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INCIDENCE RINGS OF PRE-ORDERED SETS

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Introduction. In this paper* every relation $\leqq$ on a set X is a binary relation which is transitive and reflexive. G. C. Rota [2] has defined incidence rings of partially ordered systems $\langle X, \leqq\rangle$. We generalize these rings by dropping the anti-symmetric condition on the order $\leqq$.

If X is a set and $\leqq$ a binary relation on X, then $\langle X, \leqq\rangle$ shall denote this relational system. We say that $\langle X, \leqq\rangle$ is a pre-ordered relational system if the relation $\leqq$ is transitive and reflexive. If confusion is unlikely, then we shall often take the liberty of using the relation $\leqq$ to denote the usual ordering of the natural numbers and also to denote a relation on a set X. Unless otherwise stated 0,1 should be understood to be real numbers. To each relational system $\langle X, \leqq\rangle$ there is a unique *zeta function*, ζ, mapping $X \times X$ into $\{0,1\}$. For $x, y \in X$, $\zeta(x,y) = 1$, if $x \leqq y$ and $\zeta(x,y) = 0$ otherwise. In the context of a relation system $\langle X, \leqq\rangle$, $[x,y] = \{u \in X | x \leqq u \leqq y\}$ is an *interval* and $\langle X, \leqq\rangle$ is *locally finite* iff every such interval is empty or a finite set.

We shall consider only rings R which have a multiplicative identity; rings may or may not be commutative. We do not assume any relationship between the rings R and sets X we discuss. The symbol R^* denotes the set of units of the ring R; the function det is the determinant function. If n is a positive integer, then $\mathsf{M}(n, R)$ denotes the complete ring of $n \times n$ matrices over the ring R. If X is any set, then $\mathcal{S}_X$ denotes the group of permutations of the set X; for positive integers n, $\mathcal{S}_n$ denotes $\mathcal{S}_{\{1,\ldots,n\}}$.

For a given ring R and locally finite pre-ordered system $\langle X, \leqq\rangle$, the *incidence ring* $I = \langle X, \leqq, R\rangle$ is set—theoretically the set of functions f mapping $X \times X$ into R satisfying the following *order condition*. For every $x, y \in X$, $f(x,y) \neq 0$ only if $x \leqq y$. Multiplication, addition and scalar multiplication for incidence rings are defined in section 1. If $[x,y]$ is a

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non-empty interval and if $\leqq|$ is the relation $\leqq$ restricted to the set $[x, y]$, then the *interval ring* $I[x, y]$ of I is the incidence ring $\langle [x, y], \leqq|, R\rangle$. In general $I[x, y]$ is not isomorphic to a subring of I. The zeta function ζ of $\langle X, \leqq\rangle$ is in I and if ζ has an inverse, say μ, then μ is the *Möbuis function* of I. To indicate that f is a function mapping the set X into the set Y we often write $f\colon X \to Y$. If $x \in X$, $y \in Y$ and $f(x) = y$ we often write $x \mapsto y$ to indicate the action of f. If $\langle X, \leqq\rangle$, $\langle Y, \leqq'\rangle$ are relational systems and f is a bijective map $f\colon X \to Y$ such that $x \leqq y$ iff $f(x) \leqq' f(y)$, then these relational systems are isomorphic by the function f.

Of the various sorts of orderings which may be defined on a set partial orderings are the ones which have been studied in greatest detail. The reason for this is simple. Where natural orderings arise on a mathematical structure they are often 'less than or equal to' type orderings which are partial orderings. However, there are orderings of some interest which are reflexive and transitive but not necessarily antisymmetric. Pre-ordered relational systems arise, for example, whenever a topology is defined on a finite set or whenever a topology in which arbitrary intersections of open sets are open sets is defined on a set. Incidence rings, the study of which has been made important for the foundations of Combinatorial Theory by G. C. Rota [2] are examples of structures defined originally using a partial ordering but where the structures do not collapse or become insignificant when the antisymmetry property of the order is dropped.

The study of incidence rings of a locally finite pre-ordered relational system $\langle X, \leqq\rangle$ is related to the study of enumeration problems associated with the order $\leqq$ and to the study of inversion formulas for certain functions on the sets X and $X \times X$. The zeta function ζ of an incidence ring $I = \langle X, \leqq, R\rangle$ precisely describes the relational system $\langle X, \leqq\rangle$ and the inverse of ζ, when it exists, is the Möbuis function of the ring.

By way of example consider the position integers $\mathcal{N}$ ordered by division $|$, $\langle \mathcal{N}, |\rangle$ is a locally finite partially ordered system. The zeta function of the incidence ring of $\langle \mathcal{N}, |\rangle$ over, say the real numbers, is

$$\zeta(m, n) = \begin{cases} 1 \text{ if } m|n \\ 0 \text{ otherwise.} \end{cases}$$

The classical Möbuis function is

$$\mu_c(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \text{ is divisible by the square of a prime} \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes.} \end{cases}$$

If we write

$$\mu(m, n) = \begin{cases} \mu_c\left(\dfrac{n}{m}\right) & \text{if } m|n \\ 0 & \text{otherwise} \end{cases}$$

then μ is the inverse of ζ in the incidence ring. The classical inversion formula of Möbuis becomes an inversion formula in the incidence ring. If g is a real valued function on $\mathcal{N}$ satisfying $g(n) \neq 0$ only if $n|r$ for a fixed $r \in \mathcal{N}$ and if

$$f(m) = \sum_{n|m} g(n),$$

then

$$g(m) = \sum_{n|m} f(n)\mu(n, m) = \sum_{n} f(n)\mu(n, m).$$

If we now move to the integers $\mathcal{Z}$ ordered by division, then $\langle \mathcal{Z}, |\rangle$ is no longer a partially ordered relational system; however, $\langle \mathcal{Z}, |\rangle$ is a locally finite pre-ordered relational system. Inversion formulas for $\langle \mathcal{Z}, |\rangle$ analogous to the one given above for $\langle \mathcal{N}, |\rangle$ are naturally of mathematical interest and therefore it is of interest to characterize those functions in the incidence ring of $\langle \mathcal{Z}, |\rangle$ over, say the real numbers, which are invertible. The main result of section 1 is to carry out such a characterization for an arbitrary locally finite pre-ordered system.

G. C. Rota [2] has defined incidence rings of functions subject to an order constraint mapping X^2 into the real numbers where X is a locally finite partially ordered set. It is possible to define such rings for locally finite pre-ordered systems $\langle X, \leqq\rangle$ as we do in section 1. For the study of inversion formulas in incidence rings it is desirable to know which functions in the incidence ring are invertible and in section 1 we characterize the units of an arbitrary incidence ring $I = \langle X, \leqq, R\rangle$ where $\langle X, \leqq\rangle$ is a locally finite pre-ordered relational system and R a commutative ring. A function f in I is invertible iff for each interval ring of the form $I[x, x]$ of I, the restricted function $f|[x, x]$ is invertible in $I[x, x]$. This type of local-global invertibility property can be tested using determinants for the following reason. Each interval ring $I[x, y]$ of I is isomorphic to a subring of a complete matrix algebra $\mathrm{M}(n, R)$. Using properties of the pre-order relation we show that invertibility in the subring of $\mathrm{M}(n, R)$ isomorphic to $I[x, y]$ is equivalent to invertibility in $\mathrm{M}(n, R)$ itself. Consequently f is invertible in the full incidence ring I iff a certain collection of determinants related to f are all units in R. As a corollary of this result we show that an incidence ring contains a Möbuis function iff the underlying pre-ordered system is a partially ordered system. Also, several results are developed which relate the structure of a locally finite pre-ordered system to the algebraic structure of its incidence rings.

R. P. Stanley, [3] and [5], has shown that locally finite partially ordered systems $\langle X, \leqq\rangle$ and $\langle X', \leqq'\rangle$ are isomorphic iff their respective incidence rings over any given field are isomorphic rings. In section 2 we generalize this result to pre-ordered systems. The generalization is not complete in the following sense. If $\langle X, \leqq\rangle$, $\langle X', \leqq'\rangle$ are locally finite pre-ordered systems which are isomorphic it is easy to show that their incidence rings $\langle X, \leqq, R\rangle$ and $\langle X', \leqq', R\rangle$ over a common ring R, are isomorphic. Conversely if R is a field, if either X or X' is finite and if $\langle X, \leqq, R\rangle$ and $\langle X', \leqq', R\rangle$ are isomorphic rings then $\langle X, \leqq\rangle$ and $\langle X', \leqq'\rangle$ are isomorphic relational systems. However, for the case where both X and X' are infinite sets we have had to require special conditions on a ring isomorphism ψ: $\langle X, \leqq, R\rangle \to \langle X', \leqq', R\rangle$ to ensure that the systems $\langle X, \leqq\rangle$, $\langle X', \leqq'\rangle$ will also be isomorphic.

Incidence Rings. We shall be concerned with developing the theory of incidence rings of pre-ordered sets. G. C. Rota [2] has defined incidence rings for partially ordered sets. Here we drop the antisymmetric condition on the underlying order and investigate the corresponding changes in the structure of the incidence rings.

1. *Invertibility in Incidence Rings.* In this section we completely characterize the units of an incidence ring I of a pre-ordered set over a commutative ring with identity. Several results relating the structure of the pre-order relation to invertibility are derived including a type of inverse function theorem. It is shown that I contains a Möbuis function iff the underlying order is a partial ordering.

Definition 1.1. Let $\langle X, \leqq\rangle$ be a locally finite pre-ordered system. The *incidence ring* of the relational system $\langle X, \leqq\rangle$ over the ring R is the set of functions f mapping $X \times X$ into R satisfying the following *order* condition: $f(x, y) \neq 0$ only if $x \leqq y$; for every $x, y \in X$. We refer to this set of functions as $\langle X, \leqq, R\rangle$, or where convenient, as simply I. On the set I we define operations of multiplication $\odot$, addition $\oplus$, and left and right scalar multiplication $*$. Let $+$ and . denote addition and multiplication respectively in the ring R. For every $f, g \in I$; $x, y \in X$ and $b \in R$ we have:

$$(f \odot g)(x, y) = \sum_{u \in X} f(x, u) \,.\, g(u, y)$$
$$(f \oplus g)(x, y) = f(x, y) + g(x, y)$$
$$(b * f)(x, y) = b \,.\, f(x, y) \text{ and } (f * b)(x, y) = f(x, y) \,.\, b$$

For convenience we drop the notation $\odot, \oplus, *$ and write fg, $f + g$, bf and fb respectively for multiplication, addition and left and right scalar multiplication in I.

Lemma 1.2. *Let $\langle X, \leqq\rangle$ be a locally finite pre-ordered system and R a ring. Then $I = \langle X, \leqq, R\rangle$ is a ring and a left and right R-module* (in the ring theoretic sense).

Proof. Let $u, v, x, y \in X$; $a, b \in R$ and $f, g, h \in I$. By the definition of addition in I, the members of I form a commutative group with respect to addition, provided X is non-empty, since R has this property. We neglect altogether the trivial case $X = \emptyset$. By the definition of scalar multiplication it is obvious that $af \in I$, $(a + b)f = af + bf$ and $a(f + g) = af + ag$; and similarly for scalar multiplication on the right.

Suppose $(fg)(x, y) \neq 0$, then by the definition of mulitplication in I there is a $u \in X$ such that $f(x, u)\, g(u, y) \neq 0$. Hence $f(x, u) \neq 0$, $g(u, y) \neq 0$ so $x \leqq u$ and $u \leqq y$. By the transitivity of the relation, $x \leqq y$. This shows that $fg \in I$ and that I is closed under multiplication. Furthermore, in the equation

$$(fg)(x, y) = \sum_{u \in X} f(x, u)\, g(u, y)$$

the sum on the right involves only finitely many non-zero summands, since, if $f(x, u)g(u, y) \neq 0$, then $u \in [x, y]$ and this latter set is finite. The finiteness of such summands enables us to show easily that

$$f(g + h) = fg + fh \text{ and } (g + h)f = gf + hf.$$

To show that I is a ring it suffices to show that I contains a multiplicative identity and that multiplication is associative. If we define

$$\delta(x, y) = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ if } x \neq y \end{cases}$$

then $\delta \in I$, as $\leqq$ is reflexive. Furthermore,

$$(\delta f)(x, y) = \sum_{u \in X} \delta(x, u) f(u, y) = f(x, y).$$

So $\delta f = f$ and similarly $f\delta = f$, showing that δ is the multiplicative identity of I. We have,

$$(f(gh))(x, y) = \sum_{u \in X} f(x, u)(gh)(u, y) = \sum_{u \in X} f(x, u)\Big(\sum_{v \in X} g(u, v)\, h(v, y)\Big).$$

The interval $[x, y]$ is a finite set and $[u, y]$ is a finite set for each $u \in X$ so this sum contains finitely many non-zero summands. Using the distributive and commutative addition laws in the ring R we may write this sum as

$$\begin{aligned} (f(gh))(x, y) &= \sum_{v \in X} \Big(\sum_{u \in X} f(x, u) g(u, v)\Big) h(v, y) \\ &= \sum_{v \in X} (fg)(x, y)\, h(v, y) \\ &= ((fg)h)(x, y). \end{aligned}$$

So $f(gh) = (fg)h$, thereby showing that I is a ring.

