

## A FELICITOUS FRAGMENT OF THE PREDICATE CALCULUS

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A notation without variables of quantification is provided for a fragment of the lower predicate calculus with one- and two-place predicates. It can be translated nearly symbol-by-symbol into reasonable English. The fragment is decidable. An apparently minor extension giving flexibility of quantifier scope yields a fragment that is undecidable.

Horses are animals, but not conversely. Therefore, heads of horses are heads of animals, though not conversely. But lovers of horses, if by this we mean lovers of all horses, are not necessarily lovers of animals—that is, of all animals—since they may fail to love crocodiles: in this case, it is the converse that is true, namely, that lovers of all animals are lovers of all horses. Likewise, lovers of (all) heads of animals are lovers of (all) heads of horses, and heads of lovers of (all) animals are heads of lovers of (all) horses, but in neither case conversely; and admirers of (all) lovers of (all) horses are admirers of (all) lovers of (all) animals, but not conversely. The example, in its earlier stages, is from De Morgan ([1], p. 131) and examples like it are to be found earlier in Junge [4] in discussion of the topics of genus and species. The general rule of validity for our extension of it is that the complex term containing “horses” must be in subject-position if the number of occurrences of “all” in it (and in its counterpart containing “animals”) is even, and in predicate-position if this number is odd.

The English expression of these consequences is more compact and perspicuous, provided some care is taken with quantifiers, than their predicate calculus equivalents. For example, the statement that admirers of all lovers of all horses are admirers of all lovers of all animals has the form

$$\forall x(\forall y(\forall z(az \supset gyz) \supset fxy) \supset \forall y(\forall z(bz \supset gyz) \supset fxy)) \quad (1)$$

The English expression contains no counterpart of variables of quantification. The expedient is suggested of introducing quantifiers with the grammatical function of what are called in older grammar-books *pronoun-relatives* (not to be confused with relative pronouns) like Latin *quicumque*.

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Let *A* represent “everything that”: we can use it either preceding two one-place predicate symbols, as

$$Aab$$

representing

Everything that - is *a* - is *b*,

or to link a two-place predicate symbol to a one-place predicate symbol to give a complex one-place predicate expression, as

$$fAa$$

representing

bears *f* to - everything that - is *a*.

English word-order is preserved. Formula (1) now takes the form

$$AfAgAafAgAb \tag{2}$$

“Everything that - admires - everything that - loves - everything that - is a horse - admires - everything that - loves - everything that - is an animal.”

We may use *E* similarly for “something that.” Embedded in a propositional calculus the forms *Aab* and *Eab* give us a notation for syllogistic inference in the manner of Łukasiewicz ([6], see also [8]) and the system may be seen as an extension of this. Certain other extensions immediately suggest themselves, notably to negative and compound terms. Since it is of interest to see how much of the lower predicate calculus (LPC) can be comprised, we shall also employ negative and compound two-place predicates, free individual variables and certain special symbols.

The prospect of a relatively rich language that is effectively a subset of English and is provided with both a precise syntax and a logic gives this project some connection with the work of Montague [7].\* The language to be described is richer than Montague’s in containing connectives, though poorer in making no provision for adjectives and adverbs. In another respect it is not quite comparable since its goal does not include—as Montague’s does—the modelling of some ambiguities found in English.

The notation in its main features was previously used by the author in an unpublished thesis [3]. The present paper deals with its use to represent ordinary predicate calculus but it has other possible uses and extensions, such as to many-sorted calculi, proportional quantification and probability.

*Syntax of FF* The following table gives a context-free grammar for the system to be known as **FF**. Lower-case roman words of English represent syntactical categories, and on the right of each rewriting rule items on separate lines, or on the same line but separated by a solidus, are

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\*Montague’s disturbing death was fresh news while this paper was being written. I had never met him, but held the kind of respect for his work that made me wish that I had.

alternatives. A formula is constructed by starting with the word "formula" and rewriting until no lower-case roman English words remain.

formula:	= nominal predicate	(FS)
	- formula	
	( formula connective formula	
connective:	= $\cdot   \vee   \supset   \equiv$	
nominal:	= $x   y   z   \dots$	
	quantifier predicate	
quantifier:	= $A   E   J$	
predicate:	= $a   b   c   \dots   t$	
	- predicate	
	( predicate connective predicate	
	relation nominal	
	relation s	
relation:	= $f   g   h \dots   r   =$	
	- relation	
	$\vee$ relation	
	( relation connective relation	

In the table and in subsequent discussion "predicate" means "one-place predicate." Formulae are truth-functions of formulae of the form "nominal predicate," which may take the form "nominal relation nominal": a nominal may be either an individual symbol, or a quantifier followed by a predicate. Predicates and relations may be truth-functionally complex. Explanations of the additional symbols will be given later.

