf{a})$ we have $\mathbf{q} \in A$ and $\mathbf{q} \in B$ which is a contradiction. On the other hand if $\|\mathbf{p}-\mathbf{b}\| > r_2$ we let $t' = (\|\mathbf{p}-\mathbf{b}\| - r_2)/2\|\mathbf{a}-\mathbf{b}\|$. Then we may set $\mathbf{q} = \mathbf{b} + t'(\mathbf{a}-\mathbf{b})$, $X = \{\mathbf{x} \mid \|\mathbf{x}-\mathbf{q}\| < r_2 + t'\|\mathbf{a}-\mathbf{b}\|\}$, and $Y = \{\mathbf{y} \mid \|\mathbf{y}-\mathbf{b}\| < r_2 + t'\|\mathbf{a}-\mathbf{b}\|\}$. It follows that X and Y are both disjoint from A and both contain B. Further, if we let $t_1 = (\|\mathbf{p}-\mathbf{b}\| + r_2)/2\|\mathbf{a}-\mathbf{b}\|$ and set $\mathbf{u} = \mathbf{b} + t_1(\mathbf{a}-\mathbf{b})$ and $\mathbf{v} = \mathbf{a} + t_2(\mathbf{b}-\mathbf{a})$ then we have $\mathbf{u} \in X$, $\mathbf{v} \in Y$, $\mathbf{u} \notin Y$, and $\mathbf{v} \notin X$. Thus $X \not\subset Y$ and $Y \not\subset X$. Therefore, $\|\mathbf{p}-\mathbf{b}\| = r_2$. Now assume $\mathbf{q} \neq \mathbf{p}$ satisfies the conclusion of this lemma. Then by Lemma 1 we have $\|\mathbf{b}-\mathbf{a}\| < \|\mathbf{a}-\mathbf{q}\| + \|\mathbf{q}-\mathbf{b}\|$. But

$||\mathbf{b} - \mathbf{a}|| = ||\mathbf{a} - \mathbf{p}|| + ||\mathbf{p} - \mathbf{b}||$ which produces a contradiction since $||\mathbf{a} - \mathbf{p}|| = r_1 = ||\mathbf{a} - \mathbf{q}||$ and $||\mathbf{p} - \mathbf{b}|| = r_2 = ||\mathbf{b} - \mathbf{q}||$. Therefore $\mathbf{p} = \mathbf{q}$.

Lemma 3 *Let A and B be two disjoint spheres. If a point $\mathbf{p}$ satisfies $\mathbf{p} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$, $0 < t < 1$, $||\mathbf{p} - \mathbf{a}|| = r_1$, and $||\mathbf{p} - \mathbf{b}|| = r_2$ then A is externally tangent to B.*

