Notre Dame Journal of Formal Logic
Volume XIV, Number 4, October 1973
NDJFAM

FORMALLY DEFINED OPERATIONS IN KRIPKE MODELS

LUIS E. SANCHIS

The introduction of Kripke models in intuitionistic logic has originated a growing interest in this field. There is no doubt that the relation between these models and forcing has played a role in this development. As a consequence the traditional interpretation of intuitionistic logic, namely the notion of constructiveness considered by Brower, Heyting and others, appears to have lost some of its former preponderance. Recently several logicians have been interested in defining arbitrary operations in the models, to some extent independently of the operations which are proper of intuitionistic logic. In this paper some results in this direction are presented. We consider only the propositional calculus.

We study operations which are formally defined in the following sense. Each operation is introduced first as a formal connective together with some provability rules. Then we use the same rules to give the interpretation of the connective in the model. This is possible because we use Gentzen type rules for which the subformula property holds. We give some formal conditions for the rules that are both necessary and sufficient for the system to be adequate and complete.

The most interesting property of these operations is that whenever two of them agree in one of the Kripke models then they agree in all models, hence they are actually the same operation. The usual connectives of intuitionistic logic are formally defined in our sense, hence our results apply to them. Not only that, it is easy to show that every formally defined connective can be also defined explicitly by some formula containing only the usual connectives.

On the other hand we show that there are operations defined explicitly by formulas containing only the usual connective which are not formally defined in our sense. Such operations agree with some formally defined operation in at least one model—the classical model—but for every formally defined operation there is a model in which they are different. This result generalizes the well known result that there are classical tautologies which are not provable in intuitionistic logic.

Received August 6, 1971

1 *Formal Definitions*. We consider a system in which the atomic propositional letters are the symbols: $e_1, e_2, \ldots, e_n, \ldots$. We assume several connectives are given, each one with a fixed given number of arguments. The symbol # will denote some arbitrary n-ary connective. The class of formulas is the smallest class that contains the atomic propositional letters and for every connective # contains $\#(A_1, \ldots, A_n)$ whenever it contains $A_1, \ldots, A_n$. Capital letters $A, B, C, A_1, \ldots$ will denote formulas. Letters M, N, P, Q, R, S, T will denote finite sequences of formulas, eventually the empty sequence. The formulas in the sequence M are called the components of M. If every component of M is also a component of N we say that N is an expansion of M and write $M < N$.[1] We assume the symbol $\vdash$ is not a connective. An expression $M \vdash N$ is called a sequent. M is the left side of the sequent and N the right side. The components of M are called left components of the sequent; the component of N are called right components. Whenever M and N are both empty we have the empty sequent. If all right and left components of a sequent are atomic letters we say the sequent is atomic. The sequent $M \vdash N$ is closed if there is a formula that it is both a left and a right component. The union of the sequents $M \vdash N$ and $P \vdash Q$ is the sequent $M, P \vdash Q, N$.

We assume that with every n-ary connective # we have associated two sets of atomic sequents **R**# and **L**#; the elements of these sets are called right axioms of # and left axioms of # respectively. An axiom of # is either a right axiom or a left axiom. If $P \vdash Q$ is an axiom then the components of P and Q are atomic letters from the list $e_1, e_2, \ldots, e_n$. The empty sequent can be taken as an axiom. We admit also that either of the sets **R**# or **L**# is empty. We do not exclude the possibility of both being empty but further restrictions we shall consider later will make such a case impossible.

As an example, and for future reference, we give the sets of axioms that we associate with the usual connectives: $\neg, \supset, \wedge, \vee$

$$\begin{array}{ll}
\mathbf{R}\neg = \{e_1 \vdash \ \} & \mathbf{L}\neg = \{\ \vdash e_1\} \\
\mathbf{R}\supset = \{e_1 \vdash e_2\} & \mathbf{L}\supset = \{\vdash e_1,\ e_2 \vdash\} \\
\mathbf{R}\wedge = \{\vdash e_1,\ \vdash e_2\} & \mathbf{L}\wedge = \{e_1,\ e_2 \vdash \ \} \\
\mathbf{R}\vee = \{\ \vdash e_1,\ e_2\} & \mathbf{L}\vee = \{e_1 \vdash\ ,\ e_2 \vdash\}
\end{array}$$

We assume that for every connective the right and left axioms are given in some order. Now let $\#(A_1, \ldots, A_n)$ be some formula. The right axioms of this formula are all sequents that can be obtained from the right axioms of # by simultaneous substitution of A_1 for e_1, A_2 for $e_2, \ldots, A_n$ for e_n. The left axioms of $\#(A_1, \ldots, A_n)$ are obtained in the same way from the left axioms of #. For instance $A_1 \supset A_2$ has one right axiom, the sequent $A_1 \vdash A_2$ and two left axioms, the sequents $\vdash A_1$ and $A_2 \vdash$. Clearly the right (left) axioms of # are exactly the right (left) axioms of $\#(e_1, \ldots, e_n)$. Each

1. Hence N is an expansion of M means M is a subset of N when the sequences are considered as sets.

right (left) axiom of # produces a right (left) axiom of $\#(A_1, \ldots, A_n)$ and we take these axioms in the order that corresponds to the given order of the axioms for #.