Corollary 1.3. *Let $\langle X, \leq \rangle$ be a locally finite pre-ordered system and R a ring, then $I = \langle X, \leq, R \rangle$ is an algebra over R iff R is a commutative ring.*

Proof. Since I is a ring, it is an algebra over R iff for every $b \in R$ and $f, g \in I$ we have

$$a(fg) = (af)g = f(ag).$$

Since I contains an identity this result is equivalent to

$$bf = fb$$

for $b \in R$ and $f \in I$. By definition of scalar multiplication in I this holds iff R is commutative.

By construction of incidence rings, if $I = \langle X, \leqq, R \rangle$ is an incidence ring and X a finite set, then I is isomorphic to a subring of $M(p, R)$ the complete ring of $p \times p$ matrices over R, where p is the cardinality of X.

Definition 1.4. Let $X = \{x_1, \ldots, x_p\}$ and let $\langle X, \leqq \rangle$ be a pre-ordered system and R a ring. Let $I = \langle X, \leqq, R \rangle$. A map

$$m_p \colon I \to M(p, R)$$

is defined as follows. Let $f \in I$ and let i, j be integers $1 \leqq i, j \leqq p$. $\mathsf{m}_p(f)$ is that member of $\mathsf{M}(p, R)$ whose (i, j)-th entry is

$$(\mathsf{m}_p(f))_{ij} = f(x_i, x_j).$$

For convenience we shall drop the subscript from the map m, since it will be clear from the context what the cardinality of X is.

Lemma 1.5. *Let X be a finite set, $\langle X, \leqq\rangle$ a pre-ordered system, R a ring and let $I = \langle X, \leqq, R\rangle$. Then the map $\mathsf{m}: I \to \mathsf{m}(I)$ is a ring isomorphism. If R is a commutative ring, then m is an algebra isomorphism.*

Proof. Let $X = \{x_1, \ldots, x_n\}$; $f, g \in I$ and $b \in R$. For $1 \leqq i, j \leqq n$, $(\mathsf{m}(f))_{ij} = f(x_i, x_j)$. Clearly $\mathsf{m}(f+g) = \mathsf{m}(f) + \mathsf{m}(g)$ and $\mathsf{m}(f) = 0$ iff $f = 0$. Also,

$$\begin{aligned} \mathsf{m}(fg)_{ij} &= (fg)(x_i, x_j) = \sum_{k=1}^{n} f(x_i, x_k)\, g(x_k, x_j) \\ &= \sum_{k=1}^{n} \mathsf{m}(f)_{ik}\, \mathsf{m}(g)_{kj} = (\mathsf{m}(f) \cdot \mathsf{m}(g))_{ij}. \end{aligned}$$

So $\mathsf{m}(fg) = \mathsf{m}(f)\mathsf{m}(g)$. If I_n is the identity matrix of $\mathsf{M}(n, R)$ it is easy to see that $\mathsf{m}(f) = I_n$ iff $f = \delta$. This shows that m maps I isomorphically to its image $\mathsf{m}(I)$ in $\mathsf{M}(n, R)$. Further, if R is commutative, then I and $\mathsf{M}(n, R)$ are algebras over R and m is an algebra isomorphism since $\mathsf{m}(bf) = b(\mathsf{m}(f))$.

In the preceding proof the enumeration given to the set X changes $\langle X, \leqq, R\rangle$ only by an isomorphism.

Lemma 1.6. *Let $X = \{x_1, \ldots, x_n\}$ and let $\langle X, \leqq\rangle$ be a pre-ordered system. Let $\sigma \in \mathsf{S}_n$ and let $\sigma(\leqq)$ be the pre-order relation on X defined as follows:*

$$x_i \sigma(\leqq) x_j \text{ iff } x_{\sigma(i)} \leqq x_{\sigma(j)}.$$

Let $I = \langle X, \leqq, R\rangle$ and $I^\sigma = \langle X, \sigma(\leqq), R\rangle$. For every $f \in I$ let f^σ be an element of I^σ defined as follows:

$$f^\sigma(x_i, x_j) = f(x_{\sigma(i)}, x_{\sigma(j)}).$$

Then the map $\sigma: I \to I^\sigma$ is a ring isomorphism.

Proof. Let $f, g \in I$. By the order condition on I we have $f^\sigma(x_i, x_j) \neq 0$ only if $x_{\sigma(i)} \leqq x_{\sigma(j)}$, so $f^\sigma(x_i, x_j) \neq 0$ only if $x_i \sigma(\leqq) x_j$. Thus $f^\sigma \in I^\sigma$. Clearly $(f+g)^\sigma = f^\sigma + g$ and $f^\sigma = 0$ iff $f = 0$. Also,

$$\begin{aligned} (fg)^\sigma(x_i, x_k) &= \sum_{j=1}^{n} f(x_{\sigma(i)}, x_j)\, g(x_j, x_{\sigma(k)}) \\ &= \sum_{j=1}^{n} f(x_{\sigma(i)}, x_{\sigma(j)})\, g(x_{\sigma(j)}, x_{\sigma(k)}) \\ &= (f^\sigma g^\sigma)(x_i, x_k). \end{aligned}$$

So $(fg)^\sigma = f^\sigma g^\sigma$. Clearly $f^\sigma = \delta$ iff $f = \delta$. This shows that σ is a ring isomorphism.

Definition 1.7. Let $\langle X, \leqq\rangle$ be a locally finite pre-ordered relational system and R a ring. Let $u, v \in X$ and suppose $u \leqq v$. Recall that $[u, v] = \{x \in X \mid u \leqq x \leqq v\}$. Let $\leqq_{u,v}$ be the relation $\leqq$ restricted to the subset $[u, v]$. Then

$\langle [u, v], \leqq_{u,v}, R\rangle$ is the incidence ring of the relational system $\langle [u, v], \leqq_{u,v}\rangle$ over the ring R. We call this incidence ring the *interval ring* $I[u, v]$ of the incidence ring $I = \langle X, \leqq, R\rangle$.

Definition 1.8. Let $I = \langle X, \leqq, R\rangle$ be an incidence ring and let $I[u, v]$ be an interval ring of I. If $f \in I$, let $f|_{[u,v]} \in I[u, v]$ be defined as follows: for every $x, y \in [u, v]$,

$$f|_{[u,v]}(x, y) = f(x, y).$$

Lemma 1.9. *Let $I[u, v]$ be an interval ring of an incidence ring $I = \langle X, \leqq, R\rangle$. Then every $g \in I[u, v]$ has the form $f|_{[u,v]}$ for some $f \in I$.*

Proof. Let $g \in I[u, v]$. Define $f \in I$ as follows: for every $x, y \in X$

$$f(x, y) = \begin{cases} g(x, y) & \text{if } x, y \in [u, v] \\ 0 & \text{otherwise.} \end{cases}$$

Then $f|_{[u,v]} = g$, as required.

Interval rings, for example $I[x, y]$ of an incidence ring I, are introduced because, as is later shown, a function f belonging to I is invertible in I iff $f|_{[x,y]}$ is invertible in $I[x, y]$ for each non-empty interval $[x, y]$. According to this result a global property of a function f, that is, whether or not $f \in I^*$, can be determined by examining the local properties of f, that is, whether or not $f|_{[x,y]} \in I[x, y]^*$, for each such interval $[x, y]$. The locally finite condition requires that intervals $[x, y]$ be finite sets so Lemma 1.5 shows that $I[x, y]$ is isomorphic to a subring of $\mathrm{M}(n, R)$ where n is the cardinality of $[x, y]$. In general $I[x, y]$ is not isomorphic to a subring of I as the following example shows.

Example 1.10. Let $X = \{x_1, x_2, x_3\}$. Define the relation $\leqq$ on X as follows: $x_1 \leqq x_2$, $x_2 \leqq x_1$ and for $i = 1, 2, 3$, $x_i \leqq x_i$. Let $I = \langle X, \leqq, Z_2\rangle$, where, as is usual, Z_2 is the field consisting of 2-elements. I is isomorphic to a subring of $\mathrm{M}(3, Z_2)$ by Lemma 1.5. By construction of incidence rings and by definition of this particular pre-ordered system $\langle X, \leqq\rangle$, I is a ring with $2^5 = 32$ members.

Consider the interval $[x_1, x_1] = \{x_1, x_2\}$. The interval ring $I[x_1, x_1]$ is isomorphic to $\mathrm{M}(2, Z_2)$ and has $2^4 = 16$ members. The units of $\mathrm{M}(2, Z_2)$ are the matrices:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

and the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

We note the following equation relating three of these invertible matrices:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{1}$$

By construction of I, if $\mathsf{m}(I)$ is the subring of $\mathsf{M}(3, Z_2)$ isomorphic to I according to Lemma 1.5, then the units of $\mathsf{m}(I)$ are the matrices:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. If the interval ring $I[x_1, x_1]$ is isomorphic to a subring of I, then $\mathsf{M}(2, Z_2)$ is isomorphic to a subring of $\mathsf{m}(I)$. The ring $\mathsf{M}(2, Z_2)$ has 6 units and $\mathsf{m}(I)$ has 6 units, therefore the units of $\mathsf{M}(2, Z_2)$ are mapped onto the units of $\mathsf{m}(I)$. Let i_1 be the identity matrix of $\mathsf{M}(2, Z_2)$ and let i_2 be the identity matrix of $\mathsf{m}(I)$. Then the isomorphism maps i_1 to i_2. Further, suppose that the invertible matrices $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ of $\mathsf{M}(2, Z_2)$ are mapped to invertible matrices j_1, j_2 respectively of $\mathsf{m}(I)$. Equation (1) and properties of ring isomorphisms require that

$$j_1 + j_2 = i_2. \tag{2}$$

Equation (2) is impossible to fulfill in $\mathsf{m}(I)$ because, by inspection of the units of $\mathsf{m}(I)$, the addition of any two of these units gives a matrix with zeros in its third row and i_2 has a non-zero entry in its third row.

This shows that it is impossible that $\mathsf{M}(2, Z_2)$ is isomorphic to a subring of $\mathsf{m}(I)$ and therefore, it is impossible that $I[x_1, x_1]$ is isomorphic to a subring of I.

Theorem 1.13. *Let $\langle X, \leqq\rangle$ be a locally finite pre-ordered system and R a ring. For every $f \in I = \langle X, \leqq, R\rangle$, f is invertible in I iff for every interval ring $I[u, v]$ of I, $f|_{[u,v]}$ is invertible in $I[u, v]$.*

Proof. First suppose that f is invertible in I and let $h = f^{-1}$. Let $u, v \in X$ and suppose $u \leqq v$. Let $h_1 = h|_{[u,v]}$ and $f_1 = f|_{[u,v]}$. Then $f_1^{-1} = h_1$, for if $x, y \in [u, v]$, then in $I[u, v]$

$$\begin{aligned}(h_1 f_1)(x, y) &= \sum_{s \in [u,v]} h_1(x, s)\, f_1(s, y) \\ &= \sum_{s \in X} h_1(x, s)\, f_1(s, y) \quad \text{as } \leqq \text{ is a pre-order and by the order condition.} \\ &= \sum_{s \in X} h(x, s)\, f(s, y) \quad \text{by definition of } f_1,\ h_1. \\ &= \delta(x, y)\end{aligned}$$

Thus $h_1 f_1 = \delta$ and similarly $f_1 h_1 = \delta$ in $I[u, v]$; so $f_1^{-1} = h_1$ and $f|_{[u,v]}$ is invertible in $I[u, v]$. Conversely suppose that the restriction of f is invertible in every interval ring of I. Define $h \in I$ as follows: for every $x, y \in X$

$$h(x, y) = \begin{cases} g(x, y), & \text{if } x \leqq y \text{ and } g \text{ is the inverse of } f|_{[x,y]} \text{ in } I[x, y]. \\ 0, & \text{if } x \nleqq y. \end{cases}$$

Clearly $h(x, y) \neq 0$ only if $x \leqq y$ and $h \in I$. Let $x, y, u, v \in X$ and suppose

$[x, y] \subseteq [u, v]$. Let g_1 and g_2 be the inverses of $f_1 = f|_{[x,y]}$ and $f_2 = f|_{[u,v]}$ in $I[x, y]$ and $I[u, v]$ respectively. Clearly $f_1 = f_2|_{[x,y]}$ since $[x, y] \subseteq [u, v]$; we show that $g_1 = g_2|_{[x,y]}$. Let $w, z \in [x, y]$, then $w, z \in [u, v]$ and

$$\delta(w, z) = \sum_{t \in [x,y]} g_1(w, t)\, f_1(t, z) \tag{1}$$

$$\begin{aligned} \delta(w, z) &= \sum_{t \in [u,v]} g_2(w, t)\, f_2(t, z) \\ &= \sum_{t \in [x,y]} g_2(w, t)\, f_1(t, z) \end{aligned} \tag{2}$$

by the order condition and $f_1 = f_2|_{[x,y]}$.

