448
Notre Dame Journal of Formal Logic
Volume XIV, Number 4, October 1973
NDJFAM

ON THE INTUITIONISTIC EQUIVALENTIAL CALCULUS

ROBERT E. TAX

1 *Introduction* We consider first the fragment **ICE** of the intuitionistic propositional calculus which consists of all wffs in which the only connectives are C (implication) and E (equivalence). We then consider the fragment **IE** of this system. From the Gentzen system **GCE** corresponding to **ICE**, we construct a Gentzen system **GE** corresponding to **IE**, thus obtaining a characterization of **IE** which makes no reference to an implicational system. We then look at an axiomatization and, using **GE**, show that it does indeed constitute an axiom system for **IE**.

2 *The Systems* The system **ICE** is defined as follows: The wffs of **ICE** are those constructed of propositional variables and two binary connectives, C and E. The rules of inference are substitution and Modus Ponens (from P and CPQ we can derive Q). There are five axioms:

1) $CpCqp$
2) $CCpCqrCCpqCpr$
3) $CEpqCpq$
4) $CEpqCqp$
5) $CCpqCCqpEpq$.

We define **IE** to be the equivalential fragment of **ICE**. We now construct a Gentzen system **GCE** corresponding to **ICE**: A sequent of **GCE** is to be any expression of the form $P_1, \ldots, P_n \to Q$, where $P_1, \ldots, P_n$, and Q are wffs of **ICE**, and $n \geq 0$. An axiom of **GCE** is to be any sequent of the form $P \to P$. There are nine rules of inference, as follows (where Γ and Δ represent arbitrary sequences, possibly empty, of wffs of **ICE**):

$$C \to: \frac{\Gamma \to P \quad Q, \Gamma \to R}{CPQ, \Gamma \to R} \qquad \to C: \frac{P, \Gamma \to Q}{\Gamma \to CPQ}$$

$$E \to_1: \frac{\Gamma \to P \quad Q, \Gamma \to R}{EPQ, \Gamma \to R} \qquad E \to_2: \frac{\Gamma \to Q \quad P, \Gamma \to R}{EPQ, \Gamma \to R}$$

$$\to E: \frac{P, \Gamma \to Q \quad Q, \Gamma \to P}{\Gamma \to EPQ}$$

Received August 17, 1971

$$\text{Thin: } \frac{\Gamma \rightarrow P}{Q, \Gamma \rightarrow P} \qquad \text{Cont: } \frac{P, P, \Gamma \rightarrow Q}{P, \Gamma \rightarrow Q}$$

$$\text{Int: } \frac{\Gamma, P, Q, \Delta \rightarrow R}{\Gamma, Q, P, \Delta \rightarrow R} \qquad \text{Cut: } \frac{\Gamma \rightarrow P \quad P, \Gamma \rightarrow Q}{\Gamma \rightarrow Q}$$

It is easily seen that **GCE** corresponds to **ICE**, in the sense that a sequent $P_1, \ldots, P_n \rightarrow Q$ is provable in **GCE** iff the wff $CP_1CP_2C \ldots CP_nQ$ is provable in **ICE**. Furthermore, just as in other Gentzen systems, the cut rule is optional. We define an E-wff to be a wff whose only connective is E, and an E-sequent to be a sequent which is made up of E-wffs. Suppose that S_1 is transformed by rule **L** to S_2, where S_1 and S_2 are sequents, and **L** is not the cut rule. If S_2 is an E-sequent, it is clear from the form of the rules that **L** cannot be $C\rightarrow$ or $\rightarrow C$, and that S_1 must be an E-sequent. Given a proof of an E-sequent in **GCE**, then, there is a proof of this sequent which does not use the cut rule; this proof must then consist of E-sequents, and the rules $C\rightarrow$ and $\rightarrow C$ will not appear in it. We can therefore form a Gentzen system **GE**, whose sequents and axioms are precisely those sequents and axioms of **GCE** which are E-sequents, and whose only rules of inference are $E\rightarrow_1$, $E\rightarrow_2$, $\rightarrow E$, Thin, Int, and Cont.