Proof: Let X be such that B is part of X and A is disjoint from X. We show that $||\mathbf{p} - \mathbf{x}|| = r$ and that there exists $0 < t' < 1$ such that $\mathbf{p} = \mathbf{a} + t'(\mathbf{x} - \mathbf{a})$. To establish $||\mathbf{p} - \mathbf{x}|| \geq r$ we note that since $||\mathbf{p} - \mathbf{a}|| = r_1$ there is for each $\varepsilon > 0$ a point $\mathbf{q} \in A$ such that $||\mathbf{p} - \mathbf{q}|| < \varepsilon$ and $\mathbf{q}$ satisfies $\mathbf{p} = \mathbf{a} + t''(\mathbf{q} - \mathbf{a})$ for some $0 < t'' < 1$. Applying the hypothesis X is disjoint from A we have $r \leq ||\mathbf{q} - \mathbf{x}|| \leq ||\mathbf{q} - \mathbf{p}|| + ||\mathbf{p} - \mathbf{x}|| < ||\mathbf{p} - \mathbf{x}|| + \varepsilon$. Thus $r \leq ||\mathbf{p} - \mathbf{x}||$. To establish $||\mathbf{p} - \mathbf{x}|| \leq r$ we note first that since $||\mathbf{p} - \mathbf{b}|| = r_2$ we have for each $\varepsilon > 0$ a point $\mathbf{q} \in B$ (and therefore in X) such that $||\mathbf{p} - \mathbf{q}|| < \varepsilon$. Therefore $||\mathbf{x} - \mathbf{p}|| < r + \varepsilon$ by the triangle inequality and the assumption B is part of X. Thus $||\mathbf{p} - \mathbf{x}|| \leq r$. Finally assume there does not exist $0 < t' < 1$ such that $\mathbf{p} = \mathbf{a} + t'(\mathbf{x} - \mathbf{a})$. Then by Lemma 1 $||\mathbf{a} - \mathbf{x}|| < ||\mathbf{a} - \mathbf{p}|| + ||\mathbf{p} - \mathbf{x}|| = r_1 + r$. This implies A and X are not disjoint, which is a contradiction. To finish the proof of the lemma we let Y satisfy A is disjoint from Y and B is part of Y. Then we have $||\mathbf{p} - \mathbf{y}|| = s$ and there exists $0 < t'' < 1$ with $\mathbf{p} = \mathbf{a} + t''(\mathbf{y} - \mathbf{a})$ by the argument above. Finally it follows that if $s \leq r$ then $Y \subseteq X$ and if $r \leq s$ then $X \subseteq Y$.

Lemmas 2 and 3 characterize the relation A is externally tangent to B. Proofs similar to these establish:

Lemma 4 *A is internally tangent to B iff A is part of B and there is a point $\mathbf{p}$ such that $\mathbf{p} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$, $0 < t < 1$, $||\mathbf{p} - \mathbf{a}|| = r_1$ and $||\mathbf{p} - \mathbf{b}|| = r_2$.*

We now characterize the relation externally diametrical. We shall use some elementary properties of Hilbert spaces which can be found in [3], pp. 44-46. These properties are listed in our next lemma.

Lemma 5 *Given a point $\mathbf{p} \neq \mathbf{0}$ of $\mathfrak{H}$ there is an orthonormal basis $\{\mathbf{b}_i\}$, $2 \leq i \leq n \leq \aleph_0$ such that $\mathbf{p} = (\mathbf{p}, \mathbf{b}_1)\mathbf{b}_1$, $(\mathbf{p}, \mathbf{b}_1) > 0$, and for each $\mathbf{q} \in B$ we have $\mathbf{q} = \sum_{i=1}^{n} (\mathbf{q}, \mathbf{b}_i)\mathbf{b}_i$ and $||\mathbf{q}||^2 = \sum_{i=1}^{n} (\mathbf{q}, \mathbf{b}_i)^2$.*

Lemma 6 *Let A and B be externally diametrical to C. Then $\mathbf{p}_1 - \mathbf{c} = -(\mathbf{p}_2 - \mathbf{c})$ where $\mathbf{p}_1 = \mathbf{a} + t_1(\mathbf{c} - \mathbf{a})$ and $\mathbf{p}_2 = \mathbf{b} + t_2(\mathbf{c} - \mathbf{b})$ satisfy the conclusion of* Lemma 2.