These two equations hold for every $w, z \in [x, y]$. Since $[x, y]$ is a finite set, of cardinality n say, Lemma 1.5 shows that $I[x, y]$ is isomorphic to a subring of $\mathrm{M}(n, R)$. It is well known that inverse matrices are unique if they exist. Hence inverses in $I[x, y]$ are unique if they exist. In particular $g_2|_{[x,y]} = g_1$ by equations (1), (2) and this latter remark. From this we deduce:

(3) *if $u, v \in X$, $u \leqq v$ and if g is the inverse of $f|_{[u,v]}$ in $I[u, v]$, then for every $x, y \in [u, v]$, $h(x, y) = g(x, y)$.*

Let $u, v \in X$ and let g be the inverse of $f|_{[u,v]}$.

$$\begin{aligned} (fh)(u, v) &= \sum_{t \in [u,v]} f(u, t)h(t, v) \\ &= \sum_{t \in [u,v]} f(u, t)g(t, v) \qquad \text{by (3)} \\ &= \delta(u, v). \end{aligned}$$

So $fg = \delta$ and similarly $hf = \delta$, showing that $h = f^{-1}$ and that f is invertible in I.

We shall later improve Theorem 1.13 by showing that we only need consider interval rings of the type $I[u, u]$ instead of considering all interval rings, to determine whether or not a function is invertible in the parent incidence ring I, provided R is commutative. Having shown that invertibility in an incidence ring I is equivalent to invertibility in all of the interval rings of I and hence by Lemma 1.5, equivalent to the invertibility of a collection of finite matrices in specified subrings of $\mathrm{M}(n, R)$, for various n, it is desirable to investigate invertibility criteria for interval rings. Since each interval ring of I is an incidence ring of a finite pre-ordered system, such criteria are given by the next theorem which gives an effective computational device for determining whether or not a function f belonging to $I = \langle X, \leqq, R\rangle$ is invertible whenever X is a finite set and R a commutative ring. If X has n members and m is the mapping of Definition 1.4, then invertibility in $\mathrm{m}(I)$ is equivalent to invertibility in $\mathrm{M}(n, R)$.

Theorem 1.14. *Let $X = \{x_1, \ldots, x_n\}$ and let $\langle X, \leqq\rangle$ be a pre-ordered system. Let R be a commutative ring and $I = \langle X, \leqq, R\rangle$. If $f \in I$, then $f \in I^*$ iff* $\det(\mathrm{m}(f)) \in R^*$.

Proof. If $f \in I^*$, then $\mathrm{m}(f^{-1}) = (\mathrm{m}(f))^{-1}$ since m is a ring isomorphism by Lemma 1.5. So $\mathrm{m}(f)$ is invertible in $\mathrm{m}(I)$ which is a subring of $\mathrm{M}(n, R)$. This shows that $\mathrm{m}(f)$ is invertible in $\mathrm{M}(n, R)$ and by a well known ring theoretic result $\det(\mathrm{m}(f)) \in R^*$. Conversely, suppose $\det(\mathrm{m}(f)) = u$ and $u \in R^*$. Then $\mathrm{m}(f)$ is invertible in $\mathrm{M}(n, R)$. It remains to show that $\mathrm{m}(f)^{-1} \in \mathrm{m}(I)$. Let F_{ij} be the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and j-th column of $\mathrm{m}(f)$. Then

$$(\mathrm{m}(f)^{-1})_{ij} = (-1)^{i+j} \det(F_{ji}) u^{-1}.$$

If $\mathrm{S}(j, i) = \{\sigma \in \mathrm{S}_n \mid \sigma(j) = i\}$ then $\det(F_{ji})$ is the sum of terms of the form

$$(1) \qquad \varepsilon(\sigma) f(x_1, x_{\sigma(1)}) \ldots f(x_{j-1}, x_{\sigma(j-1)}) f(x_{j+1}, x_{\sigma(j+1)}) \ldots f(x_n, x_{\sigma(n)})$$

where $\varepsilon(\sigma) = \pm 1$ depending on σ. If $(\mathrm{m}(f^{-1}))_{ij} \neq 0$ then for at least one $\sigma \in \mathrm{S}(j, i)$ there is a non-zero product of the form (1). By definition of F_{ji} no factor $f(x_p, x_q)$ where $p = j$ or $q = i$ occurs in any summand.

To show that $\mathrm{m}(f)^{-1} \in \mathrm{m}(I)$ it suffices to show that $(\mathrm{m}(f)^{-1})_{ij} \neq 0$ only if $x_i \leqq x_j$. If $i = j$ there is no problem, since $\leqq$ is reflexive. Now suppose that $i \neq j$ and that $(\mathrm{m}(f)^{-1})_{ij} \neq 0$. The factor $f(x_i, x_{\sigma(i)})$ appears in (1), so, by the order condition, $x_i \leqq x_{\sigma(i)}$. If $\sigma(i) = j$, the proof is complete. If $\sigma(i) \neq j$, the factor $f(x_{\sigma(i)}, x_{\sigma^2(i)})$ appears in (1) and $x_{\sigma(i)} \leqq x_{\sigma^2(i)}$. If $\sigma^2(i) = j$, then $x_i \leqq x_{\sigma(i)} \leqq x_{\sigma^2(i)} = x_j$ and $x_i \leqq x_j$, by the transitivity of $\leqq$. If $\sigma^2(i) \neq j$, $\sigma^3(i) \neq j$ and so on; we obtain:

$$x_i \leqq x_{\sigma(i)} \leqq \ldots \leqq x_{\sigma^{n-1}(i)}.$$

If for some p, q such that $1 \leqq p \leqq p + q \leqq n - 1$ we have $\sigma^p(i) = \sigma^{p+q}(i)$, then $i = \sigma^q(i)$ and the factor $f(x_{\sigma^{q-1}(i)}, x_i)$ occurs in (1); this is not possible. So the integers $i, \sigma(i), \ldots, \sigma^{n-1}(i)$ are distinct and for p such that $1 < p \leqq n - 1$ we have $\sigma^p(i) = j$. By the transitivity of $\leqq$, $x_i \leqq x_j$. Thus $\mathrm{m}(f)^{-1} \in \mathrm{m}(I)$ and since m is an isomorphism, f^{-1} exists in I and $f \in I^*$.

The following Theorem enables us to use Theorem 1.14 to improve Theorem 1.13 in the case where R is a commutative ring.

THEOREM 1.15. *Let $\langle X, \leqq \rangle$ be a locally finite pre-ordered system, R a commutative ring and $I = \langle X, \leqq, R \rangle$. Let Y be any finite subset of X such that Y is the disjoint union of intervals of the type $[y, y]$. Suppose that $Y = \dot{\bigcup}_{i=1}^{n} [y_i, y_i]$. Let $\leqq_Y$ be the relation $\leqq$ restricted to Y and let $I_Y = \langle Y, \leqq_Y, R \rangle$. If m is the mapping of* Definition 1.4, *then for every $f \in I_Y$ we have:*

$$\det(\mathrm{m}(f)) = \prod_{i=1}^{n} \det(f|_{[y_i, y_i]}).$$

Proof. The proof is by induction on n. If $n = 1$ then $Y = [y_1, y_1]$, $f = f|_{[y_1, y_1]}$ and the result is clearly true. Suppose, inductively, that the result holds whenever Y is the disjoint union of at most $n - 1$ intervals of the type $[y, y]$. Let $y_1, \ldots, y_n \in X$ be such that intervals $[y_i, y_i]$ are pairwise disjoint. Let $Y = \dot{\bigcup}_{i=1}^{n} [y_i, y_i]$ and $W = \dot{\bigcup}_{i=2}^{n} [y_i, y_i]$. Without loss of generality we may suppose that the y_i are enumerated such that:

(1) $y_1 \leqq y_i$ for $2 \leqq i \leqq p$
(2) $y_i \leqq y_1$ for $p + 1 \leqq i \leqq p + q$
(3) neither $y_1 \leqq y_i$ nor $y_i \leqq y_1$ for $p + q + 1 \leqq i \leqq n = p + q + r$.

The set Y is a disjoint union of intervals so that (1) and (2) imply:

(4) $y_i \nleqq y_1$ for $2 \leqq i \leqq p$.
(5) $y_1 \nleqq y_i$ for $p + 1 \leqq i \leqq p + q$.

If for particular i, j such that $2 \leqq i \leqq p$ and $p + 1 \leqq j \leqq n$ we had $y_i \leqq y_j$, then by (1) and the transitivity of $\leqq$ we would have $y_1 \leqq y_j$. This contradicts (5) if $p + 1 \leqq j \leqq p + q$ and (3) if $p + q + 1 \leqq j \leqq n$; hence

(6) $y_i \nleqq y_j$ for $2 \leqq i \leqq p$ and $p + 1 \leqq j \leqq p + q + r = n$.

If for particular i, j such that $p + q + 1 \leqq i \leqq n$ and $p + 1 \leqq j \leqq p + q$ we had $y_i \leqq y_j$, then by (2) and the transitivity of $\leqq$ we would have $y_i \leqq y_1$, a contradiction of (3). Hence:

(7) $y_i \nleqq y_j$ for $p + q + 1 \leqq i \leqq n$ and $p + 1 \leqq j \leqq p + q$.

Each interval $[y_i, y_i]$ is a subset of Y. Let us suppose that $Y = \{x_1, \ldots, x_u\}$ and that

(8) $[y_1, y_1] = \{x_1, \ldots, x_s\} = W_\alpha$

(9) $\dot{\bigcup}_{i=2}^{p} [y_i, y_i] = \{x_{s+1}, \ldots, x_{s+t}\} = W_\beta$

(10) $\dot{\bigcup}_{i=p+1}^{p+q} [y_i, y_i] = \{x_{s+t+1}, \ldots, x_{s+t+u}\} = W_\gamma$

(11) $\dot{\bigcup}_{i=p+q+1}^{n} [y_i, y_i] = \{x_{s+t+u+1}, \ldots, x_{s+t+u+v}\} = W_\epsilon$

Of course $s + t + u + v = w$. If $s = 0$, the proof is complete by induction. If $t = 0$ or $u = 0$ or $v = 0$, then one or more of the inequalities (1), (2) or (3) is vacuously true and the proof will be simplified; here we are considering the most general case when $t \neq 0$, $u \neq 0$ and $v \neq 0$. There are eight cases to be proved but the other seven cases are very similar and simpler to prove. Let $f \epsilon I_Y$, then $\mathrm{m}(f)$ is an w **x** w matrix in $\mathrm{M}(w, R)$. We have;

(12) $(\mathrm{m}(f))_{ij} = f(x_i, x_j)$. for $1 \leqq i, j \leqq w$.