Then an E-sequent $P_1, \ldots, P_n \rightarrow Q$ will be provable in **GE** iff the wff $CP_1CP_2C\ldots CP_nQ$ is provable in **ICE**. In particular, we have the following:

Theorem 1: *If P is an E-wff, then $\rightarrow P$ is provable in* **GE** *iff P is a theorem of* **IE**.

We thus have a characterization of **IE** which makes no reference to any system which uses a connective other than E. We will use this to prove that the axiom system we now construct is sufficient to prove all theorems of **IE**.

3 *The Axiom System* We now construct an axiom system for **IE**.[1] There is to be one axiom: $EEEqEqpEEqEqpEpEpErsEEpsErp$. There are to be three rules: i) substitution for propositional variables; ii) Modus Ponens (**MP**): from EPQ and P we can deduce Q; and iii) Rule *: from P we can deduce $EQEQP$.

We denote provability in this system by '$\vdash$'. It is easily seen that the axiom and the rules are provable in **ICE**, and hence hold in **IE**. Suppose $\vdash EPQ$; by rule *, $\vdash EESESREESESRERERE PQ$; by the axiom and **MP**, then, $\vdash EERQEPR$. We thus have shown the following:

1) If $\vdash EPQ$, then $\vdash EERQEPR$.

Let EPQ be any theorem; by rule *, $\vdash EpEpEPQ$; by 1), $\vdash EEpEpEPQEpp$; so, by **MP**,

2) $\vdash Epp$.

3) By 1) and 2), we have $\vdash EEqpEpq$.

Def: we write '$P \rightleftarrows Q$' to mean $\vdash EPQ$.

[1]For more about the construction of the axiom, see section **6**.

By 2), $p \rightleftarrows p$. If $P \rightleftarrows Q$, then $\vdash EPQ$, so, by 3) and **MP**, $\vdash EQP$, i.e., $Q \rightleftarrows P$. By 1) and **MP**, we see that if $P \rightleftarrows Q$ and $R \rightleftarrows Q$, then $P \rightleftarrows R$. But $R \rightleftarrows Q$ if $Q \rightleftarrows R$; so, if $P \rightleftarrows Q$ and $Q \rightleftarrows R$, then also $P \rightleftarrows R$. Thus,

4) $\rightleftarrows$ is an equivalence relation.

By 1), if $P \rightleftarrows Q$ then $ERQ \rightleftarrows EPR$; since $EQR \rightleftarrows ERQ$ and $EPR \rightleftarrows ERP$, we get, by 4),

5) If $P \rightleftarrows Q$, then $EPR \rightleftarrows EQR$ and $ERP \rightleftarrows ERQ$.

It follows that if $P \rightleftarrows Q$ and $R \rightleftarrows S$, then $EPR \rightleftarrows EQR \rightleftarrows EQS$.

Def: By an expression in $p_1, \ldots, p_n$, where each p_i is a propositional variable, we mean a wff containing no variable other than $p_1, \ldots, p_n$. If $f(p_1, \ldots, p_n)$ is an expression in $p_1, \ldots, p_n$, we denote by $f(P_1, \ldots, P_n)$ the result of substituting P_i for p_i in $f(p_1, \ldots, p_n)$, for each i between 1 and n. Similarly, if f is any expression containing p, we denote by $f(P)$ the result of substituting P for p in f.

From 5), using induction on the length of the expression, we obtain:

6) If f is any expression, and if $P \rightleftarrows Q$, then also $f(P) \rightleftarrows f(Q)$.

We will write '$P \rightleftarrows (n) Q$' to mean that '$P \rightleftarrows Q$' follows from statement number n. We will not, however, mark in this way reference to numbers 3) and 6); use of any other statement will be marked in this way.