For each connective # we introduce two rules, one called the right rule (#**R**) and the other the left rule (#**L**):

$$(\#\mathsf{R})\quad \frac{P_1, M \vdash N, Q_1, \ldots, P_k, M \vdash N, Q_k}{M \vdash N, \#(A_1, \ldots, A_n)}$$

$$(\#\mathsf{L})\quad \frac{S_1, M \vdash N, T_1, \ldots, S_m, M \vdash N, T_m}{\#(A_1, \ldots, A_n), M \vdash N}$$

In these rules M and N are arbitrary sequences of formulas, and $A_1, \ldots, A_n$ are arbitrary formulas. The sequents $P_1 \vdash Q_1, \ldots, P_k \vdash Q_k$ in (#**R**) are the right axioms of $\#(A_1, \ldots, A_n)$. The sequents $S_1 \vdash T_1, \ldots, S_m \vdash T_m$ in (#**L**) are the left axioms of $\#(A_1, \ldots, A_n)$. The components of M are called left parameters of the rule, and the components of N are called right parameters of the rule. We impose the following important restriction: In (#**R**) the sequence N of right components is empty unless all the sequences $P_1, \ldots, P_k$ are empty. In other words we allow right parameters in (#**R**) only when in every right axiom of # the set of left components is empty. In the previous example this condition is satisfied by the connectives $\wedge$ and $\vee$, but not by $\neg$ and $\supset$. There is no restriction on the left parameters of (#**R**); neither is there any restriction on the right and left parameters of (#**L**).[2] In both rules the sequents above the line are called the premises of the rule, and the sequent below the line is called the conclusion of the rule. We shall take as initial sequents all sequents $A \vdash A$ with A some atomic letter. We shall allow also the expansion rule, which has the form:

$$\frac{M \vdash N}{P \vdash Q}$$

provided that P is an expansion of M and Q is an expansion of N. We shall say that a sequent $M \vdash N$ is provable if it can be obtained from the initial sequents by using the expansion rules and the rules (#**R**) and (#**L**) for each connective #.

We shall consider systems containing one or more connectives and for each connective we assume the sets of axioms **R**# and **L**# are given. These sets determine the rules (#**R**) and (#**L**) hence the notion of provability in the system. Our aim is to define for such connectives an interpretation in Kripke models and to study under which conditions we obtain an adequate and complete formalization for such interpretation. Since we want to give conditions that are sufficient and necessary we need some results that hold for any system satisfying the very general definitions give above.

2. If some set of axioms is empty, say the right axioms of #, the rule means that the sequent $M \vdash N, \#(A_1, \ldots, A_n)$ can be taken as initial for arbitrary M and N.

Proposition 1. *Let $M \vdash N$ be provable and $N < R$, $\#(A_1, \ldots, A_n)$. Assume $P \vdash Q$ is some right axiom of $\#(A_1, \ldots, A_n)$. Then $P, M \vdash R, Q$ is also provable.*

The proof is by induction in the derivation of $M \vdash N$. The case in which $M \vdash N$ is an initial sequent or it is obtained by the expansion rule is trivial. Suppose $M \vdash N$ is $M \vdash N_1$, $\#_1(B_1, \ldots, B_m)$ where $\#(A_1, \ldots, A_n)$ and $\#_1(B_1, \ldots, B_m)$ are different formulas and $M \vdash N$ is obtained by rule $(\#_1\mathbf{R})$. Hence $\#_1(B_1, \ldots, B_m)$ is a component of R. If N_1 is empty clearly $M \vdash R$ is provable and the proposition follows by the expansion rule. If N_1 is not empty the restriction on the right parameter does not apply to $(\#_1\mathbf{R})$. Let $P_1 \vdash Q_1$ be a right axiom of $\#_1(B_1, \ldots, B_m)$. Since $P_1, M \vdash N_1, Q_1$ is provable and $N_1, Q_1 < R, Q_1, \#(A_1, \ldots, A_n)$ by the induction hypothesis (and the expansion rule), it follows that $P_1, P, M \vdash R, Q, Q_1$ is provable. Hence $P, M \vdash R, Q, \#_1(B_1, \ldots, B_m)$ is provable and by the expansion rule $P, M \vdash R$, Q is provable. Now suppose $M \vdash N$ is $M \vdash N_1$, $\#(A_1, \ldots, A_n)$ and it is obtained by rule $(\#\mathbf{R})$. It follows that $P, M \vdash N_1, Q$ is provable and since N_1, $Q < R, Q, \#(A_1, \ldots, A_n)$ by the induction hypothesis it is provable P, P, $M \vdash R, Q$ so by expansion rule $P, M \vdash R, Q$ is provable. The case in which $M \vdash N$ is obtained by some left rule is completely similar.

Corollary. *If $M \vdash N$, $\#(A_1, \ldots, A_n)$ is provable and $P \vdash Q$ is a right axiom of $\#(A_1, \ldots, A_n)$ then $P, M \vdash N, Q$ is provable.*

This result expresses some inversion property of the right rules. For the left rules a similar result can be obtained only in special cases. One of these cases is considered in the following proposition.