We express $\mathrm{m}(f)$ as a block of 16 matrices as follows:

s	t	u	v	
A_1	A_2	A_3	A_4	s
A_5	A_6	A_7	A_8	t
A_9	A_{10}	A_{11}	A_{12}	u
A_{13}	A_{14}	A_{15}	A_{16}	v

The square matrices A_1, A_6, A_{11}, A_{16} are respectively $\mathrm{m}(f|_{[y_1,y_1]})$, $\mathrm{m}(f|W_\beta)$, $\mathrm{m}(f|W_\gamma)$ and $\mathrm{m}(f|W_\epsilon)$. For example, A_7 has entries $(\mathrm{m}(f))_{ij}$ for $s+t+1 \leqq j \leqq s+t+u$ and $s+1 \leqq i \leqq s+t$; and so on for the remaining matrices. Define the $(t+u+v) \times (t+u+v)$ matrix B as follows:

$$B = \begin{array}{|c|c|c|} \hline A_6 & A_7 & A_8 \\ \hline A_{10} & A_{11} & A_{12} \\ \hline A_{14} & A_{15} & A_{16} \\ \hline \end{array}$$

Then $B = \mathrm{m}(f|_W)$ and by inductive hypothesis and definition of W we have:

$$\det(B) = \prod_{i=2}^{n} \det(\mathrm{m}(f|_{[y_i,y_i]}))$$

Of course, $\det(A_1) = \det(\mathrm{m}(f|_{[y_1,y_1]}))$; we shall show that

$$\det(\mathrm{m}(f)) = \det(A_1)\det(B)$$

and this will complete the proof by induction. The method is to show that the matrices A_3, A_4, A_5, A_7, A_8, A_{13} and A_{15} are all zero matrices and therefore that the computation of $\det(\mathrm{m}(f))$ is independent of the matrices A_2 and A_9. We may therefore set all entries of A_2 and A_9 equal to 0 to obtain a matrix C:

$$C = \begin{array}{|c|c|} \hline A_1 & 0 \\ \hline 0 & B \\ \hline \end{array}$$

such that $\det(\mathrm{m}(f)) = \det(C)$ and by a well known result in determinant theory, $\det(C) = \det(A_1)\det(B)$. To show that A_3, A_4, A_5, A_7, A_8, A_{13} and A_{15} have only zero entries we proceed as follows: By (3), (8) and (11) we have

(13) $x_i \not\leqq x_j$ and $x_j \not\leqq x_i$, for $1 \leqq i \leqq s$ and $s+t+u+1 \leqq j \leqq w$.

By (13) and the order condition on the functions in I_Y applied to f we have

(14) $f(x_i, x_j) = 0 = f(x_j, x_i)$ for $1 \leqq i \leqq s$ and $s+t+u+1 \leqq j \leqq w$.

Equations (12) and (14) show that the matrices A_4 and A_{13} are zero matrices. By (4), (8) and (9) we have:

(15) $x_i \not\leqq x_j$ for $s+1 \leqq i \leqq s+t$ and $1 \leqq j \leqq s$
(16) $f(x_i, x_j) = 0$ for $s+1 \leqq i \leqq s+t$ and $1 \leqq j \leqq s$ (order condition).

Equations (12) and (16) show that A_5 is a zero matrix. By (5), (8) and (10) we have:

(17) $x_i \not\leqq x_j$, for $1 \leqq i \leqq s$ and $s+t+1 \leqq j \leqq s+t+u$.
(18) $f(x_i, x_j) = 0$, for $1 \leqq i \leqq s$ and $s+t+1 \leqq j \leqq s+t+u$ (order condition).

Equations (12) and (18) show that A_3 is a zero matrix. By completely similar arguments we use (6), (9), (10), (12) and the order condition to show that A_7 is a zero matrix; we use (7), (11), (10), (12) and the order condition to show that A_{15} is a zero matrix. Thus A_3, A_4, A_5, A_7, A_8, A_{13} and A_{15} are zero matrices.

We now show that because the previously mentioned matrices are zero matrices, the entries of the matrices A_2 and A_9 do not affect the computation of $\det(\mathrm{m}(f))$. The ring R is commutative so we have:

(19) $$\det(\mathrm{m}(f)) = \sum_{\sigma\epsilon\mathcal{S}_w} \epsilon(\sigma) f(x_1, x_{\sigma(1)}) \ldots f(x_w, x_{\sigma(u)}).$$

Let $\sigma \epsilon \mathcal{S}_u$ and suppose that the summand corresponding to σ and appearing on the right side of equation (19) is non-zero. Referring to the diagram of $\mathrm{m}(f)$ and noting that A_5, A_7 and A_8 are zero matrices we require that

(20) $$\sigma(\{s+1, \ldots, s+t\}) = \{s+1, \ldots, s+t\}.$$

Noting that A_3 and A_4 are zero matrices we require that

(21) $$\sigma(\{1, \ldots, s\}) \subseteq \{1, \ldots, s+t\}.$$

Equations (20), (21) and the fact that σ is bijective imply that

(22) $$\sigma(\{1, \ldots, s\}) = \{1, \ldots, s\}.$$

Equation (22) shows that no entry of the matrix A_2 appears in any non-zero summand of (19). Further, noting that A_5 and A_{13} are zero matrices we require that

(23) $$\sigma(\{1, \ldots, s, s+t+1, \ldots, s+t+u\}) = \{1, \ldots, s, s+t+1, \ldots, s+t+u\}.$$

Equations (22), (23) and the fact that σ is bijective imply that

(24) $$\sigma(\{s+t+1, \ldots, s+t+u\}) = (s+t+1, \ldots, s+t+u\}.$$

Equation (24) shows that no entry of A_9 appears in any non-zero summand of (19). Hence if

$$C = \begin{array}{|c|c|}\hline A_1 & 0 \\ \hline 0 & B \\ \hline \end{array}\ ,$$

then

$$\begin{aligned}\det(\mathrm{m}(f)) &= \det(C) \\ &= \det(A_1)\det(B) \\ &= \det(\mathrm{m}(f|_{[y_1,y_1]})) \prod_{i=2}^{n} \det(\mathrm{m}(f|_{[y_i,y_i]})) \\ &= \prod_{i=1}^{n} \det(\mathrm{m}(f|_{[y_i,y_i]})).\end{aligned}$$

The proof is complete. Theorem 1.13 is improved as follows, if R is a commutative ring.

Theorem 1.16. *Let $\langle X, \leqq\rangle$ be a locally finite pre-ordered system and R a commutative ring. Let $I = \langle X, \leqq, R\rangle$. For every $f \epsilon I$, f is invertible in I iff for every $x \epsilon X$, $f|_{[x,x]}$ is invertible in $I[x,x]$.*

Proof. By Theorem 1.13 it suffices to show that the following conditions (1) and (2) are equivalent.

(1) for every $x \in X$, $f|_{[x,x]}$ is invertible in $I[x,x]$.
(2) for every u, $v \in X$ such that $u \leqq v$, $f|_{[u,v]}$ is invertible in $I[u,v]$.

Intervals are finite sets so Theorem 1.14 shows that (1) and (2) are equivalent to (3) and (4) respectively, where:

(3) for every $x \in X$, $\det(\mathrm{m}(f|_{[x,x]})) \in R^*$
(4) for every u, $v \in X$ such that $u \leqq v$, $\det(\mathrm{m}(f|_{[u,v]})) \in R^*$.

If u, $v \in X$ and $u \leqq v$, then the interval $[u,v]$ is a finite set which is a disjoint union of intervals of the form $[x,x]$. If $[u,v] = \dot{\bigcup}_{i=1}^{n} [y_i, y_i]$, then Theorem 1.15 shows that

$$\det(\mathrm{m}(f|_{[u,v]})) = \prod_{i=1}^{n} \det(\mathrm{m}(f|_{[y_i,y_i]})).$$

The group R^* is closed under multiplication so (3) and (4) are equivalent.

If $\langle X, \leqq \rangle$ is a locally finite partially ordered system, F a field and $I = \langle X, \leqq, F \rangle$, then a function f in I is invertible iff for every $x \in X$, $f(x,x) \neq 0$, according to the previous theorem, since $[x,x] = \{x\}$ and $F^* = F - \{0\}$. This result has been proved already in Smith [4]. The proof is easy to give because $\leqq$ is a partial ordering: If f is invertible, then

$$1 = (ff^{-1})(x,x) = f(x,x)f^{-1}(x,x)$$

so that $f(x,x) \neq 0$. Conversely one may show by induction on the length of intervals that if $f(x,x) \neq 0$ for all $x \in X$, then

$$f^{-1}(x,y) = -(f(y,y))^{-1}\left[\sum_{x \leqq z < y} f^{-1}(x,z)\, f(z,y)\right].$$

Accordingly, Theorem 1.16 is the precise generalization of this known result for incidence rings of partially ordered sets over a field to incidence rings of pre-ordered sets over a commutative ring. From the previous theorem we may easily derive the following corollary.

Corollary 1.17. *Let $\langle X, \leqq \rangle$ be a locally finite pre-ordered system, R a commutative ring, $I = \langle X, \leqq, R \rangle$ and let $f \in I^*$. If x, $y \in X$ are such that $x \neq y$, $x \leqq y$ and $y \leqq x$, then there are u, $v \in X$ such that*

(1) $f(x,u) \neq f(y,u)$ and $x \leqq u \leqq y$
(2) $f(v,x) \neq f(v,y)$ and $x \leqq v \leqq y$.

Proof. $[x,x] = [y,y]$ and because $f \in I^*$, Theorem 1.16 shows that if $g = f|_{[x,x]}$, then $g \in I[x,x]^*$. Let $[x,x] = \{x_1, \ldots, x_n\}$ and $x_1 = x$, $x_n = y$. The matrix $\mathrm{m}(g)$ is invertible in $\mathrm{M}(n,R)$. If for every $u \in [x,x]$ we had $f(x,u) = f(y,u)$, then the first and n-th rows of $\mathrm{m}(g)$ would be identical and $\mathrm{m}(g)$ would not be invertible. Hence (1) holds. Similarly (2) holds by considering the first and last columns of $\mathrm{m}(g)$.

Let $I = \langle X, \leqq, R \rangle$ be an incidence ring such that R is not necessarily commutative. Let Y be a subset of X and let $I_Y = \langle Y, \leqq, R \rangle$. If $f \in I$, $f|_Y$ is the restriction of f to Y, so that $f|_Y \in I_Y$, and if $f|_Y$ is invertible in I_Y, then it is possible to find a function g in I^* such that $g|_Y$ is the inverse of $f|_Y$ in I_Y.

Theorem 1.18. (Inverse function theorem for incidence rings) *Let $\langle X, \leqq\rangle$ be a locally finite pre-ordered system, R a ring, $I = \langle X, \leqq, R\rangle$, Y a subset of X, $I_Y = \langle Y, \leqq, R\rangle$ and let $f \in I$. If $f|_Y$ is invertible in I_Y, then there is a $g \in I^*$ such that $g|_Y$ is the inverse of $f|_Y$ in I_Y.*

Proof. Let h belong to I_Y and suppose h is the inverse of $f|_Y$. Define g as follows. For every $x, y \in X$,

$$g(x, y) = \begin{cases} h(x, y) \text{ if } x, y \in Y \\ \delta(x, y) \text{ otherwise.} \end{cases}$$

If $x \not\leqq y$, then $h(x, y) = 0 = \delta(x, y)$ so that g satisfies the order condition and $g \in I$. Also, $g|_Y = h$ so that $g|_Y$ is the inverse of $f|_Y$. We show that $g \in I^*$ by constructing the inverse of g. Define the function g' as follows. For every $x, y \in X$,

$$g'(x, y) = \begin{cases} f(x, y) \text{ if } x, y \in Y \\ \delta(x, y) \text{ otherwise.} \end{cases}$$

We show that $gg' = \delta = g'g$. Let $x, y \in X$.

Case (1). $x, y \in Y$.

$$(gg')(x, y) = \sum_{t \in Y} h(x, t)\, f(t, y) = \delta(x, y).$$

Case (2). $x \notin Y$ and $y \notin Y$.

$$(gg')(x, y) = \sum_{t \in X} \delta(x, t)\delta(t, y) = \delta(x, y).$$

Case (3). $x \in Y$ and $y \notin Y$.

$$\begin{aligned}(gg')(x, y) &= \sum_{t \in Y} g(x, t)\, g'(t, y) + \sum_{t \in X-Y} g(x, t)\, g'(t, y) \\ &= \sum_{t \in Y} h(x, t)\delta(t, y) + \sum_{t \in X-Y} \delta(x, t)\delta(t, y) \\ &= 0 + \delta(x, y) = \delta(x, y).\end{aligned}$$

Case (4). $x \notin Y$ and $y \in Y$.

$$\begin{aligned}(gg')(x, y) &= \sum_{t \in Y} \delta(x, t)\, f(t, y) + \sum_{t \in X-Y} \delta(x, t)\delta(t, y) \\ &= 0 + \delta(x, y) = \delta(x, y).\end{aligned}$$

In all four cases $gg' = \delta$. Similarly $g'g = \delta$ and therefore $g' = g^{-1}$ and $g \in I^*$.