7) Letting θ denote any theorem, we have, by rule *, $P \rightleftarrows EP\theta$.
8) $EEqEqpEEqEqpEpEpr \rightleftarrows (7)\ EEqEqpEEqEqpEpEpEr\theta \rightleftarrows$ $(\mathbf{Ax})\ EEp\theta Erp \rightleftarrows (7)\ EpEpr$.
9) $EpEpErs \rightleftarrows (8)\ EEqEqpEEqEqpEpEpErs \rightleftarrows (\mathbf{Ax})\ EEpsErp \rightleftarrows EEprEps$.
10) $EpEpEpq \rightleftarrows (9)\ EEppEpq \rightleftarrows (2, 7)\ Epq$.
11) $EpEpEqr \rightleftarrows (10)\ EpEpEpEpEqr \rightleftarrows (9)\ EpEpEEpqEpr \rightleftarrows (9)\ EEpEpqEpEpr$.
12) By induction, using 11), we see that if f is any expression in $p_1, \ldots, p_n$, then $EqEqf(p_1, \ldots, p_n) \rightleftarrows f(EqEqp_1, \ldots, EqEqp_n)$.
13) $EqEqEpEqr \rightleftarrows (11)\ EEqEqpEqEqEqr \rightleftarrows (10)\ EEqEqpEqr \rightleftarrows (9)$ $EqEqEEqpr \rightleftarrows EqEqEEpqr$.

$EEpqEpEqEqp \rightleftarrows (9)\ EpEpEqEqEqp \rightleftarrows (10)\ EpEpEqp \rightleftarrows EpEpEpq \rightleftarrows (10)\ Epq$; so $\vdash EEpqEEpqEpEqEqp$; by (9), $\vdash EEEpqpEEpqEqEqp$; also, $EEpqEqEqp \rightleftarrows EEqpEqEqp \rightleftarrows (9)\ EqEqEpEqp \rightleftarrows EqEqEpEpq$; so

14) $EqEqEpEpq \rightleftarrows EEqpEqEqp \rightleftarrows EpEpq$.

In this paragraph only, let R be $EpEpq$, and let S be $EqEqp$. We then see that $EEpEpqEqEqp \rightleftarrows EEEpqpEEpqq \rightleftarrows (9)\ EEpqEEpqEpq \rightleftarrows (2, 7)\ Epq$: i.e., $ERS \rightleftarrows Epq$. It follows that $ERERS \rightleftarrows EEpEpqEpq \rightleftarrows (14)\ EqEqp = S$: so a) $ERERS \rightleftarrows S$. Furthermore, we have that $EEpEpqEEpEpqp \rightleftarrows EEpEpqEpEpEpq \rightleftarrows (10)\ EEpEpqEpq \rightleftarrows (14)\ EqEqp$: i.e., b) $ERERp \rightleftarrows S$. So $EpEpEqEqr \rightleftarrows (12)\ EEpEpqEEpEpqEpEpr = ERER的EpEpr$ $\rightleftarrows (12)$ $EERERpEERERpERERr \rightleftarrows (b)\ ESESERERr$. Similarly, $EqEqEpEpr \rightleftarrows ERERESESr$. But we can also see that $ERERESESr \rightleftarrows (12)$

$EERERSEERERERSERERr \rightleftarrows$ (a) $ESESERERr$. As a result, we have proved the following:

15) $EpEpEqEqr \rightleftarrows EqEqEpEpr$.

$EqEqEEpqEEpqr \rightleftarrows$ (11) $EEqEqEpqEqEqEEpqr \rightleftarrows$ (13) $EEqEqEpqEqEqEpEqr \rightleftarrows$ (11) $EqEqEEpqEpEqr \rightleftarrows$ (9) $EqEqEpEpEqEqr \rightleftarrows$ (15) $EpEpEqEqEqEqr \rightleftarrows$ (10) $EpEpEqEqr$. Since we also know, by 10), that $EpEpEpEpEqEqr \rightleftarrows EpEpEqEqr$, we have the following:

16) $EpEpEqEqEEpqEEpqr \rightleftarrows EpEpEqEqr$.

Def: For any finite set of wffs $A = \{a_1, \ldots, a_n\}$, define a function $A\#$ by setting $A \# P = Ea_1Ea_1Ea_2Ea_2E \ldots Ea_nEa_nP$; if $A = \phi$, set $A \# P = P$. We will sometimes write '$A \# (P)$' to mean $A \# P$.

By 10) and 15) above, this expression is independent of the order and possible repetitions of the a_i, so A^* is well-defined, up to the equivalence relation $\rightleftarrows$. We will use the letters A and B to refer to finite sets of wffs. We see that for any finite sets A and B of wffs, $A \# B \# P \rightleftarrows (A \cup B) \# P$. Also, by induction on 12) above, we see that for any expression f in $p_1, \ldots, p_n$ we have $A \# f(p_1, \ldots, p_n) \rightleftarrows f(A \# p_1, \ldots, A \# p_n)$.