In forming compound formulae or predicates or relations using connectives a left-hand bracket is inserted. Right-hand brackets are redundant and are not used. Negation requires no bracketing, but in the case of relations  $\bar{\quad}$  is used instead of - to distinguish the result of concatenating " $\bar{\quad}$  relation" with "quantifier predicate" from the result of negating "relation quantifier predicate."

**FF** into canonical English Fixed nominals  $x, y, \dots$  are read as themselves, or as "John," "London," "Venus," etc.

Fixed predicates  $a, b, \dots$  are read "is  $a$ ," etc., or as "plays golf," "is green," etc.

Fixed relations  $f, g, \dots$  are read "has  $f$  to," etc., or as "loves," "likes going to," "is a parent of," etc.

Other signs are translated as follows:

- $\rightarrow$	not	(FE)
.	$\rightarrow$ and	
$\vee$	$\rightarrow$ or	
$\supset$	$\rightarrow$ then	
$\equiv$	$\rightarrow$ then and only then (thenn)	
( $\rightarrow$	$\left\{ \begin{array}{l} \text{both} \\ \text{either} \\ \text{if} \\ \text{if and only if (iff)} \end{array} \right\}$	(according as it is coupled with $\cdot, \vee, \supset, \equiv$ respectively)

$\bar{\quad}$	$\rightarrow$	non
$\bar{\quad}$	$\rightarrow$	inversely
<b>A</b>	$\rightarrow$	everything that
<b>E</b>	$\rightarrow$	something that
<b>J</b>	$\rightarrow$	the thing that
<i>t</i>	$\rightarrow$	has some property, <i>or</i> is a thing
<i>s</i>	$\rightarrow$	itself
<i>r</i>	$\rightarrow$	has some relation to, <i>or</i> is a thing and so is
=	$\rightarrow$	is identical with.

As translations of quantifiers, "something that" and "the thing that" are subject to their usual logical interpretations. The necessity of seeking alternative translations of ( could, of course, have been avoided by providing four separate bracket-symbols in the first place. It is redundant to differentiate connectives by having both four separate initial and four separate medial operators, but this corresponds partly with careful English usage: comparison with Polish notation shows that it is the medial operators, if any, that are redundant. The only translations that fail to give grammatical English are those for negation and for inverse relation. Thus "not" and "non" as initial operators give us the strained if meaningful constructions "not is *a*," "not is green," "non loves," "non is a parent of," which could be only slightly improved by using more exotic negatives such as "nohow." Normal English negative construction is just a little more complicated. In the case of the inverse relation, what is wanted is generally a passive construction, turning, say, "loves" into "is loved by"; but "inversely loves" will have to do duty. Translation is reversible: the FF formula is recoverable from its canonical English expression.

**FF into LPC** The following table gives rules for translation of FF into standard LPC notation. Expressions in square brackets [ ] which occur temporarily as parts of FF formulae are in LPC notation: if they are one-place predicates  $\xi$  marks the argument-place, and if they are two-place predicates  $\xi$  marks the left-hand argument-place and  $\eta$  the right.  $\alpha, \beta$  are arbitrary (possibly complex) one-place predicates of the LPC, and  $\phi, \psi$  two-place predicates. Translation starts with one- and two-place predicates of FF and proceeds outwards.

$a$	$\rightarrow$	$[\alpha\xi]$	(similarly $b, c, \dots$ )	(FP)
$f$	$\rightarrow$	$[f\xi\eta]$	(similarly $g, h, \dots$ )	
$\bar{[\alpha\xi]}$	$\rightarrow$	$[-\alpha\xi]$		
$([\alpha\xi] \kappa [\beta\xi])$	$\rightarrow$	$[(\alpha\xi \kappa \beta\xi)]$	( $\kappa$ any connective)	
$\bar{[\phi\xi\eta]}$	$\rightarrow$	$[-\phi\xi\eta]$		
$\bar{\bar{[\phi\xi\eta]}}$	$\rightarrow$	$[\phi\eta\xi]$		
$([\phi\xi\eta] \kappa [\psi\xi\eta])$	$\rightarrow$	$[(\phi\xi\eta \kappa \psi\xi\eta)]$	( $\kappa$ any connective)	
$[\phi\xi\eta]s$	$\rightarrow$	$[\phi\xi\xi]$		
$[\phi\xi\eta]A[\alpha\xi]$	$\rightarrow$	$[\forall\xi(\alpha\xi \supset \phi\xi\xi)]$	( $\xi$ a new variable)	

