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Notre Dame Journal of Formal Logic
Volume XIV, Number 4, October 1973
NDJFAM

A STRONGER THEOREM CONCERNING THE NON-EXISTENCE OF COMBINATORIAL DESIGNS ON INFINITE SETS

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The aim of the present paper is to show that certain combinatorial designs on infinite sets cannot exist. To be precise we introduce the following terminology. If M is a set and p a cardinal number $\leq \overline{\overline{M}}$ (the cardinal number of M) then $[M]^p$ is the collection of all subsets of M having cardinality p.

Definition 1. A family F is called a *p-tuple family* of M if and only if (i) $F \subset [M]^p$ and (ii) $x, y \in F$ and $x \subset y$ implies $x = y$.

Definition 2. Let F and G be two families of subsets of M. G is called a *Steiner cover* of F if and only if for every $x \in F$ there is exactly one $y \in G$ such that $x \subset y$.

It is now possible to state the main result of the present paper.

Theorem 3. *Let α, β and γ be ordinal numbers such that*

(i) $\alpha < \beta < \gamma$
(ii) $\mathrm{cf}(\omega_\gamma) \leq \omega_\alpha$
(iii) $\aleph_\beta^{\aleph_\alpha} \leq \aleph_\gamma$.

Then, in every set M of cardinality $\aleph_\gamma$ there exists an $\aleph_\alpha$-tuple family F of M such that there does not exist a family $G \subset [M]^{\aleph_\beta}$ which is a Steiner cover of F.

N.B. It should be noted that this result subsumes the main result of [1] (denoted there as Theorem 6) and [2] (denoted as Theorem 4) as special cases. It should also be noted that the proofs offered for both of these results contain errors. This fact was kindly pointed out to me by Professor E. C. Milner of the University of Calgary, Alberta, Canada, whom the present author wishes to take this opportunity to thank.

We begin with some preliminaries.

Definition 4. Let F and G be families of subsets of M and n a non-zero cardinal number. G is called an *n-spoiler* of F if and only if for every $x \in F$ and every $y \in [M]^n$ there is a $z \in G$ such that $z \subset x \cup y$.

Received April 2, 1972

Lemma 5. *Let k and n be infinite cardinal numbers and let F be a k-tuple family of an infinite set M. Suppose there exists subfamilies $F_1, F_2 \subseteq F$ such that* (i) *F_2 is an n-spoiler of F_1 and* (ii) $n^k\overline{\overline{F_2}} < \overline{\overline{F_1}}$. *Then, F does not possess a Steiner cover contained in $[M]^n$.*

Proof. To the contrary suppose there is a Steiner cover G of F which is contained in $[M]^n$. We will arrive at a contradiction by showing

(*) $(\exists x_0 \in F_1)$ $(\forall x' \in F_2)$ [x_0 *and* x' *are not subsets of the same member of* G]

If we can establish (*) we can derive a contradiction as follows. Let y_0 be that unique member of G such that $x_0 \subseteq y_0$. Since $x_0 \in F_1$, $y_0 \in [M]^n$ and F_2 is an n-spoiler of F_1 it must be that there exists some $x^* \in F_2$ such that $x^* \subseteq x_0 \cup y_0$. But $x_0 \subseteq y_0$. Hence $x^* \subseteq x_0 \cup y_0 = y_0$. This forces x_0 and x^* (a member of F_2) to both be subsets of the same member y_0 of G, thus contradicting (*).

To see that (*) holds let G' be defined as the set of all those members of G which contain, as a subset, at least one member of F_2. Now each member of G' is a set of cardinality n and therefore can contain at most n^k subsets of cardinality k. Thus, since each member of G' must contain as a subset a member of F_2 it follows that the total number of members of F which are contained in some member of G' cannot exceed $n^k\overline{\overline{F}}_2$. But by (ii) it then must be the case that there is some member x_0 of F_1 not contained in any member of G'. Such an x_0 will satisfy (*). This completes the proof of Lemma 5.