4 *Some Consequences* For any finite set A of wffs, we define A^* to be the smallest set containing A and which is closed under E and rule *, i.e., which satisfies the two conditions: i) if $P, Q \in A^*$, then $EPQ \in A^*$; and ii) if $P \in A^*$, then $EQEQP \in A^*$. Note that if $P \in A^*$, then $B \# P \in A^*$.

Lemma 2: *If $P \in A^*$, then $A \# EPEPQ \rightleftarrows A \# Q$.*

Proof: We use induction on the length of P. From the definition of A^*, it is clear that we must consider three cases:

Case 1: $P \in A$. Then $A \# EPEPQ \rightleftarrows A \# \{P\} \# Q \rightleftarrows (A \cup \{P\}) \# Q \rightleftarrows A \# Q$, since $A \cup \{P\} = A$.

Case 2: $P = ERS$, with $R, S \in A^*$. The lemma then holds for R and S. Then $A \# EPEPQ = A \# EERSEERSQ \rightleftarrows$ (ind. hyp.) $A \# ERERESESEERSEERSQ \rightleftarrows$ (16) $A \# ERERESESQ \rightleftarrows$ (ind. hyp.) $A \# Q$.

Case 3: $P = ERERS$, with $S \in A^*$. The lemma then holds for S. Then $A \# EPEPQ = A \# EERERSEERERSQ \rightleftarrows$ (ind. hyp.) $A \# ESESEERERSEERERSQ \rightleftarrows$ (8, 15) $A \# ESESQ \rightleftarrows$ (ind. hyp.) $A \# Q$, proving the lemma.

Lemma 3: *If $A \subset B \subset A^*$, where A and B are finite sets of wffs, then $A \# P \rightleftarrows B \# P$.*

Proof: Let $B = A \cup \{b_1, \ldots, b_n\}$, with each $b_i \in A^*$. Then, using Lemma 2 n times, $B \# P \rightleftarrows A \# Eb_1Eb_1E \ldots Eb_nEb_nP \rightleftarrows A \# P$.

Def: If A is a finite set of wffs, we write '$A > P$' to mean that $\vdash A \# P$.

Lemma 4: *The following properties of $>$ hold:*

a) *if $A > P$, then $A > B \# P$;*
b) *if $A > EPQ$ and $A > P$, then $A > Q$;*

c) *if* $A > EPQ$ *then* $A > EEPREQR$ *and* $A > EERPERQ$;
d) *if* $A > EPQ$ *and* $A > EQR$ *then* $A > EPR$;
e) *if* $A > EPQ$ *and* $A > ERS$ *then* $A > EEPREQS$;
f) *if* f *is an expression and* $A > EPQ$, *then* $A > E(fP)(fQ)$;
g) *if* $P \in A^*$, *then* $A > Ef(EQEPR)f(EEQPR)$, *for any expression* f.

Proof: a) If $A > P$, then $\vdash A \# P$; using rule *, $\vdash B \# A \# P$, so $\vdash A \# B \# P$, i.e., $A > B \# P$.

b) If $A > EPQ$, then $\vdash A \# EPQ$, so $\vdash EA \# PA \# Q$. If also $A > P$, then $\vdash A \# P$. By **MP**, $\vdash A \# Q$, i.e., $A > Q$.

c) Suppose $A > EPQ$; by a), $A > ERERERPQ$. But, by 9) of section **3**, $\vdash EERERERPQEERPERQ$, so, applying rule * several times, $A > EERERERPQEERPERQ$. By b), then, $A > EERPERQ$. Similarly, $A > EEPREQR$.

d) If $A > EPQ$, then $A > EEPREQR$ by c), so $A > EEQREPR$. If also $A > EQR$, then $A > EPR$ by b).

e) If $A > EPQ$, then $A > EEPREQR$ by c). If $A > ERS$, then $A > EEQREQS$, again by c). By d), then, if $A > EPQ$ and $A > ERS$, then $A > EEPREQS$.