$$\begin{aligned}
 [\phi\xi\eta] E [\alpha\xi] &\rightarrow [\exists\xi(\alpha\xi \cdot \phi\xi\xi)] \quad (\xi \text{ a new variable}) \\
 x[\alpha\xi] &\rightarrow [\alpha x] \quad (\text{similarly } y, z, \dots) \\
 A[\alpha\xi] [\beta\xi] &\rightarrow [\forall\xi(\alpha\xi \supset \beta\xi)] \quad (\xi \text{ a new variable}) \\
 E[\alpha\xi] [\beta\xi] &\rightarrow [\exists\xi(\alpha\xi \cdot \beta\xi)] \quad (\xi \text{ a new variable})
 \end{aligned}$$

Signs for which translations are not provided will be introduced later by definition. The last three lines yield whole LPC formulae; truth-functions of these are carried over unchanged except in respect of bracketing convention. Apart from small points concerning  $\vee$  and  $\wedge$ , translation is again reversible.

*Logic of FF* Logical properties of FF may be determined by translation into LPC equivalents but it is of interest to explore an autonomous procedure. Assuming the properties of the propositional calculus applied to formulae, we could start by adding an axiom set for syllogistic such as that of Wedberg (see [8]), but it is actually a little easier to start with the rule of singular syllogism

$$Aab \supset (xa \supset xb) \tag{3}$$

and the analogous rule

$$(xfAa \supset ya \supset xfy) \tag{4}$$

These give us in effect two forms of universal instantiation UI:

$$Aab \rightarrow (xa \supset xb) \tag{UI1}$$

$$xfAa \rightarrow (ya \supset xfy) \tag{UI2}$$

Existential quantification is definable in terms of universal

$$Eab = \neg Aa\neg b \tag{DE1}$$

$$fEa = \neg fAa \tag{DE2}$$

Furthermore, negations and connectives standing over or between predicates or relations can be converted, when the arguments are individual, to negations and connectives standing over or between formulae:

$$x\neg a = \neg xa \tag{DN1}$$

$$x\neg f y = \neg xfy \tag{DN2}$$

$$x(a\kappa b) = (xa \kappa xb) \tag{DC1}$$

$$x(f\kappa g y) = (xfy \kappa xgy) \tag{DC2}$$

where  $\kappa$  is any connective. Consequently, we can derive corresponding forms of EG:

$$(xa \cdot xb) \rightarrow Eab \tag{EG1}$$

$$(ya \cdot xfy) \rightarrow xfEa \tag{EG2}$$

Rules of UG and EI, which are, of course, subject to the usual restrictions, are converses of those for UI and EG.

The inverse symbol and reflexive can be eliminated in individual contexts:

$$(x\check{f}y \equiv yfx) \quad (\text{AI1})$$

$$(\check{\neg}f \equiv \neg\check{f}) \quad (\text{AI2})$$

$$(\check{f}\kappa g \equiv (\check{f}\kappa\check{g})) \quad (\text{AI3})$$

$$(xfs \equiv xfx) \quad (\text{AR})$$

Although UI and EI can be used formally on any formula to replace quantifiers by individual letters, it is not in general possible to conduct the reverse process by UG and EG. Thus EG, for example, can be performed only on formulae that can be expressed as conjunctions of the form of the left-hand side of (EG1), or that of (EG2). In spite of this limitation we can prove

**Theorem 1.** *Natural deduction is a complete proof procedure within FF.*

The proof is rather trivial. No difficulty attaches to truth-functional procedures or to conditionalisation and it is necessary only to consider the availability of the usual procedure of stripping of quantifiers and their possible replacement. But the operation of stripping quantifiers can always be performed; and the operation of replacing them can be performed whenever the object formula is expressible. The logic of formulae not containing quantifiers is contained in the rules DN1-2, DC1-2, AI1-3 and AR.

We prove by way of example in essentially Quine's system [9] that if all horses are animals then everything that is the head of some horse is the head of some animal.