Proof: (see Figure 1 below) Assume $\mathbf{c} = \mathbf{0}$. The conclusion then becomes $\mathbf{p}_1 = -\mathbf{p}_2$. If $\mathbf{p}_1 \neq \pm\mathbf{p}_2$ we derive a contradiction as follows: Let $\mathbf{b}_1, \ldots, \mathbf{b}_n$ be an orthonormal basis having the properties of Lemma 5 where $\mathbf{p}$ is taken to be $\mathbf{p}_1$. Since $\mathbf{p}_1 \neq \pm\mathbf{p}_2$ it is easily shown that $|(\mathbf{p}_2, \mathbf{b}_1)| < (\mathbf{p}_1, \mathbf{b}_1)$. Let $\mathbf{q} = 1/2\,\{[(\mathbf{p}_1, \mathbf{b}_1) + (\mathbf{p}_2, \mathbf{b}_1)]\mathbf{b}_1 + (\mathbf{p}_2, \mathbf{b}_2)\mathbf{b}_2 + \ldots + (\mathbf{p}_2, \mathbf{b}_n)\mathbf{b}_n\}$, let $t = 4(\mathbf{p}_1, \mathbf{b}_1)/[(\mathbf{p}_1, \mathbf{b}_1) + (\mathbf{p}_2, \mathbf{b}_1)]$, and let $u = 1/t_1 + 1/t_2 + [t^2||\mathbf{q}||^2 + 3r_3^2]/2r_3^2$. Now let X and Y be the spheres such that X has center $u\mathbf{p}_1$ and radius $(u-1)||\mathbf{p}_1||$ and Y has center $u\mathbf{p}_2$ and radius $(u-1)||\mathbf{p}_2||$. Using Lemma 3 it follows that X

and Y are externally tangent to C. Furthermore A is part of X and B is part of Y and $t\mathbf{q} \in X$ and $t\mathbf{q} \in Y$, a contradiction. Therefore $\mathbf{p}_1 = \pm\mathbf{p}_2$. If $\mathbf{p}_1 = \mathbf{p}_2$ then it is easily shown that A is not disjoint from B, again a contradiction. Thus $\mathbf{p}_1 = -\mathbf{p}_2$. If we now remove the restriction $\mathbf{c} = \mathbf{0}$ the result follows easily since a translation will preserve disjointness and nondisjointness.

As a consequence of this lemma and Lemma 2 we note that $\mathbf{c} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$ for some $0 < t < 1$.

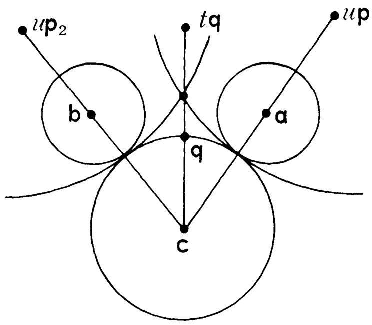

Figure 1.

As a converse to Lemma 6 we have:

Lemma 7 *If the spheres A, B, and C satisfy: A is externally tangent to C, B is externally tangent to C, and $\mathbf{c} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$ with $0 < t < 1$ then A and B are externally diametrical to C.*

Proof: By Lemma 2 we have the existence of $\mathbf{p}_1$ and $\mathbf{p}_2$ such that $\mathbf{p}_1 = \mathbf{c} + t_1(\mathbf{a} - \mathbf{c})$, $\mathbf{p}_2 = \mathbf{c} + t_2(\mathbf{b} - \mathbf{c})$, $0 < t_1 < 1$, $0 < t_2 < 1$, $||\mathbf{p}_1 - \mathbf{a}|| = r_1$, $||\mathbf{p}_1 - \mathbf{c}|| = r_3$, $||\mathbf{p}_2 - \mathbf{b}|| = r_2$, and $||\mathbf{p}_2 - \mathbf{c}|| = r_3$. Now let X and Y be open balls satisfying A is part of X, B is part of Y, and C is disjoint from X and Y. From the proof of Lemma 3 we can find t_1' and t_2' such that $0 < t_1' < 1$, $0 < t_2' < 1$, $\mathbf{p}_1 = \mathbf{c} + t_1'(\mathbf{x} - \mathbf{c})$, and $\mathbf{p}_2 = \mathbf{c} + t_1'(\mathbf{y} - \mathbf{c})$. The hypothesis $\mathbf{c} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$ and the previous sentence now imply the existence of $0 < t'' < 1$ such that $\mathbf{c} = \mathbf{x} + t''(\mathbf{y} - \mathbf{x})$.

Lemma 1 then implies $||\mathbf{x} - \mathbf{y}|| = s + 2r_3 + t > s + t$. It now follows that X is disjoint from Y.