f) If f is an expression and $A > EPQ$, then $\vdash A \# EPQ$. Let $f(p)$ be $g(p, q_1, \ldots, q_n)$, where g is an expression in $p, q_1, \ldots, q_n$. Then $f(P) = g(P, q_1, \ldots, q_n)$, so, by an obvious induction applied to 12) above, $A \# fP \rightleftarrows g(A \# P, A \# q_1, \ldots, A \# q_n)$. Similarly, $A \# fQ \rightleftarrows g(A \# Q, A \# q_1, \ldots, A \# q_n)$. Then, using 9) above, $A \# E(fP)(fQ) \rightleftarrows EA \# fPA \# fQ \rightleftarrows Eg(A \# P, A \# q_1, \ldots, A \# q_n)g(A \# Q, A \# q_1, \ldots, A \# q_n)$. Since $\vdash A \# EPQ$, also $\vdash EA \# PA \# Q$, so this last wff in the chain is a theorem, by property 6) above; so $\vdash A \# E(fP)(fQ)$, i.e., $A > E(fP)(fQ)$.

g) Suppose $P \in A^*$. By Lemma 2, $A \# EPEPEEQEPREEQPR \rightleftarrows A \# EEQEPREEQPR$. But by 13) above, $\vdash EPEPEEQEPREEQPR$, so $\vdash A \# EPEPEEQEPREEQPR$, and hence $\vdash A \# EEQEPREEQPR$, i.e., $A > EEQEPREEQPR$. The result then follows by f).

Notation: we write '$\sum_{i=1}^{n} P_i$' to mean $EP_1EP_2E \ldots EP_{n-1}P_n$. We set this equal to P_1 if $n = 1$, and to any theorem if $n = 0$: we will often omit the limits of the index when clear from the context, writing $\sum_i P_i$ or even $\sum P_i$.

We note that $EpEpEqEqEpEqr \rightleftarrows$ (13) $EpEpEqEqEEpqr \rightleftarrows$ (15, 3) $EqEqEpEpEEqpr \rightleftarrows$ (13) $EqEqEpEpEqEpr \rightleftarrows$ (15) $EpEpEqEqEqEpr$. Using this and Lemmas 4b, 4f, 4g and some of the results from section **3**, the following additional properties of $>$ are easily seen:

Lemma 4′: *Let* $Q_1, \ldots, Q_n$ *be a permutation of* $P_1, \ldots, P_n$, *where each* $P_i \in A^*$, *and let* f *be any expression. Then*

a) $A > f\left(\sum P_i\right)$ *iff* $A > f\left(\sum Q_i\right)$;
b) $A > f(EP_1EP_2E \ldots EP_nR)$ *iff* $A > f\left(E \sum P_iR\right)$;
c) $A > f(EP_1EP_2E \ldots EP_nR)$ *iff* $A > f(EQ_1EQ_2E \ldots EQ_nR)$.

5 *Completeness of the Axiom System* Our major goal is to show that $\vdash P$ iff P is a theorem of **IE**. We have already noted that the axiom and rules of our system hold in **IE**, so that P is a theorem of **IE** whenever $\vdash P$. By Theorem 1, it suffices to show that if $\rightarrow P$ is a theorem of **GE**, then $\vdash P$. We do this by defining a relation $P_1, \ldots, P_n \Vdash Q$ in our system, with the property that $\vdash P$ iff $\Vdash P$. The desired result is then a special case of the fact that if $P_1, \ldots, P_n \rightarrow Q$ is a theorem of **GE**, then $P_1, \ldots, P_n \Vdash Q$. To show this, we show that the axioms of **GE**, when interpreted in this way, become provable in our system, and that this property is preserved by all rules of inference of **GE**. The only difficulties will be the rules $E \rightarrow_1$ and $\rightarrow E$, which we dispose of in Theorems 8 and 9. We will then be able to conclude that $\vdash P$ iff P is a theorem of **IE**, as desired.

Def: For any finite set A of wffs, we set $A' = \{B \# a \mid a \in A, B$ a finite set of wffs$\}$.

We note that $A \subset A' \subset A^*$. Furthermore, $(A \cup B)' = A' \cup B'$.

Def: If A is any finite set of wffs, we say '$A \Vdash P$' to mean that there are wffs $Q_1, \ldots, Q_n$, with each $Q_i \in A'$, such that $\vdash EQ_1EQ_2E \ldots EQ_nP$. We allow $n = 0$ in this definition; thus, if $\vdash P$, then $A \Vdash P$.