Proposition 2. *Let $M \vdash N$ be provable and suppose $M < R$, $\#(A_1, \ldots, A_n)$. Assume all formulas $A_1, \ldots, A_n$ are atomic letters and assume also that the components of N and R are atomic letters. Let $S \vdash T$ be some left axiom of $\#(A_1, \ldots, A_n)$. Then $S, R \vdash N, T$ is provable.*

The proof by induction in the derivation of $M \vdash N$ is similar to that given in Proposition 1.

Corollary. *Let $\#(A_1, \ldots, A_n)$, $M \vdash N$ be provable, where the components of M and N are atomic letters and the formulas $A_1, \ldots, A_n$ are also atomic letters. Let $S \vdash T$ be some left axiom of $\#(A_1, \ldots, A_n)$. Then $S, M \vdash N, T$ is provable.*

2 *Models.* A model $\mathcal{M}$ is a pair $\langle G, R\rangle$ where G is a non empty set and R is a reflexive, transitive relation on G. The elements of G are called points. If xRy holds we say that y is an extension of x. An assignment $t(x)$ in the model $\mathcal{M}$ is a function defined for all points x of the model, and the values of $t(x)$ are sets of atomic letters, and whenever xRy holds then $t(x)$ is a subset of $t(y)$. The pair $\langle x, t\rangle$ where x is some point of $\mathcal{M}$ and t is some assignment in $\mathcal{M}$ is called a realization in $\mathcal{M}$. If y is an extension of x then $\langle y, t\rangle$ is an extension of $\langle x, t\rangle$. Now we define the value given by a realization $\langle x, t\rangle$ in some model $\mathcal{M}$ to every formula of our system. Such value is always a truth value: $\mathbf{T}$ (true) or $\mathbf{F}$ (false). The realization $\langle x, t\rangle$ gives to e_i the value $\mathbf{T}$ if and only if e_i is an element of $t(x)$.

Suppose A is the formula $\#(A_1, \ldots, A_n)$ for some connective $\#$. Then $\langle x, t\rangle$ gives to A the value **T** if and only if the following condition is satisfied: *If $P \vdash Q$ is a right axiom of A, and $\langle y, t\rangle$ is some extension of $\langle x, t\rangle$ then either $\langle y, t\rangle$ gives value* **F** *to some component of P or value* **T** *to some component of Q.*[3] It follows immediately from the assumptions on t and R that whenever $\langle x, t\rangle$ gives to A value **T** then any extension $\langle y, t\rangle$ also gives to A value **T**.

We say that a realization $\langle x, t\rangle$ in $\mathcal{M}$ accepts a sequent $M \vdash N$ if every extension $\langle y, t\rangle$ of $\langle x, t\rangle$ either gives value **F** to some component in M or value **T** to some component in N. We say that $\langle x, t\rangle$ refutes $M \vdash N$ if $\langle x, t\rangle$ gives value **T** to every component of M and value **F** to every component of N. It follows that $\langle x, t\rangle$ accepts $M \vdash N$ if and only if no extension of $\langle x, t\rangle$ refutes $M \vdash N$. It is clear that $\langle x, t\rangle$ gives value **T** to the formula $\#(A_1, \ldots, A_n)$ if and only if $\langle x, t\rangle$ accepts all right axioms of the formula.

A model with exactly one point is called a classical model. We identify such models with the model $\langle\{0\}, R\rangle$ which we call $\mathcal{C}$. An assignment t in $\mathcal{C}$ is essentially a subset of the set of all atomic letters, $t(0)$ denoting such subset. We say that t gives value **T** or **F** to a formula if $\langle 0, t\rangle$ gives such value. In the same manner we say that t accepts or refutes a sequent whenever $\langle 0, t\rangle$ accepts or refutes the sequent. In the model $\mathcal{C}$ the truth value of a formula given by an assignment t is a truth function of the values given by t to the subformulas. We say that a sequent $M \vdash N$ is valid in a model $\mathcal{M}$ if every realization in $\mathcal{M}$ accepts $M \vdash N$. A valid sequent is a sequent which is valid in every model. We say that a system is consistent if every provable sequent is valid.

The consistency property for a connective $\#$ in a model $\mathcal{M}$ is the following: *Whenever a realization $\langle x, t\rangle$ in $\mathcal{M}$ accepts all right axioms of $\#$ then it refutes some left axiom of $\#$.*

Theorem 1. *If the consistency property fails for some connective $\#$ in some model $\mathcal{M}$, the system is not consistent.*

Suppose $\langle x, t\rangle$ is some realization in $\mathcal{M}$ that accepts all right axioms of $\#$ and refutes no left axiom. Consider all left axioms of $\#$, say

$$S_1 \vdash T_1, S_2 \vdash T_2, \ldots, S_k \vdash T_k$$

We define sequents $M_1 \vdash N_1, M_2 \vdash N_2, \ldots, M_k \vdash N_k$ as follows. Suppose $\langle x, t\rangle$ gives value **F** to some component in S_j say the atomic letter e_i. In this case the sequent $M_j \vdash N_j$ is $S_j \vdash T_j, e_i$. If this is not the case some component of T_j is given value **T** by $\langle x, t\rangle$ since $\langle x, t\rangle$ does not refute $S_j \vdash T_j$. Let e_u be such component and we put in this case $M_j \vdash N_j$ to be the sequent $e_u, S_j \vdash T_j$. It is clear that all sequents $M_j \vdash N_j$ are closed, hence provable, then by rule ($\#$**L**) we can get

$$e_{u_1}, \ldots, e_{u_s}, \#(e_1, \ldots, e_n) \vdash e_{i_1}, \ldots, e_{i_r}$$

which is refuted by $\langle x, t\rangle$.