Having characterized the notion of externally diametrical in Lemmas 6 and 7 we note that similar proofs establish the following characterization for internally diametrical:

Lemma 8 *If A and B are internally tangent to C then A and B are internally diametrical to C iff there exists $0 < t < 1$ such that $\mathbf{c} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$.*

Our next two lemmas show that A is concentric to B iff A and B have the same center.

Lemma 9 *Given spheres A and B if $\mathbf{a} \neq \mathbf{b}$ then A and B are not concentric.*

Proof: (see Figure 2 below) Assume A and B are concentric and without loss of generality we take B to be properly contained in A. First form lines l_1 and l_2 as follows: l_1 is the line determined by $\mathbf{a}$ and $\mathbf{b}$; l_2 is taken to be any line distinct from l_1 such that $\mathbf{b} \in l_2$ (recall the dimension of $\mathfrak{H}$ is ≥ 2). Since $\mathfrak{H}$ is a Hilbert space there exists a point $\mathbf{d}$ on l_2 such that for all points $\mathbf{p}$ on l_2 (except $\mathbf{d}$) we have $||\mathbf{d} - \mathbf{a}|| < ||\mathbf{p} - \mathbf{a}||$ and $||\mathbf{p} - \mathbf{a}||^2 = ||\mathbf{p} - \mathbf{d}||^2 + ||\mathbf{d} - \mathbf{a}||^2$ [3], chap. 2, §4. We now construct on l_2 the points $\mathbf{q}$, $\mathbf{q}'$, $\mathbf{e}$, $\mathbf{e}'$, $\mathbf{g}$, and $\mathbf{g}'$ as follows: There exists $\mathbf{q} \neq \mathbf{q}'$ such that $||\mathbf{a} - \mathbf{q}|| = r_1 = ||\mathbf{a} - \mathbf{q}'||$; there exists $\mathbf{e} \neq \mathbf{e}'$ such that $||\mathbf{b} - \mathbf{e}|| = r_2 = ||\mathbf{b} - \mathbf{e}'||$; finally there exists $\mathbf{g} \neq \mathbf{g}'$ such that $||\mathbf{e} - \mathbf{g}|| + ||\mathbf{g} - \mathbf{a}|| = r_1 = ||\mathbf{e}' - \mathbf{g}'|| + ||\mathbf{g}' - \mathbf{a}||$ (the existence of $\mathbf{g}$ (and $\mathbf{g}'$) follows from the fact that the real function $f(t) = t||\mathbf{q} - \mathbf{d}|| + ||(\mathbf{d} - \mathbf{a}) + t(\mathbf{q} - \mathbf{d})||$ is continuous and so there exists $0 < t_0 < 1$ such that $f(t_0) = r_1$. So let $\mathbf{g} = \mathbf{d} + t_0(\mathbf{q} - \mathbf{d})$).

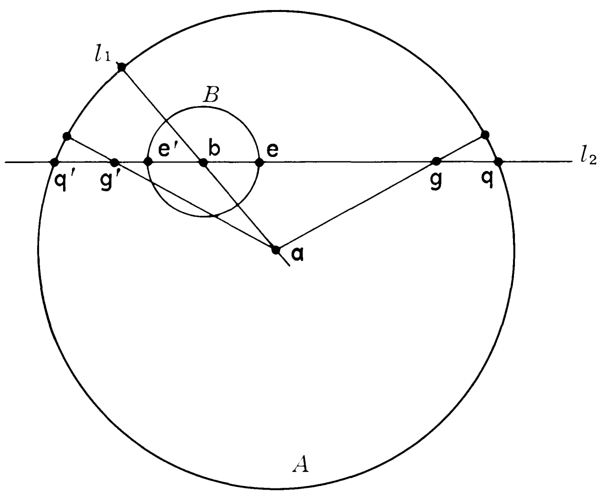

Figure 2.