1.  $Aab$  Assumption
2.  $xfEa$  Assumption
3.  $\xi a \cdot xf\xi$  EI2
4.  $\xi a \supset \xi b$  1, UI1
5.  $\xi b \cdot xf\xi$  3, 4, truth-functionally
6.  $xfEb$  EG2
7.  $AfEafEb$  2, 6, conditionalisation and UG1

Hence 1.  $\supset$  7. by a further conditionalisation. The variable  $\xi$  introduced by EI in 3. is extinguished in 6. and the assumptions in 1. and 2. are cancelled by the conditionalisations.

Identity cannot be introduced by prefixing of axioms into FF as it can into LPC. This is because the axioms cannot all be represented in FF. However, if identity is introduced as a primitive having its usual properties when standing between individual symbols, its properties in quantified formulae follow. The reflexive symbol  $s$  is definable in terms of it

$$fs = (f \cdot =Et) \quad (\text{DS})$$

where  $t$  is the universal predicate, defined

$$t = (a \vee \neg a) \quad (\text{DT})$$

An identity following a quantifier is always redundant together with the quantifier; thus

$$(A=xa \equiv xa) \tag{5}$$

$$(E=xa \equiv xa) \tag{6}$$

results which are rather concise compared with their **LPC** counterparts. Identity also permits definition of the definite description quantifier, thus

$$Ja = E(a.=Aa) \tag{DJ}$$

“the *a*,” or “the thing that is *a*.” Compare in this connection Montague’s treatment of “the” as a quantifier [7]. In expressions of the form *Jab*, *J* has the properties of Leśniewski’s  $\varepsilon$  (see [8]); but it can also be used in other contexts, such as in *xfJa*.

It is hardly an exaggeration to say that **FF** is adequate to all the material of an elementary course in **LPC**. We turn now to its limitations.

*Limitations of FF* In considering expressibility of **LPC** formulae in **FF** we assume tacitly that they are confined to one- and two-place predicates.

**Theorem 2.** *Every two-quantifier LPC formula has an equivalent expression in FF.*

Let the quantifiers  $Q_1 \xi Q_2 \eta$  be prenex and arrange the basic formulae of the matrix in the forms  $\alpha \xi$ ,  $\beta \eta$ ,  $\phi \xi \eta$  and  $\psi \eta \xi$ : the last of these may be rewritten  $\psi \xi \eta$ . Any term  $\lambda \xi x$  or  $\omega \eta x$  containing a free individual variable *x* may be regarded as being of the form  $\alpha \xi$ ,  $\beta \eta$ . Let  $Q_2$  be  $\exists$  (or proceed dually if it is  $\forall$ ): put the matrix in disjunctive normal form and distribute it over disjunctions, then contract the scope of each  $\exists \eta$  as far as possible. Each term containing  $\exists \eta$  is now of the form

$$\exists \eta (\beta \eta \cdot \phi \xi \eta)$$

where  $\beta$ ,  $\phi$  may now be complex: write this (in **FF** notation within the **LPC** formula) as

$$[\xi \phi E \beta]$$

Now let  $Q_1$  be  $\exists$  (or again proceed dually if it is  $\forall$ ): drive inwards as before and conjoin predicates of  $\xi$  in **FF** notation to get terms of form

$$\exists \xi (\gamma \xi)$$

written

$$[E t \gamma].$$

The result is essentially in **FF** notation. On the other hand we can prove

**Theorem 3.** *There exist three-quantifier LPC formulae that have no equivalent expression in FF.*

Such a formula is

$$\exists \xi \exists \eta \exists \zeta (f \xi \eta \cdot g \eta \zeta \cdot h \zeta \xi)$$

Thus in absorbing an **LPC** quantifier as an **FF** one we can proceed only by the inverse of the quantifier rules in (**FP**) above. In first absorbing  $\exists \zeta$  we can use either rule containing **E**, getting

$$\exists \xi \exists \eta (f \xi \eta \cdot [\eta g \mathbf{E} h \xi])$$

or

$$\exists \xi \exists \eta (f \xi \eta \cdot [\mathbf{E} h \xi \vee g \eta])$$

or trivially different forms. The second cannot be further reduced: in the first  $\exists \eta$  can be absorbed giving