3. It is clear that this definition is the same used by Kripke for $\neg$ and $\supset$. The formulation is different for $\vee$ and $\wedge$ but it is easy to show they are equivalent.

This result shows that the consistency property is necessary to have a consistent system. We prove now it is also a sufficient condition. We need first the following result.

Proposition 3. *Suppose the consistency property for a connective # holds in a model* $\mathcal{M}$. *If* $\langle x, t\rangle$ *is a realization in* $\mathcal{M}$ *that gives value* **T** *to a formula* $\#(A_1, \ldots, A_n)$ *then it refutes some left axiom of the same formula.*

Define a new assignment $t_1(y)$ as follows: for $i = 1, \ldots, n$, e_i is an element of $t_1(y)$ if and only if $\langle y, t\rangle$ gives value **T** to the formula A_i. For $j > n$, e_j is not an element of $t_1(y)$. Clearly $\langle x, t_1\rangle$ accepts all right axioms of #, hence refutes some left axiom of #, and it follows that $\langle x, t\rangle$ refutes the corresponding left axiom of $\#(A_1, \ldots, A_n)$.

Theorem 2. *Let* $M \vdash N$ *be a provable sequent and assume that* $\mathcal{M}$ *is a model such that the consistency property for every connective # in* $M \vdash N$ *holds in* $\mathcal{M}$. *Then* $M \vdash N$ *is valid in* $\mathcal{M}$.

The proof is by induction in the derivation of $M \vdash N$. The initial sequents are trivial. It is clear that whenever a realization $\langle x, t\rangle$ accepts the premise of the expansion rule it also accepts the conclusion of the rule. The same situation can be checked for the rules (#**R**) and for this it is not necessary to assume the consistency property. But it is essential to assume that the set of right parameters is empty; this restriction can be dropped for those connectives with the property that in every right axiom the set of left components is empty. For the rules (#**L**) we have a similar situation: whenever a realization accepts the premise it also accepts the conclusion. But the proof of this requires a straight application of the consistency property we assume in this theorem.

From this result it follows that the system is consistent provided the consistency property holds for every connective in every model. We can prove that actually it is sufficient that it holds in at least one model. At the same time we can give a formal characterization of the consistency property. We recall that the cut rule can be formulated in the following way:

$$\frac{M \vdash N, A \qquad A, M \vdash N}{M \vdash N}$$

Theorem 3. *The following conditions are equivalent for a given connective #:*

a) *There is a model* $\mathcal{M}$ *such that every realization* $\langle x, t\rangle$ *in* $\mathcal{M}$ *refutes at least one axiom (right or left) of #.*

b) *There is a model* $\mathcal{M}$ *such that the consistency property holds for # in* $\mathcal{M}$.

c) *The empty sequent can be obtained from the set of all axioms of # (right and left) using only the cut rule and the extension rule.*

d) *For every model* $\mathcal{M}$ *every realization* $\langle x, t\rangle$ *in* $\mathcal{M}$ *refutes at least one axiom (right or left) of #.*

e) *The consistency property for # holds in every model.*

f) *In the classical model C every assignment t refutes at least one axiom (right or left) of* $\#$.

The implication from a) to b) is trivial. Assume b) to prove c). For this we show that whenever $M \vdash N$ is an atomic sequent in which all components are taken from the list $e_1, \ldots, e_n$ then either $M \vdash N$ is closed or it can be obtained from the set of all axioms of $\#$ using only the cut rule and the expansion rule. This is proved by induction on k = the number of atomic letters in the list $e_1, \ldots, e_n$ which are not components of $M \vdash N$. Assume first that $k = 0$ and $M \vdash N$ is not closed. In the model given by b) define the following assignment: $t(y)$ = the set of all components of M for every y. Take some point x of the model. Assume $\langle x, t\rangle$ does not accept some right axiom $P \vdash Q$ of $\#$. This means that some extension $\langle y, t\rangle$ refutes $P \vdash Q$. Hence $M \vdash N$ can be obtained by expansion from $P \vdash Q$. Assume $\langle x, t\rangle$ accepts all right axioms of $\#$; by the consistency property some left axiom of $\#$ is refuted by $\langle x, t\rangle$. Again this means that $M \vdash N$ is obtained by expansion of such left axiom. Assume now that $k > 0$ and $M \vdash N$ is not closed. Let e_i be some letter in the list which is not a component of $M \vdash N$. By the induction hypothesis both $M \vdash N, e_i$ and $e_i, M \vdash N$ can be obtained from the set of all axioms using cut rule and induction rule. Then we can obtain $M \vdash N$ by one application of the cut rule. This proves that $M \vdash N$ in all cases can be obtained using cut rule and expansion rule. Taking $M \vdash N$ to be the empty sequent we get c).