Proof of Theorem 3. On the strength of hypotheses (i)-(iii) it is possible to construct an increasing sequence of ordinal numbers $\{\alpha_\xi\}_{\xi < \mathrm{cf}(\omega_\gamma)}$ such that

(1) $\aleph_\gamma = \sum_{\xi < \mathrm{cf}(\omega_\gamma)} \aleph_{\alpha_\xi}$

and

(2) $\aleph_\beta \leq \aleph_{\alpha_\xi} < \aleph_\gamma$ for all $\xi < \mathrm{cf}(\omega_\gamma)$.

Thus we can find sets $\{M_\xi\}_{\xi < \mathrm{cf}(\omega_\gamma)}$ such that

(3) $M = \bigcup\{M_\xi \mid \xi < \mathrm{cf}(\omega_\gamma)\}$

(4) $\overline{\overline{M}}_\xi = \aleph_{\alpha_\xi}$

and

(5) $M_{\xi_1} \cap M_{\xi_2} = 0$ whenever $\xi_1 \neq \xi_2$.

For each $\xi < \mathrm{cf}(\omega_\gamma)$ let F'_ξ be any collection of pairwise disjoint subsets of M_ξ such that

(6) $F'_\xi \subseteq [M_\xi]^{\aleph_\alpha}$

and

(7) $\overline{\overline{F'_\xi}} = \overline{\overline{M_\xi}} = \aleph_{\alpha_\xi}$.

That such F'_ξ exist is guaranteed by the fact that $\aleph_{\alpha_\xi} \aleph_{\alpha_\xi} = \aleph_{\alpha_\xi}$.

Definition 6. Two elements p and q of M_ξ are said to be *independent* if and only if they are not members of the same element of F'_ξ.

Definition 7. For each $\xi < \mathrm{cf}(\omega_\gamma)$ let

$$P_\xi = \{\{p, q\} \in [M_\xi]^2 \mid p \text{ and } q \text{ are independent}\}.$$

Clearly for each $\xi < \mathrm{cf}(\omega_\gamma)$

(8) $\overline{\overline{P_\xi}} \leq \overline{\overline{M_\xi}} = \aleph_{\alpha_\xi}$.

Definition 8. For each $\xi < \mathrm{cf}(\omega_\gamma)$ let

$$F_\xi = \{x \cup x' \mid x \in P_\xi \text{ and } x' \in F'_{\xi+1}\}.$$

Lemma 9. *For each $\xi < \mathrm{cf}(\omega_\gamma)$, F_ξ is an $\aleph_\alpha$-tuple family of M.*

Proof. If $y \in F_\xi$ then we have $\overline{\overline{y}} = \aleph_\alpha + 2 = \aleph_\alpha$. Let $x \cup x'$ and $y \cup y'$ be elements of F_ξ. Now suppose $(x \cup x') \subset (y \cup y')$. But this would imply $x' \subset y'$ and $x \subset y$. Since $x', y' \in F'_{\xi+1}$ and x and y are finite sets of the same cardinality it must follow that $x' = y'$ and $x = y$. This proves Lemma 9.

Lemma 10. *For each $\xi < \mathrm{cf}(\omega_\gamma)$, $\overline{\overline{F_\xi}} = \aleph_{\alpha_{\xi+1}}$.*

Proof. By (8) we have $\overline{\overline{F_\xi}} = \overline{\overline{P_\xi}}\ \overline{\overline{F'_{\xi+1}}} = \overline{\overline{P_\xi}}\ \aleph_{\alpha_{\xi+1}} = \aleph_{\alpha_{\xi+1}}$.

Definition 11. $F^* = \bigcup\{F_\xi \mid \xi < \mathrm{cf}(\omega_\gamma)\}$.

Lemma 12. *F^* is an $\aleph_\alpha$-tuple family of M.*

Proof. It is sufficient to show that any two distinct members of F^*, say x_1 and x_2, are not nested. Suppose $x_1 \in F_{\xi_1}$ and $x_2 \in F_{\xi_2}$. Without loss of generality we may assume $\xi_1 \leq \xi_2$. If $\xi_1 = \xi_2$, x_1 and x_2 cannot be nested since F_{ξ_1} is an $\aleph_\alpha$-tuple family of M. Now suppose $\xi_1 < \xi_2$. Then there are at least two elements of x_1 which lie outside x_2 (*viz.* the members of $x_1 \cap M_{\xi_1}$); moreover, there are at least $\aleph_\alpha$ elements of x_2 which lie outside x_1 (*viz.* the members of $x_2 \cap M_{\xi_2+1}$). Thus x_1 and x_2 cannot be nested. This proves Lemma 12.