$$\exists \xi ([\xi f \mathbf{E} g \mathbf{E} h \xi])$$

or

$$\exists \xi ([\mathbf{E} \vee f \xi g \mathbf{E} h \xi])$$

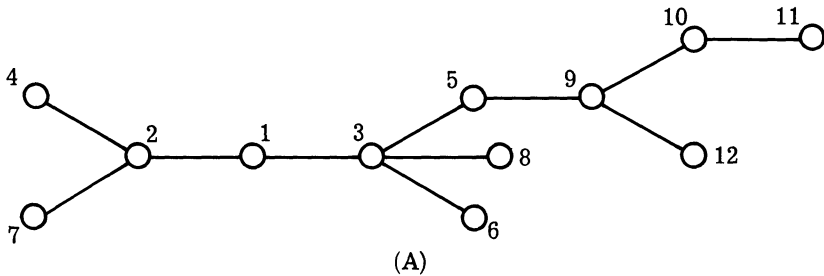
but neither of these is further reducible.

In consequence of Theorems 2 and 3 it is possible to express in **FF** the numerical quantifier  $\exists_1$  (as implied in the definition of **J**) but not  $\exists_2$  or any higher one. Thus to express "there are at most two things that" it is necessary to use an expression resembling the negation of the example in Theorem 3.

Now given a prenex formula  $F$  represent bound variables (equivalently, quantifiers) as points on a diagram and join by a single line every pair of points whose variables occur as the respective arguments of a two-place predicate in  $F$ . The result is a *relation graph* for  $F$ . If the graph divides into two or more unconnected parts the formula can be divided into separately quantified formulae.

**Theorem 4.** *Every LPC formula whose relation graph consists of branching lines without closed paths, and whose prenex quantifiers can be arranged in an order consistent with the distance of their points from an arbitrary starting-point, has an equivalent expression in **FF**.*

For example, a formula whose relation graph is as in (A), and whose quantifiers are in the order indicated by the numbers, is expressible. Note that two variables whose points are not adjacent, and in particular those whose points are on different branches, can be isolated into separate sub-formulae, and hence that their quantifiers can be permuted; also that this necessarily applies to points of equal distance from the starting-point.



Since the last quantifier must correspond with the end of a branch, there is only one other bound variable to which it is related by a two-place



predicate. Drive the quantifier inwards and absorb it as in Theorem 2. Then take the next last, and so on.

Theorem 5. **FF** is decidable.

We shall say that a predicate letter  $a$  at a particular occurrence stands under a quantifier  $Q$  if  $a$  is or is part of the predicate  $\alpha$  in the expression

$$\phi Q\alpha$$

in which  $Q$  appears. Now consider a formula  $F$  in which no predicate stands under more than  $n$  quantifiers altogether: we say  $F$  is of level  $n$ . We show that  $F$  may be analysed in terms of statements expressing the instantiation or otherwise of certain predicates which may be called *Venn predicates*, comparable with the predicates that characterise regions of a Venn diagram. Venn predicates are not independently instantiable but it is possible that they should all be instantiated.

Let  $F$  contain elementary predicates  $a_i$ , relations  $f_j$ , which may include =, and individual variables  $x_k$ . The *basic predicates of level 0* will be the predicates  $a_i$  together with  $f_j x_k$  and  $\check{f}_j x_k$  (for all  $j$  and  $k$ ). *Venn predicates of level 0* are state-descriptions (s.ds) of these, namely, lexically ordered conjunctions each containing just each predicate or its negation. We now form *basic predicates of level 1* by adding to the basic predicates of level 0 all predicates of the form  $\phi E\pi$  where  $\phi$  is an s.d. of the relations  $f_j$  and their inverses  $\check{f}_j$  and  $\pi$  is a Venn predicate of level 0. *Venn predicates of level 1* are s.ds of these, and so on. We now have the

Lemma. Any formula of level  $n$  has an equivalent expression as a truth-function of the formulae of the forms

$$Et\pi \text{ and } x_k\pi$$

in which  $\pi$  is a Venn predicate of level  $n$ .

Let  $F$  be such a formula and express all its quantifiers as existential ones by DE1-2: for  $Ea\beta$  write  $Et(\alpha.\beta)$ . Now at each level of each subformula starting at the lowest, express each predicate or relation or truth-function of either in complete disjunctive normal form and distribute quantifiers using

$$\begin{aligned} (fE(a \vee b) &\equiv (fEa \vee fEb) \\ ((f \vee g)Ea &\equiv (fEa \vee gEa) \\ (Ea(b \vee c) &\equiv (Eab \vee Eac) \\ (Eab &\equiv Eba \\ (x(a \vee b) &\equiv (xa \vee xb) \end{aligned}$$

Repeated application yields the form specified. This proves the lemma.