We write '$\Vdash P$' to mean $\phi \Vdash P$. Thus, it is clear that $\vdash P$ iff $\Vdash P$. We write '$P_1, \ldots, P_n \Vdash Q$' to mean $\{P_1, \ldots, P_n\} \Vdash Q$. As noted above, we will show that $P_1, \ldots, P_n \Vdash Q$ whenever $P_1, \ldots, P_n \rightarrow Q$ is a theorem of **GE**. Clearly, $P \Vdash P$, for any wff P. Equally clearly, the rules Cont and Int of **GE** preserve $\Vdash$. If $A \subset B$, and $A \Vdash P$, then it is clear from the definition of $\Vdash$ that $B \Vdash P$; thus, the rule Thin also preserves $\Vdash$. The rule $E \rightarrow_2$ is an easy consequence of the rule $E \rightarrow_1$, when interpreted in terms of $\Vdash$, since we know that EPQ can be substituted for EQP anywhere, in our system. Thus, we have merely to show that $\Vdash$ is preserved by the rules $E \rightarrow_1$ and $\rightarrow E$ in order to prove that we do have an axiomatization of **IE**. This is what we now do, after some preliminary lemmas.

Lemma 5: *Suppose $A > EPQ$, and $Q \in A^*$. Then $A \Vdash P$.*

Proof: We use complete induction on the length of Q; suppose the lemma is true for all wffs shorter than Q. Let $A = \{a_1, \ldots, a_n\}$.

Case 1: $Q \in A$; since $\vdash Ea_1Ea_1E \ldots Ea_nEa_nEQP$, with $a_i, Q \in A'$, $A \Vdash P$.

Case 2: $Q = ERS$, with $R, S \in A^*$; then the lemma is true for R and S by induction hypothesis. We have $A > EERSP$, with $R, S \in A^*$; by Lemma 4′, $A > ERESP$; by ind. hyp., $A \Vdash ESP$. Then there are wffs $c_1, \ldots, c_k \in A' \subset A^*$ such that $\vdash Ec_1E \ldots Ec_kESP$. We then also have $A > Ec_1E \ldots Ec_kESP$. Since $c_i, S \in A^*$, we can permute, by Lemma 4′, getting $A > ESEc_1E \ldots Ec_kP$. By induction hypothesis, then, $A \Vdash Ec_1E \ldots Ec_kP$. Then there are wffs $b_1, \ldots, b_m \in A'$ such that $\vdash Eb_1E \ldots Eb_mEc_1E \ldots Ec_kP$. Since each b_i, $c_i \in A'$, this shows that $A \Vdash P$, as desired.

Case 3: $Q = B \# ERS$, with $R, S \in A^*$. Then $A > EPEB \# RB \# S$, with $B \# R$ and $B \# S$ in A^* and shorter than Q. By Case 2, $A \Vdash P$.

Lemma 6: *Suppose* $A \Vdash P$. *Then there is a wff* $Q \epsilon A^*$ *such that* $A > EQP$.

Proof: By definition, there are $a_1, \ldots, a_n \epsilon A' \subset A^*$ such that $\vdash Ea_1E \ldots Ea_nP$. Let $Q = \sum a_i$. By Lemma 4′, $A > E\, Ea_1E \ldots Ea_nPEQP$; also, $A > Ea_1E \ldots Ea_nP$. By Lemma 4b, $A > EQP$. Also, each $a_i \epsilon A^*$, so $Q \epsilon A^*$, as desired.

Lemma 7: *Suppose* $A, P \Vdash Q$. *Then there is a wff* $R \epsilon A^*$ *and finite sets* $B_1, \ldots, B_n$ *such that* $A > EPEPERE\left(\sum B_i \# P\right) Q$ *and* $A > EPEPEQER \sum B_i \# P$.