Assume now c) to prove d). Take any model $\mathcal{M}$ and any realization $\langle x, t\rangle$ in $\mathcal{M}$. Suppose $\langle x, t\rangle$ does not refute any axiom. It is easy to check that the expansion rule and the cut rule whenever applied to sequents that are not refuted by $\langle x, t\rangle$ produce sequents with the same property. It follows that the empty sequent is not refuted by $\langle x, t\rangle$ and this is a contradiction.

The implication from d) to e) is trivial. To prove f) assuming e) it is sufficient to note that in the classical model every sequent is either accepted or refuted by any assignment t. The implication from f) to a) is of course trivial.

3 *Completeness.* In this section we consider which conditions must be imposed in the system in order that every valid sequent can be proved. The union property for the connective $\#$ is the following: *The union of a left axiom of $\#$ with a right axiom of $\#$ is a closed sequent.*

Proposition 4. *If the sequent $\#(e_1, \ldots, e_n) \vdash \#(e_1, \ldots, e_n)$ is provable the union property for $\#$ holds.*

Assume the sequent is provable. Let $P \vdash Q$ be some right axiom of $\#$. By the corollary to Proposition 1 it follows that $P, \#(e_1, \ldots, e_n) \vdash Q$ is provable. Now let $S \vdash T$ be some left axiom of $\#$. By the corollary to Proposition 2 it follows that $S, P \vdash Q, T$ is provable. Since this is an atomic sequent this is possible only if it is closed.

This result shows that the union property for every connective is a necessary condition for the system to be complete. To prove it is a sufficient condition we need several definitions.

A block of a sequent $M \vdash N$ is a finite sequence of sequents, say $M_1 \vdash N_1$, $M_2 \vdash N_2, \ldots, M_k \vdash N_k$, constructed using the following rules:

B1 $M_1 \vdash N_1$ is $M \vdash N$.

B2 If $M_i \vdash N_i$ is $A, M_i' \vdash N_i$ where A is an atomic letter but not every component of M_i' is an atomic letter, then $M_{i+1} \vdash N_{i+1}$ is the sequent M_i', $A \vdash N_i$.

B3 If $M_i \vdash N_i$ is $\#(A_1, \ldots, A_n)$, $M_i' \vdash N_i$ for some connective $\#$, then $M_{i+1} \vdash N_{i+1}$ is any of the sequents $S, M_i' \vdash N_i, T$ where $S \vdash T$ is any of the left axioms of $\#(A_1, \ldots, A_n)$.

B4 In the last sequent of the block all left components are atomic letters.

We shall use letters **U** and **V** for blocks. The left components of the block **U** are the left components of the sequents in **U**, and the right components of **U** are the right components of the sequents in **U**. The notation **U*** indicates all left components of **U** in some conventional order. The residuals of the block **U** are those formulas that are right components **U** and are not atomic letters. We say that the block **U** is closed if there is some atomic letter that it is both a right component and a left component of **U**.

Proposition 5. *Let $M \vdash N$ be a sequent and R some given sequence of formulas. Assume that every block* **U** *of $M \vdash N$ is either closed or there is a residual B of* **U** *such that* **U***, $R \vdash B$ *is provable. Then M, $R \vdash N$ is provable.*

The proof is by induction on k = the number of connectives occurring in M. If $k = 0$ the assumptions imply that there is a component B of N such that M, $R \vdash B$ is provable. It follows that M, $R \vdash N$ is provable. Assume now that $k > 0$. Then by 0 or more applications of **B2** in every block **U** we must reach a sequence $M_i \vdash N_i$ of the form $\#(A, \ldots, A)$, $M_i' \vdash N_i$. Let $S \vdash T$ be some left axiom of $\#(A_1, \ldots, A_n)$ and assume **V** is some block of S, $M_i' \vdash N_i$, T. By the assumptions of the proposition it follows that either **V** is closed or there is a residual B of **V** such that **V***, $\#(A_1, \ldots, A_n)$, $R \vdash B$ is provable. By the induction hypothesis (but with $R_1 = \#(A_1, \ldots, A_n)$, R in place of R) we get that S, M_i', $R_1 \vdash N_i$, T is provable. Since this holds for all left axioms of $\#$, we have that $\#(A_1, \ldots, A_n)$, M_i, $R_1 \vdash N_i$ is provable, hence M, $R \vdash N$ is also provable.

Corollary. *Let $M \vdash N$ be a sequent such that every block* **U** *is either closed or there is a residual B of* **U** *such that* **U*** $\vdash B$ *is provable. Then $M \vdash N$ is provable.*

This result gives the relation between blocks and provability which we need to prove the completeness of the system. For some applications in the next section we need a similar result that we state and prove now. We say that a connective $\#$ is regular if there is at most one left axiom of $\#$ with the property that the set of left components of the axioms is non empty.

For example, the connective $\vee$ is not regular, but the other usual connectives are regular. A sequent $M \vdash N$ is regular if in every block **U** whenever rule **B3** is applied the connective $\#$ is regular.