We now let X and Y be spheres with $\mathbf{x} = \mathbf{g}$, $\mathbf{y} = \mathbf{g}'$, $r = ||\mathbf{b} - \mathbf{g}||$, $s = ||\mathbf{b}' - \mathbf{g}'||$. It is easily established using Lemma 7 and the above that X and Y are externally diametrical to B. Using the definition of $\mathbf{g}$ and $\mathbf{g}'$ and Lemma 4 we have that X and Y are each internally tangent to A. However if we assume A and B are concentric then Lemma 8 implies $\mathbf{a} = \mathbf{g} + t(\mathbf{g}' - \mathbf{g})$ for some $0 < t < 1$. This implies $\mathbf{a} \in l_2$, a contradiction.

Lemma 10 *Let A and B be open balls with $\mathbf{a} = \mathbf{b}$. Then A and B are concentric.*

Proof: We assume without loss of generality $r_1 < r_2$. Assume X and Y satisfy the hypothesis of the definition of concentric (ii). X and Y externally diametrical to A implies by the note to Lemma 6 that there exists $0 < t < 1$ such that $\mathbf{a} = \mathbf{x} + t(\mathbf{y} - \mathbf{x})$. Since $\mathbf{a} = \mathbf{b}$ we have $\mathbf{b} = \mathbf{x} + t(\mathbf{y} - \mathbf{x})$ which implies by Lemma 8 that X and Y are internally diametrical to B.

We now conclude the proof of the Theorem. We use $[A]$ to denote the set of open balls concentric to A.

Lemma 11 *There exists a bijection from* $\mathfrak{H}$ *to the set of sets* $[A]$ *where* A *is an open ball of the separable real Hilbert space* $\mathfrak{H}$ *of dimension* ≥ 2.

Proof: Given $\mathfrak{a} \in \mathfrak{H}$ let A be any sphere with center $\mathfrak{a}$. Let $f(\mathfrak{a}) = [A]$. Lemmas 9 and 10 easily imply that f is well defined and that it is a bijection.

We conclude this paper with a counter example to our theorem in the case where $\mathfrak{B}$ is an arbitrary Banach space [4] of dimension at least 2. For our space $\mathfrak{B}$ we take the space $\mathbf{R} \times \mathbf{R}$ whose elements will be denoted by (x, y) with $x \in \mathbf{R}$ and $y \in \mathbf{R}$ and whose norm is defined by: $\|(x, y)\| = \max\{|x|, |y|\}$. To show that the bijection of Lemma 11 does not hold we shall prove that the relation of concentric fails to be an equivalence relation. Our method of proof will be to show that no sphere is externally tangent to the sphere U whose center is $(0, 0)$ and whose radius is 1. Concentric then becomes just the subset relation. To this end let S be a sphere with center (u, v) and radius $r > 0$ disjoint from U. We observe that the points (x, y) of S satisfy either $|x| > 1$ for all x or $|y| > 1$ for all y; otherwise, there is a point $(t, w) \in (S \cap U)$ with $t = x(1 - s)$ and $w = y(1 - s)$ where $|x| \leq 1$, $|y| \leq 1$, and $s = 1/2 \min\{r - |x - u|, r - |y - v|, 1\}$. Without loss of generality we shall assume $x > 1$ for all x where $(x, y) \in S$. To show S is not externally tangent to U let X be the sphere with center $(u + (|u|r/u), v + r)$ and radius $2r$ and let Y be the open ball with center $(u + (|u|r/u), v - r)$ and radius $2r$. Then the inequalities $|u + (|u|r/u) - x| \leq |u - x| + r$, $|v + r - y| \leq |v - y| + r$, and $|v - r - y| \leq |v - y| + r$ establish: (1) S is part of X and (2) S is part of Y. Further X is disjoint from U for if $(x, y) \in X$ then $|u + (|u|r/u) - x| < 2r$ which implies $|x| > 1$ since $|u| > 1$ by our hypothesis about S. Similarly Y is disjoint from U. Finally since the norm of the difference between the centers of X and Y is $2r$ it follows that the center of X is not an element of Y and the center of Y is not an element of X.

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