Lemma 13. *$\overline{\overline{F^*}} = \aleph_\gamma$.*

Proof. $\overline{\overline{F^*}} = \sum_{\xi < \mathrm{cf}(\omega_\gamma)} \overline{\overline{F_\xi}} = \sum_{\xi < \mathrm{cf}(\omega_\gamma)} \aleph_{\alpha_{\xi+1}} = \aleph_\gamma$.

Definition 14. $F\# = \{y \subset M \mid (y \cap M_\xi) \in F'_\xi \text{ for each } \xi < \mathrm{cf}(\omega_\gamma)\}$.

Lemma 15. *$F\#$ is an $\aleph_\alpha$-tuple family of M.*

Proof. Every element y of $F\#$ may be written as $y = \bigcup\{y_\xi \mid \xi < \mathrm{cf}(\omega_\gamma)\}$ where $y_\xi \in F'_\xi$ for each $\xi < \mathrm{cf}(\omega_\gamma)$. By (6) we know $\overline{\overline{y_\xi}} = \aleph_\alpha$. Hence $\overline{\overline{y}} = \aleph_\alpha\ \overline{\mathrm{cf}(\omega_\gamma)} = \aleph_\alpha$: the last equality being justified by hypothesis (ii).

Now let y and y' be two distinct members of $F\#$. There must exist a $\xi_0 < \mathrm{cf}(\omega_\gamma)$ such that $(y \cap M_{\xi_0}) \neq (y' \cap M_{\xi_0})$. But the sets $(y \cap M_{\xi_0})$ and $(y' \cap M_{\xi_0})$ both are members of the family F'_{ξ_0} whose members are pairwise disjoint. Hence $(y \cap M_{\xi_0}) \cap (y' \cap M_{\xi_0}) = 0$. Thus there are members of y which lie outside y' and vice versa. Thus y and y' are not nested. This proves Lemma 15.

Lemma 16. $\overline{\overline{F\#}} > \aleph_\gamma$.

Proof. Since the sequence $\{\alpha_\xi\}$ is increasing it follows from a theorem of J. König that

$$\overline{\overline{F\#}} = \prod_{\xi < \mathrm{cf}(\omega_\gamma)} \overline{\overline{F'_\xi}} = \prod_{\xi < \mathrm{cf}(\omega_\gamma)} \aleph_{\alpha_\xi} > \sum_{\xi < \mathrm{cf}(\omega_\gamma)} \aleph_{\alpha_\xi} = \aleph_\gamma.$$

Definition 17. $F = F^* \cup F\#$.

Lemma 18. *F is an $\aleph_\alpha$-tuple family of M.*

Proof. It is sufficient to show that any two distinct members of F, say x_1 and x_2, are not nested. Clearly, if $x_1, x_2 \in F^*$ or $x_1, x_2 \in F\#$ then they could not be nested since F^* and $F\#$ are both $\aleph_\alpha$-tuple families. Now suppose $x_1 \in F^*$ and $x_2 \in F\#$. We may assume further that $x_1 \in F_{\xi_0}$. Clearly x_1 is a subset of $M_{\xi_0} \cup M_{\xi_0+1}$ which makes it impossible for x_2, which intersects each of the M_ξ, to be a subset of x_1. To show x_1 is not a subset of x_2 we may write $x_1 = \{p, q\} \cup x_0$ where $\{p, q\} \in P_{\xi_0}$ and $x_0 \in F'_{\xi_0+1}$. If $x_1 \subset x_2$ then it would follow that $\{p, q\} \subset (x_2 \cap M_{\xi_0})$. But $(x_2 \cap M_{\xi_0})$ is a member of F'_{ξ_0}, since x_2 is a member of $F\#$, and thus the fact that p and q are independent is contradicted. Thus x_1 and x_2 are not nested. This proves Lemma 18.