Proposition 6. *Let $M \vdash N$ be a regular sequent with N non empty, and R some given sequence of formulas. Assume that for every block* **U** *of $M \vdash N$ there is a right component B of* **U** *such that* $\mathbf{U}^*$, $R \vdash B$ *is provable. Then there is a component A of N such that M, $R \vdash A$ is provable.*

The proof is again by induction on the number of connectives in M. The initial case is trivial, so we may assume that by using 0 or more times **B2** in every block **U** we have reached a sequent $M_i \vdash N_i$ of the form $\#(A_1, \ldots, A_n)$, $M_i^! \vdash N_i$. In this case $\#$ is a regular connective. If $S_j \vdash T_j$ is a left axiom of $\#(A_1, \ldots, A_n)$ by the induction hypothesis we get that S_j, $M_i^!$, $R_1 \vdash A_j$ is provable, where A_j is a component of N_i, T_j. We have here three cases. i) For all j, A_j is a component of T_j. By using ($\#$**L**) we get $\#(A_1, \ldots, A_n)$, $M_i^!$, $R_1 \vdash$ hence M, $R \vdash A$ is provable for any component A of N. ii) Some A_j is a component of N_i and S_j is empty. In this case we take $A = A_j$ and clearly M, $R \vdash A$ is provable. iii) Otherwise for every j such that S_j is empty A_j is a component of T_j and there is a unique j such that S_j is not empty but A_j is a component of N_i. In this case we take $A = A_j$ and get M, $R \vdash A$ by rule ($\#$**L**) and expansion.

We return to the proof of the completeness theorem. We define a special model $\mathfrak{B}^* = \langle G^*, R^* \rangle$ where G^* is the set of all blocks **U** such that **U** is not closed and there is no residual B of **U** such that $\mathbf{U}^* \vdash B$ is provable. The relation **U** R^* **V** means that $\mathbf{U}^* < \mathbf{V}^*$. We have the following consequence of Proposition 5:

If the sequent $M \vdash N$ is not provable then it has a block **U** *that belongs to G^*.*

Let $t^*(\mathbf{U})$ be the following assignment in $\mathfrak{B}^*$. For every block **U** $t^*(\mathbf{U})$ is the set of atomic letters that are left components of **U**.

Theorem 4. *Assume the union property holds for all connectives and let* **U** *be some block in G^*. Then if a formula A is a left (right) component of* **U** *the realization* $\langle \mathbf{U}, t^* \rangle$ *gives value* **T** (**F**) *to A.*

We note that this result can be stated in the following way: *if* **U** *belongs to G^* then* $\langle \mathbf{U}, t^* \rangle$ *refutes every sequent in* **U**. The proof of the theorem is by induction on the number of connectives in the formula A. The case in in which A is some atomic letter is clear from the definition of t^* and the fact that **U** is not closed.

Suppose A is $\#(A_1, \ldots, A_n)$ for some connective $\#$. We consider first the case in which A is a left component of **U**. Let **V** be some element of G^* such that **U** R^* **V** holds. We must show that $\langle \mathbf{V}, t^* \rangle$ does not refute any right axiom of A. Since A is a left component of **V** by the induction hypothesis some left axiom of A, say $S \vdash T$, is refuted by $\langle \mathbf{V}, t^* \rangle$. Hence given any right axiom of A, say $P \vdash Q$, the sequent S, $P \vdash Q$, T is closed, hence $\langle \mathbf{V}, t^* \rangle$ does not refute $P \vdash Q$.

We consider now the case in which A is a right component of **U**. Since

A is a residual of U the sequent $\mathsf{U}^* \vdash A$ is not provable. This means that for some right axiom of A, say $P \vdash Q$, the sequent P, $\mathsf{U}^* \vdash Q$ is not provable. Let V be some block of P, $\mathsf{U}^* \vdash Q$ which is an element of G^*. By the induction hypothesis $\langle \mathsf{V}, t^* \rangle$ refutes $P \vdash Q$ and since $\langle \mathsf{V}, t^* \rangle$ is an extension of $\langle \mathsf{U}, t^* \rangle$ this means that $\langle \mathsf{U}, t^* \rangle$ gives value F to A.

Theorem 5. *The following conditions are equivalent:*

a) *Every valid sequent is provable.*
b) *The union property holds for all connectives.*
c) *Every sequent valid in the model* $\mathfrak{B}^*$ *is provable.*

The implication from a) to b) follows from Proposition 4. Assume b) and to prove c) assume $M \vdash N$ is valid in $\mathfrak{B}^*$. If $M \vdash N$ is not provable then there is some U in G^* such that $\langle \mathsf{U}, t^* \rangle$ refutes $M \vdash N$; so $M \vdash N$ is provable. The implication from c) to a) is trivial.

This proof of the completeness theorem is a generalization of the proof given by Fitting in [2]. There is another proof given by Schutte in [4] that can be also be generalized to the situation considered in this paper. The argument given by this generalization is actually more elegant but it is also less informative. For this reason we have preferred to use Fitting's construction that is also Kripke's original proof.[4] A connective for which both the consistency property and the union property holds is called a Gentzen connective. A system in which all connectives are Gentzen connectives is called a Gentzen system. There are some differences between the original treatment given by Gentzen and the one presented here, specially in connection with the possibility of allowing more than one right component in sequents. This seems to us to be a consequence of the fact that Gentzen considered only the four usual connectives of intuitionistic logic. A general treatment as the one presented here seems to require a different notion of sequent.[5]

It is possible that two connectives with different sets of axioms are related, and eventually are the same. The following result provides a method to compare different connectives.