A question of some interest is the following. If R is a ring, $\langle X', \leqq'\rangle$ a locally finite pre-ordered system, X a subset of X' and $\leqq$ the relation $\leqq'$ restricted to X, what are the relationships between the algebraic structures $\langle X', \leqq', R\rangle$ and $\langle X, \leqq, R\rangle$? Example 1.10 shows that if $I = \langle X, \leqq, R\rangle$ and $I' = \langle X', \leqq', R\rangle$, then I is not necessarily a subring of I'. However, the multiplicative group I^* is isomorphic to a subgroup of $(I')^*$.

Theorem 1.19. *Let $\langle X', \leqq'\rangle$ be a locally finite pre-ordered system, R a* *ring*, $X \subseteq X'$ *and* $\leqq$ *the relation* $\leqq'$ *restricted to* X. *Let* $I = \langle X, \leqq, R\rangle$ *and* $I' = \langle X', \leqq', R\rangle$. *Then* I^* *is isomorphic to a subgroup of* $(I')^*$.

Proof. We define an injective group homomorphism $\alpha: I^* \to (I')^*$ as follows. If $f \in I^*$, then for every $x, y \in X'$

$$\alpha(f)(x, y) = \begin{cases} f(x, y) \text{ if } x, y \in X \\ \delta(x, y) \text{ otherwise.}\end{cases}$$

By construction of $\alpha(f)$ and the proof of Theorem 1.18, $\alpha(f) \in (I')^*$. Let $f, g \in I^*$ and $x, y \in X'$. Then,

$$\alpha(fg)(x, y) = \begin{cases} (fg)(x, y) \text{ if } x, y \in X \\ \delta(x, y) \quad \text{otherwise.}\end{cases}$$

We show that $\alpha(fg) = \alpha(f)\alpha(g)$.
Case (1). $x, y \in X$.

$$\begin{aligned}\alpha(f)\alpha(g)(x, y) &= \sum_{t \in X'} \alpha(f)(x, t)\alpha(g)(t, y)\\ &= \sum_{t \in X} f(x, t)g(t, y)\\ &= (fg)(x, y) = \alpha(fg)(x, y)\end{aligned}$$

Case (2). $x \notin X$ and $y \notin X$.

$$\begin{aligned}\alpha(f)\alpha(g)(x, y) &= \sum_{t \in X'} \delta(x, t)\delta(t, y)\\ &= \delta(x, y) = \alpha(fg)(x, y)\end{aligned}$$

Case (3). $x \in X$ and $y \notin X$.

$$\begin{aligned}\alpha(f)\alpha(g)(x, y) &= \sum_{t \in X} \alpha(f)(x, t)\alpha(g)(t, y) + \sum_{t \in X' - X} \alpha(f)(x, t)\alpha(g)(t, y)\\ &= \sum_{t \in X} f(x, t)\delta(t, y) + \sum_{t \in X' - X} \delta(x, t)\delta(t, y)\\ &= 0 + \delta(x, y) = \alpha(fg)(x, y).\end{aligned}$$

Case (4). $x \notin X$ and $y \in X$. The computation is the same as Case (3). This shows that $\alpha(fg) = \alpha(f)\alpha(g)$ for every f, g in I^*.

If $\alpha(f) = \delta$, then $X = \emptyset$ or $f = \delta$; but if $X = \emptyset$ then $I = \emptyset$, so $f = \delta$. Conversely it is clear that $\alpha(\delta) = \delta$. This shows that α is an injective group homomorphism, and that I^* is isomorphic to $\alpha(I^*)$.

The map α of the previous theorem is not a homomorphism of the additive group of I into the additive group of I' because $\alpha(0) \neq 0$.

Definition 1.20. If $\leqq$ is any pre-order relation on a set X, then $\leqq^c$ is the converse relation on X defined as follows: for every $x, y \in X$, $x \leqq^c y$ iff $y \leqq x$.

It is easy to see that if $\langle X, \leqq\rangle$ is a locally finite pre-ordered relational system, then $\langle X, \leqq^c\rangle$ is a locally finite pre-ordered relational system and conversely. Let R be a commutative ring, $I = \langle X, \leqq, R\rangle$ and $I^c = \langle X, \leqq^c, R\rangle$. We shall give an example at the end of section 2 to show that I and I^c are not necessarily isomorphic rings. However it is true that the groups of units of I and I^c are isomorphic as we now show.

Theorem 1.21. *Let $\langle X, \leqq\rangle$ be a locally finite pre-ordered system, R a commutative ring, $I = \langle X, \leqq, R\rangle$ and let $I^c = \langle X, \leqq^c, R\rangle$. Then the multiplicative groups I^* and I^{c*} are isomorphic.*

Proof. $\leqq^c$ is the converse relation of $\leqq$ and we shall denote by $[x, y]^c$ the interval $\{z | x \leqq^c z \leqq^c y\}$ of $\langle X, \leqq^c\rangle$. It is easy to see that for every $x, y \in X$, $[x, y] = [y, x]^c$. We shall define a function c: $I^* \to I^{c*}$ as follows: for every $f \in I^*$ and every $x, y \in X$

$$f_c(x, y) = f^{-1}(y, x).$$

Then $f_c(x, y) \neq 0$ only if $x \leqq^c y$ by the order condition on I, the fact that $f^{-1} \in I$ and by definition of $\leqq^c$. Therefore, c is a function from I^* into I^c. If $x \in X$, then $[x, x]^c = [x, x]$. If $[x, x] = \{x_1, \ldots, x_n\}$ and if m is the map of Definition 1.4, then for $1 \leqq i, j \leqq n$ we have

$$(\mathrm{m}(f_c|_{[x,x]^c}))_{ij} = f_c(x_i, x_j) = f^{-1}(x_j, x_i) = (\mathrm{m}(f^{-1}|_{[x,x]}))_{ji}.$$

In other words,

$$\mathrm{m}(f_c|_{[x,x]^c}) = (\mathrm{m}(f^{-1}|_{[x,x]}))^t.$$

By a well known result in determinant theory

$$\det(\mathrm{m}(f_c|_{[x,x]^c})) = \det(\mathrm{m}(f^{-1}|_{[x,x]})^t) = \det(\mathrm{m}(f^{-1}|_{[x,x]})).$$

Since $f^{-1} \in I^*$, this computation yields a member of R^*, so by Theorem 1.16, f_c is invertible in I^{c*}. Hence the mapping c takes I^* into I^{c*}. If g is any member of I^{c*} let h be the member of I defined as follows: for every $x, y \in X$

$$h(x, y) = g^{-1}(y, x).$$

From the previous discussion it is clear that $h \in I^*$ and that $h_c = g$, so the map c is surjective. Furthermore it is clear that for every $f \in I^*$, $f_c = \delta$ iff $f = \delta$ and the map is injective. It remains to show that $(fg)_c = f_c g_c$. Let $f, g \in I^*$ and let $x, y \in X$. Then

$$\begin{aligned}
(fg)_c(x, y) &= (fg)^{-1}(y, x) = (g^{-1}f^{-1})(y, x) \\
&= \sum_{y \leqq z \leqq x} g^{-1}(y, z) f^{-1}(z, x) \\
&= \sum_{x \leqq^c z \leqq^c y} g_c(z, y) f_c(x, z) \\
&= \sum_{x \leqq^c z \leqq^c y} f_c(x, z) g_c(z, y) && R \text{ is commutative} \\
&= (f_c g_c)(x, y).
\end{aligned}$$

Thus $f_c g_c = (fg)_c$ and c: $I^* \to I^{c*}$ is a group isomorphism, as required.

Definition 1.22. If R is a ring, $Z(R)$ denotes the center of R.

Theorem 1.23. *$\langle X, \leqq\rangle$ is a locally finite pre-ordered system, R a ring and $I = \langle X, \leqq, R\rangle$. Then,*

$Z(I) = \{f \in I \mid$ *for every* $x, y \in X$; $f(x, y) = 0$ *if* $x \neq y$ *and for every* $h \in I$; $h(x, y) f(y, y) = f(x, x) h(x, y)\}$.

Proof. Let $f \in Z(I)$, $x, y \in X$, $x \leqq y$ and $x \neq y$. Define a function h as follows. For every $u, v \in X$

$$h(u, v) = \begin{cases} 1 \text{ if } u = v = y \\ 0 \text{ otherwise.} \end{cases}$$

Then $h \in I$ and $hf = fh$. However,

$$(fh)(x, y) = f(x, y)h(y, y) = f(x, y)$$
$$(hf)(x, y) = h(x, x)f(x, y) = 0.$$

So $f(x, y) = 0$ if $x \neq y$. Let g be any member of I; since $fg = gf$ we have, for every $x, y \in X$

$$(fg)(x, y) = f(x, x)g(x, y)$$
$$(gf)(x, y) = g(x, y)f(y, y)$$

as required. Conversely suppose that $f \in I$ and f satisfies the conditions: for every $x, y \in X$ and $g \in I$,

$$f(x, y) = 0 \text{ if } x \neq y, \text{ and } f(x, x)g(x, y) = g(x, y)f(y, y).$$

Then,

$$(fg)(x, y) = f(x, x)g(x, y) = g(x, y)f(y, y) = (gf)(x, y)$$

showing that $gf = fg$ and hence, $f \in Z(I)$.

We show that in only very simple cases is I a commutative ring or a field. Namely:

Corollary 1.25. *$\langle X, \leqq \rangle$ is a locally-finite pre-ordered system, R a ring and $I = \langle X, \leqq, R \rangle$. Then I is a commutative ring iff R is commutative and the relation $\leqq$ is the equality relation on X.*

Proof. I is commutative iff $Z(I) = I$. Suppose I is commutative, then $Z(I) = I$. If $x, y \in X$ and $x \leqq y$, there is a function $f \in I$ such that $f(x, y) = 1$. By Theorem 1.21, $f(x, y) = 0$ if $x \neq y$, hence $x = y$. So the relation $\leqq$ is the equality relation on X.

Let $b, c \in R$ and let $x \in X$. There are functions $f, g \in I$ such that $f(x, x) = b$ and $g(x, x) = c$. However, $gf = fg$ so that

$$(fg)(x, x) = f(x, x)g(x, x) = (gf)(x, x) = g(x, x)f(x, x).$$

Thus, $bc = cb$ and R is therefore commutative. Conversely, suppose that R is commutative and that $\leqq$ is the equality relation on X. For every $f \in I$ and $x, y \in X$, $f(x, y) = 0$ if $x \neq y$ by the order condition. Also if $g \in I$ then $f(x, x)g(x, x) = g(x, x)f(x, x)$, as R is commutative. By Theorem 1.23, $f, g \in Z(I)$ and therefore $Z(I) = I$.

Corollary 1.25. *Let $I = \langle X, \leqq, R \rangle$. Then I is a field iff $X = \{x\}$ and R is a field.*

Proof. It is clear that if R is a field and $X = \{x\}$, then I is isomorphic to R, and I is a field. Conversely suppose that I is a field, in particular I is a commutative ring, $\leqq$ is the equality relation on X and R is a commutative ring; by the previous theorem. If X contains two distinct members, say x and y, let h be defined as follows. For every $u, v \in X$

$$h(u, v) = \begin{cases} 1 \text{ if } u = v = x \\ 0 \text{ otherwise.} \end{cases}$$

$h \in I$ and $h|_{[y,y]} = 0$. Since R is commutative, Theorem 1.16 shows that $h \notin I^*$; but $h \neq 0$ so I is not a field. This contradiction shows that $X = \{x\}$, a singleton. If $f \neq 0$, then $f(x,x)f^{-1}(x,x) = 1$ so that $f(x,x)$ is invertible in R if $f(x,x)$ is non-zero. Every non-zero member of R is $f(x,x)$ for some $f \in I$ so that R is a field.

According to Corollary 1.29, some interesting inversion formulas may be derived whenever the zeta function of an incidence ring is invertible. For this reason we turn our attention to showing that the zeta function of an incidence ring is invertible iff the underlying pre-order relation is a partial order. The 'if' part of the following theorem has been proved in Rota [2]. The 'only if' part has been proved in Tainiter [6] provided that X is a finite set. The following theorem does not require X to be a finite set and its proof differs from both of the latter proofs. First we give a useful Lemma.