Proof: There are $a_1, \ldots, a_k \epsilon (A \cup \{P\})' = A' \cup \{P\}'$ such that $\vdash Ea_1E \ldots Ea_kQ$. Then $A, P > Ea_1E \ldots Ea_kQ$, with each $a_i \epsilon (A \cup \{P\})^*$. Let the a_i's which are in A' be $b_1, \ldots, b_r$, and let $R = \sum_{i=1}^{r} b_i$; then $R \epsilon A^*$. Let the other a_i's, which are then in $\{P\}'$, be $B_1 \# P, \ldots, B_n \# P$. By Lemma 4′, we can permute the a_i's, getting $A, P > Eb_1E \ldots Eb_rEB_1 \# PE \ldots EB_n \# PQ$; by Lemma 4′, we can now reassociate the a_i's, getting $A, P > ERE\left(\sum B_i \# P\right)Q$, which is equivalent to the first desired form. Now, since $R, \sum B_i \# P \epsilon (A \cup \{P\})^*$, we can, by Lemma 4′, rearrange the terms, getting $A, P > EQER \sum B_i \# P$, which is equivalent to the other desired form.

Theorem 8: *Suppose* $A \Vdash P$ *and* $Q, A \Vdash R$. *Then* $EPQ, A \Vdash R$.

Proof: By Lemmas 6 and 7 we get

i) $A > EQEQEa_1E \sum B_i \# QR$, with $a_1 \epsilon A^*$,
ii) $A > Ea_2P$, with $a_2 \epsilon A^*$.

We apply $\{P\} \#$ to i), by Lemma 4a, and distribute, letting $a_3 = EPEPa_1$; $a_3 \epsilon A^*$:

iii) $A > EEPEPQEEPEPQEa_3E \sum B_i \# (EPEPQ)\ EPEPR$.

By ii) and iii) and Lemmas 4f and 4b, we get

v) $A > EEa_2EPQEEa_2EPQEa_3E \sum B_i \# (Ea_2EPQ)\ Ea_2Ea_2R$.

Now let $a = EEa_2EPQEEa_2EPQEa_3E \sum B_i \# (EA_2EPQ)\, Ea_2a_2$; then, since each of Ea_2EPQ, a_3, a_2, $B_i \# (Ea_2EPQ) \epsilon (A \cup \{EPQ\})^*$, we see that $a \epsilon (A \cup \{EPQ\})^*$, and also that we can reassociate v) to get the following, using rule * and the definition of $>$:

vi) $A, EPQ > EaR$.

By Lemma 5, then, we see that $A, EPQ \Vdash R$, as desired.

Theorem 9: *Suppose* $A, P \Vdash Q$ *and* $A, Q \Vdash P$. *Then* $A \Vdash EPQ$.

Proof: In the following, let i and k run from 1 to m; j from 1 to n. Let $B = A \cup \{P, Q\}$. We have, by Lemma 7, since $P \epsilon B$ and $Q \epsilon B$,

i) $B > EQEa_1 \sum_i C_i \# P$ where $a_1 \epsilon A^*$,

ii) $B > EPEa_2 \sum_j D_j \# Q$ where $a_2 \epsilon A^*$.

For each j, we apply $D_j\#$ to i), getting

iii) $B > ED_j\#\ QED_j\#\ a_1 \sum_i (C_i \cup D_j)\ \#\ P$, for each j.

By ii) and iii) and Lemmas 4b and 4f, we can 'substitute,' getting

iv) $B > EPEa_2 \sum_j \left(ED_j\#\, a_1 \sum_i (C_i \cup D_j)\ \#\ P\right)$;

so, since P, $a_1 \epsilon B^*$, we get, by Lemma 4′, $B > EPEa_2E\sum_j D_j\ \#\ a_1 \sum_{i,j} (C_i \cup D_j)\#P$; letting $a_3 = Ea_2 \sum_j D_j\ \#\ a_1 \epsilon A^*$, we get, by Lemma 4′,

v) $B > EPEa_3 \sum_{i,j} (C_i \cup D_j)\#P$.

Applying $C_k\#$, we get: For each k, $B > EC_k\#PEC_k\#a_3 \sum_{i,j} (C_i \cup D_j \cup C_k)\#P$, i.e.,

vi) $B > EC_k\#PEC_k\#a_3E\sum_j (C_k \cup D_j)\#P \sum_{\substack{i,j\\ i\neq k}} (C_i \cup D_j \cup C_k)\#P$, for each k.