Theorem 6. *Let* $\#_1$ *and* $\#_2$ *be two n-ary Gentzen connectives. The following conditions are equivalent:*

a) *The union of a left axiom of* $\#_1$ *with a right axiom of* $\#_2$ *is always a closed sequent.*
b) *The sequent* $\#_1(e_1, \ldots, e_n) \vdash \#_2(e_1, \ldots, e_n)$ *is provable.*
c) *There is a model in which* $\#_1(e_1, \ldots, e_n) \vdash \#_2(e_1, \ldots, e_n)$ *is valid.*

---

4. The main change in the proof is the treatment of the right rules. Note that what we call here a block is not the same notion that Fitting introduces in the predicate calculus.

5. A theory of Gentzen rules for the usual connectives allowing more than one right component in sequents was given by Curry in [1].

The implication from a) to b) and from b) to c) is clear. To go from c) to a) assume $\mathcal{M}$ is a model in which the sequent is valid. To get a contradiction assume a) is not true so there is a left axiom of $\#_1$ say $S \vdash T$ and a right axiom of $\#_2$, say $P \vdash Q$ such that $S, P \vdash Q, T$ is not closed. In this case define the following assignment $t(y)$ in $\mathcal{M}$: for every point y, $t(y) =$ the set of all components of S and P. If x is any point in the model then $\langle x, t\rangle$ gives value **F** to $\#_2(e_1, \ldots, e_n)$ hence also gives value **F** to the formula $\#_1(e_1, \ldots, e_n)$. This means that there is some right axiom of $\#_1$, say $P_1 \vdash Q_1$, and some extension $\langle y, t\rangle$ of $\langle x, t\rangle$ such that $\langle y, t\rangle$ refutes $P_1 \vdash Q_1$. Since $\langle y, t\rangle$ refutes $S \vdash T$ it follows that it also refutes $S, P_1 \vdash Q_1, T$ which is impossible since this sequent is closed.

We shall say that the connectives $\#_1$ and $\#_2$ are equivalent in the model $\mathcal{M}$, if the two following sequents are valid in $\mathcal{M}$:

$$\#_1(e_1, \ldots, e_n) \vdash \#_2(e_1, \ldots, e_n)$$
$$\#_2(e_1, \ldots, e_n) \vdash \#_1(e_1, \ldots, e_n)$$

We say the connectives are equivalent if they are equivalent in all models.

Let $\#_1$ and $\#_2$ be two n-ary Gentzen connectives. It is an easy consequence of Theorem 6 that they are equivalent if and only if they are equivalent in some model, and also if and only if the union of a left axiom of one connective with the right axiom of the other connective is always a closed sequent. A consequence of the previous result is the following:

If two Gentzen connectives have the same set of right axioms, or the same set of left axioms, they are equivalent.

We note also the following transformation. Suppose a Gentzen connective is given by means of the sets of right and left axioms. Suppose some axioms are dropped from both sets and the reduced sets still satisfy the consistency property. Then the connective defined by the new sets is equivalent to the original. A situation in which the consistency property is preserved is the case in which closed sequents are dropped from the sets of axioms. Finally we note that whenever one set of axioms contains the empty sequent then all other axioms can be dropped and the new connective is equivalent.

4 *Applications.* In the remainder of the paper we consider only Gentzen connectives. We recall that the usual connectives of intuitionistic logic are Gentzen connectives.

Theorem 7. *If $M \vdash N, A$ and $A, P \vdash Q$ are both provable sequents then $M, P \vdash N, Q$ is also provable.*

Since both sequents are provable they are valid. It follows that $M, P \vdash N, Q$ is also valid, hence it is provable.

Theorem 8. *Let $M \vdash N$ be a provable regular sequent where N is non empty. Then there is a component A of N such that $M \vdash A$ is provable.*

Since $M \vdash N$ is provable every block $\mathbf{U}$ is either closed or there is a residual B such that $\mathbf{U}^* \vdash B$ is provable. In either case there is a right component B of $\mathbf{U}$ such that $\mathbf{U}^* \vdash B$ is provable. The conclusion follows then by Proposition 6.

We assume now that the usual connectives have been introduced with the axioms given in the first section. We note that the property of a sequent $M \vdash N$ being regular depends only on the components of M. We shall say that a formula A is regular if the sequent $A \vdash$ is regular. Suppose the sequent $A \vdash B \vee C$ is provable, where A is regular. It follows that $A \vdash B, C$ is provable and by Theorem 8 either $A \vdash B$ is provable or $A \vdash C$ is provable. In the case that A contains only the usual connective the result is equivalent to the theorem proved by Harrop in [3] for the intuitionistic propositional calculus. We shall consider now the possibility of defining some connective by means of a formula containing other connectives. We shall say that a formula A is elementary if it is of one of the forms $B \supset C$, $\neg B$ or C where B is a conjunction of atomic letters and C is a disjunction of atomic letters. An atomic letter alone is considered both a conjunction and a disjunction of atomic letters. We say that a formula A is in normal form if it is a conjunction of elementary formulas. We say that the formulas A and B are equivalent if the sequents $A \vdash B$ and $B \vdash A$ *are both provable.* Let A be some formula in which all atomic letters are in the list $e_1, \ldots, e_n$ and the connective $\#$ does not occur in A. We say that A defines $\#$ if A is equivalent to the formula $\#(e_1, \ldots, e_n)$.