Lemma 1.26. *Let X be a finite set, $\langle X, \leqq \rangle$ a pre-ordered system, $I = \langle X, \leqq, R \rangle$ and let ζ be the zeta function of I. Then*

$$\det(\mathrm{m}(\zeta)) = \prod_{x \in X} \zeta(x, x) = 1,$$

if $\leqq$ is a partial order and $\det(\mathrm{m}(\zeta)) = 0$ if $\leqq$ is not a partial order.

Proof. Let $X = \{x_1, \ldots, x_n\}$. For $1 \leqq i, j \leqq n$, $\mathrm{m}(\zeta)_{ij} = \zeta(x_i, x_j)$. This is 0 or 1 according as $x_i \nleqq x_j$ or $x_i \leqq x_j$. If $\leqq$ is not a partial ordering, there are i, j such that $1 \leqq i, j \leqq n$, $i \neq j$ and $\zeta(x_i, x_j) = 1 = \zeta(x_j, x_i)$. This shows that the i-th and j-th rows of $\mathrm{m}(\zeta)$ are identical. Hence $\det(\mathrm{m}(\zeta)) = 0$. On the other hand suppose that $\leqq$ is a partial ordering. We have:

(1) $\det(\mathrm{m}(\zeta)) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \zeta(x_1, x_{\sigma(1)}) \ldots \zeta(x_n, x_{\sigma(n)})$.

Because the relation $\leqq$ is reflexive we have $\zeta(x_1, x_1) \ldots \zeta(x_n, x_n) = 1$. We show that this is the only non-zero summand of (1). For suppose $\sigma \in S_n$, σ is not the identity permutation and that

(2) $\zeta(x_1, x_{\sigma(1)}) \ldots \zeta(x_n, x_{\sigma(n)}) \neq 0$.

Let $S = \{i \mid \sigma(i) \neq i\} = \{i_1, \ldots, i_r\}$. S is non-empty because σ is not the identity permutation, so that $1 < r \leqq n$. If $j \in S$ and $\sigma(j) \notin S$ then $\sigma^2(j) = \sigma(j)$ and therefore $\sigma(j) = j$, a contradiction. Therefore, if $j \in S$ then $\sigma(j) \in S$. For $1 \leqq t \leqq r$ we have $x_{i_t} \leqq x_{\sigma(i_t)}$ and

(3) $x_{i_1} \leqq x_{\sigma(i_1)} \leqq \ldots \leqq x_{\sigma^r(i_1)}$.

For $0 \leqq t \leqq r$, $\sigma^t(i_1) \in S$. The set S contains r elements and (3) lists $r + 1$ elements so for some p, q such that $1 \leqq p < p + q \leqq r$ and $q > 1$ we have

(4) $\sigma^{p}(i_1) = \sigma^{p+q}(i_1)$ and $i_1 = \sigma^{q}(i_1)$.

By the transitivity of $\leqq$, (4) and (3) we have $x_{i_1} \leqq x_{\sigma(i_1)} \leqq x_{i_1}$, which contradicts the antisymmetry of $\leqq$. So if σ is not the identity permutation, then equation (2) is false and

$$\det(\mathrm{m}(\zeta)) = \prod_{x \in X} \zeta(x,x) = 1.$$

Theorem 1.27. *Let $\langle X, \leqq\rangle$ be a locally finite pre-ordered system, R a commutative ring and $I = \langle X, \leqq, R\rangle$. If ζ is the zeta function of I then ζ is invertible iff $\leqq$ is a partial order.*

Proof. If $\langle X, \leqq\rangle$ is a partially ordered system, then for every $x \in X$, $[x,x] = \{x\}$. Also, $\zeta(x,x) = 1$, so ζ is invertible in $I[x,x]$ for every $x \in X$. By Theorem 1.16 ζ is invertible in I. Conversely suppose that $\langle X, \leqq\rangle$ is not a partially ordered system. Then there are $u, v \in X$ such that $u \leqq v$, $v \leqq u$ and $v \neq u$. For every $x, y \in [u,u]$, $\zeta(x,y) = 1$ so that $\det(\mathrm{m}(\zeta|_{[u,u]})) = 0$ and $\zeta|_{[u,u]}$ is not invertible in $I[u,u]$ by Theorem 1.14. By Theorem 1.16, ζ is not invertible in I.

Recall that the Möbuis function of an incidence ring is the inverse of the zeta function. The previous theorem shows that an incidence ring of a locally finite pre-ordered set over a commutative ring contains a Möbuis function iff the pre-order relation is a partial ordering.

Interest in functions which are invertible in incidence rings is motivated by the following theorem and its corollary. The theorem is the corresponding generalization of Proposition 2 and Corollary 1 of section 3 in Rota [2]. In that paper incidence rings are of the form $\langle X, \leqq, R\rangle$ where R is the field of real numbers and $\langle X, \leqq\rangle$ is a locally finite partially ordered system; however, the proof is the same as the proof in Rota [2], since both are statements of the equation $ff^{-1} = \delta$ for a function $f \in I^*$. The usefulness of the following theorem is increased due to the fact that the previous results allow us to construct all functions which are units in the incidence ring of a locally finite pre-ordered set over a commutative ring, whereas previously, only the units of incidence rings over locally finite partially ordered sets were known. The important Möbuis inversion theorem does not generalize to incidence rings of pre-ordered sets, as we have shown that an incidence ring contains a Möbuis function iff the underlying order is a partial order.

Theorem 1.28. *Let $\langle X, \leqq\rangle$ be a locally finite pre-ordered system, R a ring, $I = \langle X, \leqq, R\rangle$ and let $g, h \in I^*$. If P_0 and P_1 are functions mapping X into R such that for given $x_0, x_1 \in X$, $P_0(x) \neq 0$ only if $x \leqq x_0$ and $P_1(x) \neq 0$ only if $x_1 \leqq x$, and if Q_0 and Q_1 are defined by*

$$Q_0(x) = \sum_{y \in X} h(x,y)\, P_0(y),\quad Q_1(x) = \sum_{y \in X} P_1(y) g(y,x)$$

then

$$P_0(x) = \sum_{y \in X} h^{-1}(x,y)\, Q_0(y) \text{ and } P_1(x) = \sum_{y \in X} Q_1(y) g^{-1}(y,x).$$

Proof. Rota [2].

Corollary 1.29. (Möbuis Inversion) *Let $\langle X, \leqq\rangle$ be a locally finite partially ordered system, R a commutative ring and $I = \langle X, \leqq, R\rangle$. Let μ be the Möbuis function of I. If P_0 and P_1 are functions mapping X into R such that for given $x_0, x_1 \in X$ we have $P_0(x) \neq 0$ only if $x \leqq x_0$ and $P_1(x) \neq 0$ only if $x_1 \leqq x$; and if*

$$Q_0(x) = \sum_{x \leqq y} P_0(y), \; Q_1(x) = \sum_{y \leqq x} P_1(y)$$

then

$$P_0(x) = \sum_{y \in X} \mu(x, y)\, Q_0(y) \text{ and } P_1(x) = \sum_{y \in X} Q_1(y)\, \mu(y, x).$$

Proof. Rota [2].

2. *The Isomorphism Problem for Incidence Rings.* This section gives a partial solution to the following problem: if $\langle X, \leqq\rangle$ and $\langle X', \leqq'\rangle$ are locally finite pre-ordered systems, R a ring, $I = \langle X, \leqq, R\rangle, I' = \langle X', \leqq', R\rangle$ and if I and I' are isomorphic rings is it necessarily the case that the pre-ordered systems $\langle X, \leqq\rangle$, $\langle X', \leqq'\rangle$ are isomorphic? R. P. Stanley has shown in [3] and [5] that if R is a field, $\langle X, \leqq\rangle$ and $\langle X', \leqq'\rangle$ partially ordered systems and if I and I' are isomorphic rings, then the partially ordered systems are isomorphic. Parts of Stanley's proof make essential use of the anti-symmetric property of the partial orderings; for this reason a simple generalization of his proof does not seem possible. However, using a similar proof technique we have the following results:

Let F be a field and a topological space such that if $t \in F - \{0\}$ there is an open set U such that $0 \in U$ and $t \notin U$. Let $\langle X, \leqq\rangle$ and $\langle X', \leqq'\rangle$ be locally finite pre-ordered systems, $I = \langle X, \leqq, F\rangle$ and $I = \langle X', \leqq', F\rangle$. If ψ: $I \to I'$ is a ring isomorphism such that whenever K is a closed maximal 2-sided ideal in I, then $\psi(K)$ is a closed set in I', then the pre-ordered systems $\langle X, \leqq\rangle$ and $\langle X', \leqq'\rangle$ are isomorphic. The sets I, I' are given a topological structure related to the topology on F.

If one of the sets X, X' is finite the previous result can be improved as follows:

If either X or X' is a finite set and ψ: $I \to I'$ is a ring isomorphism, then the pre-ordered systems $\langle X, \leqq\rangle$ and $\langle X', \leqq'\rangle$ are isomorphic.

The following conventions shall be observed in this section. $\langle X, \leqq\rangle$, $\langle X', \leqq'\rangle$ are locally finite pre-ordered systems, F is a field and a topological space such that if $t \in F - \{0\}$, there is an open set U in F such that $0 \in U$ and $t \notin U$. $I = \langle X, \leqq, F\rangle$ and $I' = \langle X', \leqq', F\rangle$. The topological condition on F is always possible; the discrete topology on F has this property.

Definition 2.1. $\widetilde{X} = \{[x, x] \mid x \in X\}$.

We use the variables a, b, c for members of $\widetilde{X}$, a', b', c' for members of $\widetilde{X}'$ and u, v, r, x, y, z as members of X. For convenience we allow that the relation $\leqq$ is also defined on $\widetilde{X}$ as follows: $[x, x] \leqq [y, y]$ iff $x \leqq y$. Then $\langle \widetilde{X}, \leqq \rangle$ is a partially ordered system.

Definition 2.2. For every $a \in \widetilde{X}$, f_a is that member of I defined as follows: for every $x, y \in X$:

$$f_a(x, y) = \begin{cases} \delta(x, y) & \text{if } x, y \in a \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.3. For every $a \in \widetilde{X}$.

$$J_a = \{g \in I \mid g(x, y) = 0 \text{ for every } x, y \in a\}.$$

Definition 2.4. For every $x, y \in X$, e_{xy} is that member of I defined as follows: for every $u, v \in X$

$$e_{xy}(u, v) = \begin{cases} 1 \text{ if } u = x,\ v = y \text{ and } x \leqq y \\ 0 \text{ otherwise.} \end{cases}$$

We assume that $f_{a'}$, $J_{a'}$, $e_{x'y'}$ are defined similarly. Recall that f_aIf_b is the set of functions f_agf_b, where $g \in I$.

Lemma 2.5. *For every $a, b \in \widetilde{X}$, $a \leqq b$ iff $f_aIf_b \neq \{0\}$.*

Proof. If $a \leqq b$, $x \in a$, $y \in b$, then $x \leqq y$ and

$$(f_ae_{xy}f_b)(x, y) = f_a(x, x)e_{xy}(x, y)\, f_b(y, y) = 1.$$

Thus f_aIf_b contains a non-zero function. Conversely suppose that $f_aIf_b \neq \{0\}$. Let $g \in I$ be such that $f_agf_b \neq 0$. Let $x, y \in X$ be such that $(f_agf_b)(x, y) \neq 0$. Then for some $u, v \in X$ we have $f_a(x, u)\, g(u, v)\, f_b(v, y) \neq 0$. This shows that $x = u$, $x \in a$, $a = [x, x]$, $y = v$, $y \in b$, $b = [y, y]$, $x \leqq y$ and therefore $a \leqq b$.

Definition 2.6. (1) For each $a \in \widetilde{X}$, $\bar{\bar{a}}$ denotes the cardinality of a.

(2) $R = \bigcap \{J_a \mid a \in \widetilde{X}\}$

(3) $R' = \bigcap \{J_{a'} \mid a' \in \widetilde{X}'\}$.