Lemma 19. *F^* is an $\aleph_\beta$-spoiler of $F\#$.*

Proof. Let $x \in F\#$ and $y \in [M]^{\aleph_\beta}$. It is now necessary to produce a $z \in F^*$ such that $z \subset x \cup y$. Since $\overline{\overline{y}} = \aleph_\beta > \aleph_\alpha$ and $\mathrm{cf}(\omega_\gamma) \leq \omega_\alpha$ (hypothesis (ii)) there must exist an $\xi_0 < \mathrm{cf}(\omega_\gamma)$ such that $\overline{\overline{(y \cap M_{\xi_0})}} > \aleph_\alpha$. Thus there must exist two distinct elements p and q of $y \cap M_{\xi_0}$ which are independent (i.e., that $\{p, q\} \in P_{\xi_0}$). This is so since a subset of M_{ξ_0}, every two elements of which are not independent, can have cardinality at most $\aleph_\alpha$: such a subset would have to be completely contained within a single member of F'_{ξ_0}. Let $z = \{p, q\} \cup (x \cap M_{\xi_0+1})$. Since $x \in F\#$, $(x \cap M_{\xi_0+1}) \in F'_{\xi_0+1}$ which implies that $z \in F_{\xi_0}$ and hence $z \in F^*$. Now since $\{p, q\} \subset (y \cap M_{\xi_0}) \subset y$ and $(x \cap M_{\xi_0+1}) \subset x$ it must be that $z \subset x \cup y$. This completes the proof of Lemma 19.

Lemma 20. $\aleph_\beta^{\aleph_\alpha} \overline{\overline{F^*}} < \overline{\overline{F\#}}$.

Proof. Using hypothesis (iii) and Lemmas 13 and 16 we have $\aleph_\beta^{\aleph_\alpha} \overline{\overline{F^*}} \leq \aleph_\gamma \aleph_\gamma = \aleph_\gamma < \overline{\overline{F\#}}$.

Lemmas 18, 19 and 20 together with Lemma 5 establish that the $\aleph_\alpha$-tuple family F, given in Definition 17, does not possess a Steiner cover contained in $[M]^{\aleph_\beta}$. This completes the proof of Theorem 3.

With the aid of Theorem 3 it is possible to exhibit a whole range of situations where Steiner covers are not available.

Definition 21. Let p be some fixed cardinal number. Then for any ordinal number α the cardinal number $\beth_\alpha(p)$ is defined inductively by the following conditions: (i) if $\alpha = 0$ then $\beth_\alpha(p) = p$; (ii) if $\alpha = \beta + 1$ then $\beth_\alpha(p) = 2^{\beth_\beta(p)}$; and (iii) if α is a limit ordinal then $\beth_\alpha(p) = \sum_{\beta < \alpha} \beth_\beta(p)$.

Corollary 22. *Let* $\aleph_\alpha$ *and* $\aleph_\beta$ *be any cardinal numbers such that* $\aleph_\alpha < \aleph_\beta$. *Then there exists an infinite set* M *and an* $\aleph_\alpha$*-tuple family* F *of* M *which does not possess a Steiner cover* $G \subset [M]^{\aleph_\beta}$.

Proof. Let M be any set of cardinality $\beth_{\omega_\alpha}(\aleph_\beta)$ and let γ be such that $\beth_{\omega_\alpha}(\aleph_\beta) = \aleph_\gamma$. Since $\mathrm{cf}(\omega_\gamma) \leq \omega_\alpha$ and $\aleph_\beta^{\aleph_\alpha} \leq 2^{\aleph_\beta} < \beth_{\omega_\alpha}(\aleph_\beta) = \aleph_\gamma$ it follows that all the hypotheses of Theorem 3 are satisfied. Corollary 22 is thus proved.

Theorem 6 of [1] is the special case of the above corollary where $\alpha = 0$ and $\beta = 1$ while Theorem 4 of [2] follows *a fortiori* from Theorem 3 of the present paper.

REFERENCES

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