Theorem 9. *Every connective # different from the usual connectives can be defined by some formula A in normal form.*

Let $P_1 \vdash Q_1, \ldots, P_k \vdash Q_k$ be all right axioms of $\#$. If $k = 0$ we take as A the formula $e_1 \supset e_1$. If one of the axioms is the empty sequent we take as A the formula $e_1 \wedge \neg e_1$. Otherwise we associate with $P_i \vdash Q_i$ an elementary formula A_i as follows. Let B_i be the conjunction of the components of P_i (provided this set is non empty) and C_i the disjunction of the components of Q_i (provided also it is non empty). If neither set is empty then A_i is $B_i \supset C_i$. If P_i is empty A_i is C_i. Otherwise it is $\neg B_i$. Let A be the conjunction of $A_1, \ldots, A_k$. Since for every i the sequent $A, P_i \vdash Q_i$ is provable it follows that $A \vdash \#(e_1, \ldots, e_n)$ is provable. We know also that for every i the sequent $\#(e_1, \ldots, e_n), P_i \vdash Q_i$ is provable. It follows that $\#(e_1, \ldots, e_n) \vdash A_i$ is provable, hence $\#(e_1, \ldots, e_n) \vdash A$ is also provable.

A formula which is equivalent to some formula in normal form is called a Gentzen formula.

Theorem 10. *Let A be some Gentzen formula. Then it is possible to introduce a Gentzen connective # which is defined by A.*

We may assume A is in normal form. The procedure of Theorem 8 can be reversed so we can get from A a set of axioms such that if $\#$ is a connective with such set as right axioms then A defines $\#$. We need then only to show that given a set of atomic sequents it is possible to construct another set such that when the first is taken as right axioms and the second as left axioms of some connective # the consistency and union properties are satisfied.

Let L be such set of atomic sequents. Then L_1 is the set of all sequents that can be obtained by the following method: from every sequent in L take exactly one component and put it as a right (left) component provided it was a left (right) component in the sequent from which it was taken. The same component may be taken from different sequents in L. By construction it is clear that the union of a sequent in L and a sequent in L_1 is closed, so the union property holds. To prove the consistency property consider some assignment t in the classical model; assume t accepts all sequents in L. This means that in every sequent of L there is some left component which is given value **F** by t or some right component which is given value **T**. By selecting precisely those components we get a sequent in L_1 which is refuted by t. The conjunction of two Gentzen formulas is again a Gentzen formula. We give now some examples that show that the other operations do not produce Gentzen formulas when applied to Gentzen formulas.

Consider first the formula $\neg e_1 \vee e_2$. Suppose it defines some binary connective #. Since the sequent $e_1 \supset e_2 \vdash \#(e_1, e_2)$ is valid in the classical model it is provable by Theorem 6. It follows that $e_1 \supset e_1 \vdash \neg e_1 \vee e_1$ is provable, hence $\vdash \neg e_1 \vee e_1$ is provable, and this is impossible.

Our second example is the formula $(e_1 \supset e_2) \supset e_2$. Suppose again it defines some binary connective #. By Theorem 6 the sequent $(e_1 \supset e_2) \supset e_2 \vdash e_1 \vee e_2$ is provable, hence by Theorem 8 one of the sequents $(e_1 \supset e_2) \supset e_2 \vdash e_1$ or $(e_1 \supset e_2) \supset e_2 \vdash e_2$ is provable and this is impossible.

By a similar argument if $\neg(\neg e_1 \wedge \neg e_2)$ is a Gentzen formula then one of the sequents $\neg(\neg e_1 \wedge \neg e_2) \vdash e_1$ or $(\neg e_1 \wedge \neg e_2) \vdash e_2$ is provable and this is impossible.

On the other hand a conditional of Gentzen formulas which is a classical tautology it is provable intuitionistically. We state this result as a theorem.

Theorem 11. *If the sequent $A \vdash B$ is valid in the classical model and A and B are Gentzen formulas then it is provable.*

This is a direct consequence of Theorem 6.

REFERENCES

[1] Curry, H. B., *Foundations of Mathematical Logic*, McGraw Hill Book Co., New York (1963).

[2] Fitting, M. C., *Intuitionistic Logic Model Theory and Forcing*, North Holland. Amsterdam (1969).

[3] Harrop, R., "Concerning formulas of the types $A \rightarrow B \vee C$, $A \rightarrow (Ex)B(x)$ in intuitionistic formal systems," *The Journal of Symbolic Logic*, vol. 25 (1960), pp. 27-32.

[4] Schutte, K., "Zur Semantik der intuitionistischen Aussagenlogic," in *Contributions to Mathematical Logic*, North Holland, Amsterdam (1968), pp. 231-235.

[5] Kripke, S., "Semantical analysis of intuitionistic logic I," in *Formal Systems and Recursive Functions*, North Holland, Amsterdam (1965), pp. 92-130.

Syracuse University
Syracuse, New York