Lemma 2.7. *R is a 2-sided ideal of I and for every $a \in \widetilde{X}$, J_a is a maximal 2-sided ideal of I.*

Proof. By definition of J_a it is clear that J_a is an additive subgroup of I. Let $g \in I$, $f \in J_a$ and $x, y \in a$. Then

$$\begin{aligned}(fg)(x, y) &= \sum_{u \in X} f(x, u)\, g(u, y) \\ &= \sum_{u \in a} f(x, u)\, g(u, y) \quad \text{by the order condition as } f \in J_a. \\ &= 0\end{aligned}$$

Similarly $(gf)(x, y) = 0$. This shows that $fg, gf \in J_a$ and therefore J_a is a 2-sided ideal of I. Now suppose that K is any 2-sided ideal of I which properly contains J_a. Then J_a is maximal if $K = I$, which holds if $\delta \in K$. By definition of J_a, and since a is a finite set

$$\delta - \sum_{x \in a} e_{xx} \in J_a .$$

To show that $\delta \in K$ it suffices to show that $e_{xx} \in K$ for every $x \in a$. Let $x \in a$ and $f \in K - J_a$. For some $u, v \in a, f(u, v) \neq 0$. Let $l = f(u, v)$, then l^{-1} exists because $l \neq 0$ and F is a field. Also, $l^{-1} e_{xu} \in I$. The members x, u, v of X are in a so $x \leqq u$ and $v \leqq x$. The set K is a 2-sided ideal so that $(l^{-1} e_{xu})(f e_{vx}) \in K$. If $r, s \in X$ we have

$$\begin{aligned}(l^{-1} e_{xu})(f e_{vx})(r, s) &= \sum_{y, z \in X} l^{-1} e_{xu}(r, y)\, f(y, z) e_{vx}(z, s) \\ &= \delta(r, x)\delta(s, x) = e_{xx}(r, s).\end{aligned}$$

Thus $(l^{-1} e_{xu})(f e_{vx}) = e_{xx}$ and $e_{xx} \in K$, for every $x \in a$. Therefore $\delta \in K$, $K = I$ and J_a is a maximal 2-sided ideal of I. R is therefore an intersection of 2-sided ideals of I so R is a 2-sided ideal of I.

Notation. For an integer $n \geqq 1$, R^n is the smallest ideal of I which contains all functions $f_1 f_2 \ldots f_n$ where $f_1, \ldots, f_n$ are members of R.

Lemma 2.8. $\bigcap_{n \geqq 1} R^n = \{0\}$.

Proof. It is clear that the function 0 is in this intersection. For every $x, y \in X$, let

$$\mathrm{l}[x, y] = \mathrm{card}\{a \in \widetilde{X} \mid a \subseteq [x, y]\}.$$

For every $x, y \in X$, $\mathrm{l}[x, y]$ is a non-negative integer by the locally finite condition on $\langle X, \leqq \rangle$. If $f \in R$ and $\mathrm{l}[x, y] \leqq 1$, then either $x \nleqq y$, in which case $f(x, y) = 0$, or $[x, y] = [x, x]$ and if $a = [x, x]$, then $f(x, y) = 0$ because $x, y \in a$ and $f \in J_a$. Suppose inductively that if $f \in R^n$ and $\mathrm{l}[x, y] \leqq n$, then $f(x, y) = 0$. We shall show that if $f \in R^{n+1}$ and $\mathrm{l}[x, y] \leqq n + 1$ then $f(x, y) = 0$. Each function f belonging to R^{n+1} may be expressed as an F-sum of functions of the type $g \cdot h$, where $g \in R^n$ and $h \in R$; therefore it suffices to show that $(gh)(x, y) = 0$ whenever $\mathrm{l}[x, y] \leqq n + 1$.

$$\begin{aligned}(gh)(x, y) &= \sum_{z \in X} g(x, z) h(z, y) \\ &= \sum_{z \in [x, y]} g(x, z) h(z, y) \\ &= \sum_{z \in S \cup [y, y]} g(x, z) h(z, y)\end{aligned}$$

where $S = [x, y] - [y, y]$. For every $z \in S$, $\mathrm{l}[x, z] \leqq n$ so by our inductive hypothesis $g(x, z) = 0$. For any $z \in [y, y]$, $\mathrm{l}[z, y] \leqq 1$ so that $h(z, y) = 0$ by our inductive hypothesis. Therefore $(gh)(x, y) = 0$. This shows that for every positive integer n, if $f \in R^n$ and $\mathrm{l}[x, y] \leqq n$, then $f(x, y) = 0$. Now suppose that $f \in \bigcap_{n \geqq 1} R^n$. Let $x, y \in X$ and let $\mathrm{l}[x, y] = r$. If $r = 0$, then $x \nleqq y$ and $f(x, y) = 0$. If $r > 0$, then $f \in R^r$ and $f(x, y) = 0$. So $f = 0$.

We shall now introduce a topology to the ring I, which is identical to the standard topology for incidence ring in Doubilet, Rota and Smith [3]. For the definition of topological terms used but not defined we refer the reader to Kelley [1].

Definition 2.9. A convergence class for I is a set $\boldsymbol{K}$ consisting of pairs

$((\varphi, D), f)$ where $f \in I$, D is a directed set and (φ, D) is a net in I, satisfying the following conditions:

(a) If (φ, D) is a net in I such that $\varphi(d) = f$ for every $d \in D$, then $((\varphi, D), f) \in \boldsymbol{K}$.
(b) If $((\varphi, D), f) \in \boldsymbol{K}$ and (φ', D') is a subnet of (φ, D), then $((\varphi', D'), f) \in \boldsymbol{K}$.
(c) If $((\varphi, D), f) \notin \boldsymbol{K}$, there is a subnet (ψ, E) of (φ, D) such that if (ψ', E') is a subnet of (ψ, E), then $((\psi', E'), f) \notin \boldsymbol{K}$.
(d) Let D be a directed set, E_d a directed set for each $d \in D$ and let $G = D \times \mathsf{X}\{E_d \mid d \in D\}$. If $(d, \alpha) \in G$ let $\varphi(d, \alpha) = (d, \alpha(d))$. G is a directed set and if $(\psi, D \times E_d)$ is a net in I such that $((\psi(d, \ldots), E_d), f_d) \in \boldsymbol{K}$ for every $d \in D$ and if $\psi'(d) = f_d$ and $((\psi', D), f) \in \boldsymbol{K}$, then $((\psi \circ \varphi, G), f) \in \boldsymbol{K}$.

For the incidence ring I we define the convergence class $\boldsymbol{K}$ as follows: let D be any directed set and let (φ, D) be a net in I, then $\varphi(d) \in I$ for every $d \in D$. For every $x, y \in X$, $\{\varphi(d)(x, y) \mid d \in D\}$ is a net in F. Then $((\varphi, D), f) \in \boldsymbol{K}$ iff for every $x, y \in X$, the net $\{\varphi(d)(x, y) \mid d \in D\}$ converges to $f(x, y)$ in the topology on F. The class $\boldsymbol{K}$ is a convergence class for I as is easily shown and by Theorem 9 of Chapter 2 in Kelley [1], $\boldsymbol{K}$ may be used to define a closure operator on the subsets of I and hence a unique topology on I. Following Doubilet-Rota and Stanley [3] we refer to this topology as the standard topology on I.

Definition 2.10. If A is a subset of $X \times X$ and $f \in I$, then f_A is that member of I defined as follows: for every $x, y \in X$

$$f_A(x, y) = \begin{cases} f(x, y) & \text{if } x, y \in A \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.11. *Let $\mathfrak{A}$ be a set of subsets of $X \times X$ such that $\bigcup \mathfrak{A} = X \times X$ and if $A, B \in \mathfrak{A}$ then $A \cup B \in \mathfrak{A}$. Let Y be a subset of I which is closed in the standard topology. If $f \in I$ and if for every $A \in \mathfrak{A}$, $f_A \in Y$, then $f \in Y$.*

Proof. The set $\mathfrak{A}$ is directed by inclusion. If $A \in \mathfrak{A}$, let $f(A) = f_A$. Then $(f, \mathfrak{A})$ is a net in Y. Because Y is closed in the standard topology, if $((f, \mathfrak{A}), f) \in \boldsymbol{K}$, then $f \in Y$. Let $x, y \in X$, then, since $\bigcup \mathfrak{A} = X^2$ there is an $A \in \mathfrak{A}$ such that $(x, y) \in A$. For every $B \in \mathfrak{A}$ such that $A \subset B$, $(x, y) \in B$ and $f_B(x, y) = f(x, y)$. Therefore the net $\{f_C(x, y) \mid C \in \mathfrak{A}\}$ converges to $f(x, y)$. Hence $((f, \mathfrak{A}), f) \in \boldsymbol{K}$ and $f \in Y$.

Lemma 2.12. *For every $a \in \widetilde{X}$, J_a is closed in the standard topology on I.*

Proof. It is enough to show that if (φ, D) is a net in J_a and $((\varphi, D), f) \in \boldsymbol{K}$, then $f \in J_a$. Assume this situation holds. Let $x, y \in a$; then $\varphi(d)(x, y) = 0$ for every $d \in D$. If $f(x, y) = t$ and $t \neq 0$, there is an open set U in F such that $0 \in U$ and $t \notin U$. Therefore the net $\{\varphi(d)(x, y) \mid d \in D\}$ does not converge to $f(x, y)$ and $((\varphi, D), f) \notin \boldsymbol{K}$. This contradiction shows that $f(x, y) = 0$ for every $x, y \in a$ and therefore J_a is closed in the standard topology on I.

The previous lemma is the only preliminary result in this section requiring the restriction on the topology of F. The reason we introduced the standard topology on I is to facilitate the following lemma. In Stanley's proof given in [3] and [5] such recourse to the standard topology is not necessary.

Lemma 2.13. *If K is a maximal 2-sided ideal in I and closed in the standard topology, then $K = J_a$ for some $a \in \widetilde{X}$.*

Proof. Lemmas 2.7 and 12 show that the J_a are closed maximal 2-sided ideals in I. Suppose K is not a subset of any of the J_a, we show that this implies $\delta \in K$ so that $K = I$ and K is not a proper ideal. So $K \subseteq J_a$ for some $a \in \widetilde{X}$ and by the maximality of K, $K = J_a$. Let $b \in \widetilde{X}$, $K \not\subseteq J_b$ so for some $x, y \in b$, there is an $f \in K$ such that $f(x, y) = l \neq 0$. If $u \in b$, then e_{ux} and $e_{yu} \in I$ and therefore $(l^{-1}e_{ux})(fe_{yu})$ is in K, because K is a 2-sided ideal. But $l^{-1}e_{ux}fe_{yu} = e_{uu}$ so that $e_{uu} \in K$ for every $u \in b$. The set b was an arbitrary member of $\widetilde{X}$, and since $\bigcup \widetilde{X} = X$ we see that $e_{uu} \in K$ for every $u \in X$. This implies that for every finite subset A of $X \times X$, $\delta_A = \sum_{(x,x) \in A} e_{xx} \in K$. Now let $\mathfrak{A} = \{A | A$ is a finite subset of $X \times X\}$; $\mathfrak{A}$ is directed by inclusion, $\bigcup \mathfrak{A} = X \times X$ and $(\delta, \mathfrak{A})$ is a net in K where $\delta(A) = \delta_A$. Clearly $((\delta, \mathfrak{A}), \delta) \in K$, by definition of K. By Lemma 2.11, $\delta \in K$. The proof is complete.

Definition 2.14. A function ψ: $I \to I'$ is closed on maximal 2-sided ideals iff for every closed maximal 2-sided ideal K of I, $\psi(K)$ is closed in the standard topology on I'.