But by i), $B > EQEa_1 \sum_k C_k\#P$; summing over k, using vi) and Lemmas 4′, 4b, 4f, we see that $B > EQEa_1E \sum_k C_k\#\ a_3E \sum_{j,k} (C_k \cup D_j)\#P \sum_{\substack{i,j,k\\ i\neq k}} (C_i \cup D_j \cup C_k)\#P$. But $B > \sum_{\substack{i,j,k\\ i\neq k}} (C_i \cup D_j \cup C_k)\#P$, by Lemma 4′, since each term appears exactly twice. By Lemma 4a, then, $B > EE \sum_{j,k} (C_k \cup D_j)\#P \sum_{\substack{i,j,k\\ i\neq k}} (C_i \cup D_j \cup C_k)\#P\sum_{j,k} (C_k \cup D_j)\#P$, and hence, by Lemmas 4′, 4b, 4f, letting $a_4 = Ea_1 \sum_k C_k\#a_3 \epsilon A^*$, we get $B > EQEa_4 \sum_{j,k} (C_k \cup D_j)\#P$. Using v) and Lemma 4e, we see that $B > EEPQEEa_3 \sum_{j,k} (C_k \cup D_j)\#PEa_4 \sum_{j,k} (C_k \cup D_j)\#P$. Since a_3, a_4, $\sum_{k,j} (C_k \cup D_j)\#P \epsilon B^*$, we can reassociate, by Lemma 4′, getting $B > EEPQEa_3a_4$. Letting $a_5 = Ea_3a_4 \epsilon A^*$, we see that $B > Ea_5EPQ$. Since $B = A \cup \{P, Q\}$, we have $A > EPEPEQEQEa_5EPQ$, so by Lemma 4b, $A > EEPEPEQEQa_5EPEPEQEQEPQ$. But

$EPEPEQEQEPQ \rightleftarrows EPEPEQEQEQP \rightleftarrows$ (10 above) $EPEPEQP \rightleftarrows EPEPEPQ \rightleftarrows$ (10) EPQ.

Thus, letting $a = EPEPEQEQa_5$, we see that $a \epsilon A^*$ and $A > EaEPQ$. By Lemma 5, then, $A \Vdash EPQ$, as desired.

As noted before, Theorems 8 and 9 give us the following results:

Theorem 10: $P_1, \ldots, P_n \Vdash Q$ *iff* $P_1, \ldots, P_n \rightarrow Q$ *is a theorem of* **GE**.
Theorem 11: *For any E-wff* P, $\vdash P$ *iff* P *is a theorem of* **IE**.

6 *Further Remarks* To help the intuition, we note that $EEpEpEqEqrEKpqEKpqr$ is a theorem of the full intuitionistic propositional calculus; many of the wffs we used follow quite easily from this. For instance, since *CCpqEpKpq* is intuitionistically valid, it follows that *CCpqEEpEpEqEqrEpEpr* is also. Since *CpEqEqp* is also intuitionistically valid, for instance, then, so is *EEpEpEEqEqpEEqEqprEpEpr*. And, since *EKpqKqp* is a theorem of the intuitionistic system, so is *EEpEpEqEqrEEqEqEpEpr*.

Our axiom is essentially built up of three wffs: i) *EEpqEqp*; ii) *EEpEpEqrEEpqEpr*; and iii) *EEEqEqpEEqEqpEpEprEpEpr*. If we were to take i) and ii) as axioms, with the same rules as before, we would get a very large subsystem of **IE**; in fact, all of the numbered wffs in section 3 would be provable with the exception of number 8. That iii) is actually independent of i) and ii) can be shown using the following matrix: The values are 0, a, 1+, 1-, 2+, 2-, . . ., with 0 the designated value. For any values x and y, $Exy = Eyx$; $Exx = 0$; and $E0x = Ex0 = x$. For $n = 1$, 2, . . ., $Ea(n\pm) = n\mp$, and $E(n\pm)(n\mp) = ((n+1)-)$. Also, if $m < n$, then $E(m\pm)(n\pm) = (m+)$, and $E(m\pm)(n\mp) = (m-)$. It seems to me that this subsystem would be of great value in any search for a shortest sole axiom of **IE**. I conjecture that rule * is necessary, in the sense that there is no finite axiomatization of **IE** in which the only rules are substitution and **MP**. I have not succeeded in proving this, however.

University of Chicago
Chicago, Illinois