Lemma 2.15. *Let ψ: $I \to I'$ be a ring isomorphism which is closed on maximal 2-sided ideals. Then for every $a \in \widetilde{X}$ there is an $a' \in \widetilde{X}'$ such that $\psi(J_a) = J_{a'}$, $\mathrm{card}(a) = \mathrm{card}(a')$, $\mathrm{card}(X) = \mathrm{card}(X')$ and $\psi(R) = R'$.*

Proof. Let $a \in \widetilde{X}$, then $\psi(J_a)$ is a maximal 2-sided ideal in I' because ψ is a ring isomorphism. ψ is closed on maximal 2-sided ideals so that $\psi(J_a)$ is closed in the standard topology on I'. Therefore, by Lemma 2.13 applied to I', $\psi(J_a) = J_{a'}$ for some $a' \in \widetilde{X}'$. Let $n = \mathrm{card}(a)$ and $n' = \mathrm{card}(a')$. We show that $I | J_a \cong \mathrm{M}(n, F)$. Let θ: $I \to \mathrm{M}(n, F)$ be defined as follows: let $x \in a$, for every $f \in I$

$$\theta(f) = \mathrm{m}(f|_{[x,x]}).$$

It is easy to show that θ is a ring homomorphism (*cf.* Lemma 1.7) onto $\mathrm{M}(n, F)$ and that $\theta(f) = (0)$ iff $f \in J_a$. Therefore $I | J_a \cong \mathrm{M}(n, F)$. Since $I | J_a \cong I' | J_{a'}$, we have $\mathrm{M}(n, F) \cong \mathrm{M}(n', F)$ and therefore $n = n'$, since both matrix rings have the same dimension as vector spaces over F. Thus $\mathrm{card}(a) = \mathrm{card}(a')$. If X is a finite set, then

$$\mathrm{card}(X) = \sum_{a \in X} \mathrm{card}(a) = \sum_{a' \in X'} \mathrm{card}(a') = \mathrm{card}(X').$$

If X is an infinite set, then $\mathrm{card}(X) = \mathrm{card}(\widetilde{X})$ because each $a \in \widetilde{X}$ is a finite subset of X. However, $\mathrm{card}(\widetilde{X}) = \mathrm{card}(\widetilde{X}')$ by the bijective correspondence between the closed maximal 2-sided ideals of I and I'. Thus, $\mathrm{card}(X) = \mathrm{card}(X')$. Finally,

$$\begin{aligned}\psi(R) &= \psi\Big(\bigcap \{J_a | a \in \widetilde{X}'\}\Big) \\ &= \bigcap \{\psi(J_a) | a \in \widetilde{X}\} = \bigcap \{J_{a'} | a' \in \widetilde{X}'\} = R'.\end{aligned}$$

Lemma 2.16. *$I|R$ is isomorphic to $\prod_{a\in \tilde{X}} I|J_a$ by the isomorphism*

$$\theta(f + R) = (f + J_a)_{a\in \tilde{X}}.$$

Proof. To show that θ is surjective let $(h_a + J_a)_{a\in X}$ be a member of $\prod I|J_a$. Define $h \in I$ as follows: for every $x, y \in X$,

$$h(x, y) = \begin{cases} h_a(x, y) & \text{if } x, y \in a \\ 0 & \text{otherwise.} \end{cases}$$

This definition is possible because the members of $\tilde{X}$ are pairwise disjoint sets. Clearly $\theta(h + R) = (h_a + J_a)_{a\in \tilde{X}}$. The proof that θ is an isomorphism is a standard result given that $R = \bigcap\{J_a \mid a \in \tilde{X}\}$.

Theorem 2.17. *Let ψ: $I \to I'$ be a ring isomorphism, closed on maximal 2-sided ideals. Let $\psi(J_a) = J_{a'}$ for every $a \in \tilde{X}$, $a' \in \tilde{X}'$, be the bijective correspondence between closed maximal 2-sided ideals. Then for every $a \in \tilde{X}$, $\psi(f_a) - f_{a'} \in R'$.*

Proof. Consider the following diagram.

$$\begin{tikzcd}
 & I \arrow[r, "\psi"] \arrow[dl, "\beta"'] \arrow[d, "\varepsilon"] & I' \arrow[d, "\varepsilon'"'] \arrow[dr, "\beta'"] & \\
I|R \arrow[r, "\gamma"'] & \prod_{a\in X} I|J_a \arrow[r, "\tilde{\psi}"] & \prod_{a'\in X'} I'|J_{a'} & I'|R'. \arrow[l, "\gamma'"]
\end{tikzcd}$$

The functions β, γ, ε, $\tilde{\psi}$ are defined as follows: for $f \in I$, $\beta(f) = f + R$, $\gamma(f + R) = (f + J_a)_{a\in X}$, $\varepsilon(f) = (f + J_a)_{a\in X}$. Clearly $\gamma \circ \beta = \varepsilon$; β is an isomorphism by Lemma 2.16.

$$\tilde{\psi}(f + J_a)_{a\in X} = (\psi(f) + J_{a'})_{a'\in X'}.$$

$\tilde{\psi}$ is an isomorphism by Lemma 2.15. β', ψ' and ε' are defined similarly to β, γ and ε respectively. If $a \in \tilde{X}$ then

$$\begin{aligned}
\beta' \circ \psi(f_a) &= \psi(f_a) + R' && \text{definition of } \beta' \\
&= (\gamma')^{-1}(\psi(f_a) + J_{b'})_{b'\in X'} && \text{definition of } \gamma' \\
&= (\gamma')^{-1} \circ \tilde{\psi}(f_a + J_b)_{b\in X} && \text{definition of } \tilde{\psi} \\
&= (\gamma')^{-1} \circ \tilde{\psi}(0, \ldots, 0, f_a + J_a, 0, \ldots) && \text{definition of } f_a \\
&= (\gamma')^{-1} \circ \tilde{\psi}(0, \ldots, 0, 1_a, 0, \ldots) && \text{definition of } f_a, J_a \\
&= (\gamma')^{-1}(0, \ldots, 0, f_{a'}, 0, \ldots) && \text{definition of } \tilde{\psi} \\
&= (\gamma')^{-1}(f_{a'} + J_{b'})_{b'\in X'} && \text{definition of } f_{a'}, J_{a'} \\
&= f_{a'} + R' && \text{definition of } \gamma' \\
&= \beta'(f_{a'}) && \text{definition of } \beta'.
\end{aligned}$$

Thus, $\beta'(\psi(f_a) - f_{a'}) = 0$ and $\psi(f_a) - f_{a'} \in R'$.

We are now able to use Stanley's Lemma of [3] and [5] to show that the partially ordered systems $\langle \tilde{X}, \leq\rangle$ and $\langle \tilde{X}', \leq'\rangle$ are isomorphic under suitable isomorphism conditions between the incidence algebras I and I', and thus conclude that the pre-ordered systems $\langle X, \leq\rangle$, $\langle X', \leq'\rangle$ are isomorphic.

Lemma 2.18. (Stanley's Lemma). *Let I_0 be a ring and let e, f, e', f' be idempotent members of I_0 such that $e - e' \in R_0$ and $f - f' \in R_0$ where R_0 is a 2-sided ideal in I_0 such that $\bigcap_{n \geq 1} R_0^n = \{0\}$. Then*

$$eI_0 f = \{0\} \text{ iff } e'I_0 f' = \{0\}.$$

Proof. Stanley [5] and Doubilet, Rota and Stanley [3].

Theorem 2.19. *Let $\langle X, \leqq\rangle$, $\langle X', \leqq'\rangle$ be locally finite pre-ordered relational systems. Let F be a field and a topological space such that if $t \in F - \{0\}$, then there is an open set U in F such that $0 \in U$ and $t \notin U$. Let $I = \langle X, \leqq, F\rangle$ and $I' = \langle X', \leqq', F\rangle$. If ψ: $I \to I'$ is a ring isomorphism which is closed on maximal 2-sided ideals with respect to the standard topologies on I and I', then the relational systems $\langle X, \leqq\rangle$, $\langle X', \leqq'\rangle$ are isomorphic.*

Proof. In the set $\widetilde{X}'$ we have $a' \leqq b'$ iff $f_{a'}I'f_{b'} \neq \{0\}$ by Lemma 2.5, and in $\widetilde{X}$, $a \leqq b$ iff $f_a I f_b \neq \{0\}$. Let $a \longleftrightarrow a'$ be the bijective correspondence between the sets $\widetilde{X}$, $\widetilde{X}'$ set up by ψ according to Lemma 2.15. The function ψ is an isomorphism so that $f_a I f_b \neq \{0\}$ iff $\psi(f_a)I'\psi(f_b) \neq \{0\}$. However $\bigcap_{n \geq 1}(R')^n = \{0\}$ by Lemma 2.8 and $\psi(f_a) - f_{a'} \in R'$ by Theorem 2.17 so we may use Stanley's Lemma to conclude that $\psi(f_a)I'\psi(f_b) \neq \{0\}$ iff $f_{a'}I'f_{b'} \neq \{0\}$. Therefore $a \leqq b$ in $\widetilde{X}$ iff $a' \leqq b'$ in $\widetilde{X}'$. In particular the partially ordered systems $\langle \widetilde{X}, \leqq\rangle$ and $\langle \widetilde{X}', \leqq'\rangle$ are isomorphic by the function $\Psi(a) = a'$.

Lemma 2.15 shows that the map Ψ preserves cardinality. For each $a \in \widetilde{X}$ there is a function Ψ_a: $a \to a'$ which is bijective. We may now define a bijective function φ: $X \to X'$ which is a binary isomorphism. For every $x \in X$,

$$\varphi(x) = \Psi_a(x), \text{ where } x \in a.$$

If $x, y \in X$, $x \in a$, $y \in b$ and $x \leqq y$, then $a \leqq b$, $a' \leqq' b'$, therefore $\Psi_a(x) \leqq' \Psi_b(y)$ and $\varphi(x) \leqq' \varphi(y)$. Similarly, if $\varphi(x) \leqq' \varphi(y)$, then $x \leqq y$. Hence the systems $\langle X, \leqq\rangle$ and $\langle X', \leqq'\rangle$ are isomorphic.

If one of the sets X or X' are finite, then both are finite and the topological conditions on ψ may be eliminated, as the following theorem shows.

Theorem 2.20. *Let $\langle X, \leqq\rangle$ and $\langle X', \leqq'\rangle$ be locally finite pre-ordered sets, F a field, $I = \langle X, \leqq, F\rangle$ and $I' = \langle X', \leqq', F\rangle$. Let I and I' be isomorphic rings and suppose one of the sets X, X' are finite, then the pre-ordered systems $\langle X, \leqq\rangle$ and $\langle X', \leqq'\rangle$ are isomorphic.*

Proof. Without loss of generality suppose that X' is finite and suppose that ψ: $I \to I'$ is a ring isomorphism. Let F be given the discrete topology, then F satisfies the topological condition required by the hypothesis of Theorem 2.19. In this case the function ψ is closed on maximal 2-sided ideals; in fact, for every set $K \subseteq I$, $\psi(K)$ is closed in the standard topology on I', because the standard topology on I' is the discrete topology on I' and every subset of I' is both open and closed.

To show that the topology on I' is the discrete topology it suffices to show that for every $g \in I'$ the set $I' - \{g\}$ is closed. Let (φ, D) be any net in I' and suppose that (φ, D) converges, in the standard topology, to a function f in I'. It must be shown that $g \neq f$. Suppose that $g = f$. Then for every $x, y \in X'$ the net $\{\varphi(d)(x, y) \mid d \in D\}$ converges to $g(x, y)$ in F. The set $\{g(x, y)\}$ is open in F, since F has the discrete topology so that there exists d_{xy} in D such that if $\leqq$ is the relation which directs D, for every d in D such that $d_{xy} \leqq d$,

$$\varphi(d)(x, y) = g(x, y).$$

The set $X' \times X'$ is finite; by the properties of a directed set there is a d_0 in D such that $d_{xy} \leqq d_0$ for every $x, y \in X'$. In particular $\varphi(d_0)(x, y) = g(x, y)$ for every $x, y \in X'$. Thus $\varphi(d_0) = g$; this contradicts our assumption that (φ, D) is a net in $I' - \{g\}$. Hence $f \neq g$ and the set $I' - \{g\}$ is closed and $\{g\}$ is an open set in I'. This shows that whenever F has the discrete topology and X' is a finite set, the standard topology on I' is, in fact, the discrete topology on I'.

The conditions of Theorem 2.19 are now fulfilled, hence $\langle X, \leqq\rangle$ and $\langle X', \leqq'\rangle$ are isomorphic and X is also a finite set.

There are several other topologies which could be put on incidence rings and used in the previous two theorems with precisely the same results. These topologies need not concern us here; it is desirable to remove the topological condition on the ring isomorphism in Theorem 2.19 or to show that it is necessary. I have not been able to do this.

To end this chapter we give an example of a pre-ordered system $\langle X, \leqq\rangle$ whose incidence ring I is not isomorphic to the incidence ring I^c of the converse relational system $\langle X, \leqq^c\rangle$.

Example 2.21. Let $X = \{x, y, z\}$ have the relation $\leqq$ defined $x \leqq x$, $y \leqq y$, $z \leqq z$, $x \leqq y$, $x \leqq z$. Then $x \leqq^c x$, $y \leqq^c y$, $z \leqq^c z$, $y \leqq^c x$ and $z \leqq^c x$. By inspecting the 6 members of the permutation group S_X one may verify that the pre-ordered relational systems $\langle X, \leqq\rangle$ and $\langle X, \leqq^c\rangle$ are not isomorphic. Thus, if F is any field and $I = \langle X, \leqq, F\rangle$ and $I^c = \langle X, \leqq, F\rangle$ then I and I^c are not isomorphic rings by Theorem 2.20. However, as Theorem 1.21 shows the multiplicative groups I^* and I^{c*} are isomorphic groups.

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