Turning to the theorem, let  $F$  be of level  $n$  and let the number of Venn predicates given the elementary predicates, relations and individuals of  $F$  be  $m$ . Since multiple instantiation of any Venn predicate is non-significant compared with single instantiation, the maximum number of individuals,

exclusive of the  $x_k$ , needed to achieve all instantiation-possibilities of the Venn predicates is  $m$ . The  $x_k$  will instantiate at least one of these and decision of  $f$  can be achieved in a universe of  $m + k' - 1$  individuals, where  $k'$  is the number of the  $x_k$ .  $F$  is valid if it is true on all valuations of elementary one- and two-place predicates of the  $m + k' - 1$  individuals, subject only to conditions on =.

*The augmented system FF+* We noticed that although expressions such as

$$xfAg y$$

are syntactically well-formed in **FF**,  $fAg$  cannot be regarded as a relation. In particular given say

$$(ya . xfAg y)$$

we cannot use EG2 to derive

$$xfAgEa$$

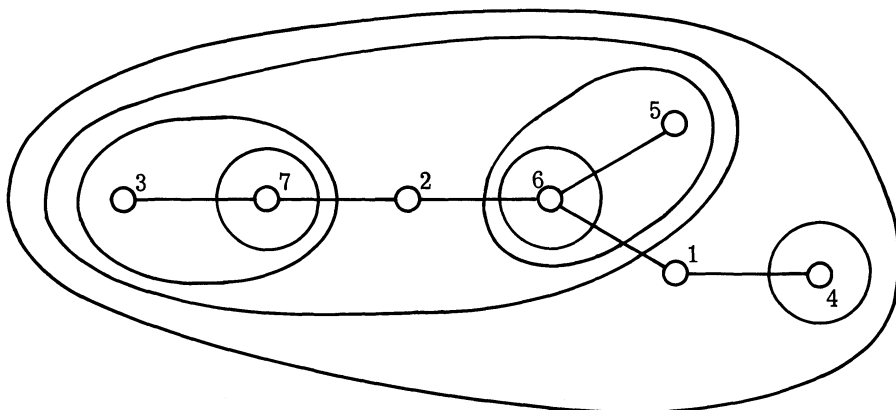
From the fact that  $x$  admires everything written by Conan Doyle and that Conan Doyle is a spiritualist, it does not follow that  $x$  admires everything written by a spiritualist. We can, however, contemplate an augmented system in which bracketed  $fQg$ —we shall actually use right braces as  $fAg\}$ ,  $fEg\}$ —count as relations. Note that  $fEg\}$  is what is usually called a relational product. This supplementation of the system turns out to be, on its own, a half-measure, and it is more satisfactory to treat  $\psi\}$ , where  $\psi$  is a relation, as itself a relation that may be joined by logical connectives to other relations of the same kind, as in the expression

$$\phi Q(\psi\} \kappa \chi\} \quad (\kappa \text{ a connective})$$

or to predicates, as

$$\phi Q(\alpha \kappa \psi\}$$

where in the latter case the addition of an individual symbol or quantified predicate will complete  $\psi\}$  and convert  $(\alpha \kappa \psi\}$  into logical connection of predicates. The genesis of this notation is as follows: Consider a relation graph in respect of the variables of quantification of a matrix and imagine these variables initially free but bound one by one. As a variable is bound, draw a circle round the point representing it, enclosing also any circles that already enclose adjacent points. The final circle encloses the whole graph as in the example (B). The last point circled is the first quantifier of the prefix and the corresponding **FF** formula traces a path from it to each other point. We can suppose at first that a left brace is inserted whenever a circle is entered, and a right brace when one is left, with appropriate right braces at the end of a branch: the enclosure circles resemble contour lines and are recoverable from the pattern of braces, which determine altitudes of the quantifiers. But left braces may be omitted since they are recoverable by tracing backwards (from the end of each branch) and locating a counterpart of each right brace. And braces at the end of a



(B)

branch may be omitted since we can always assume we have an inexhaustible supply there. It is further possible to economise braces on open-ended branches by systematically omitting those that result from enclosure of points on independent branches. In the case of formulae of **FF** all braces disappear and the notation is the same as before. Braces, when they occur, are associated with relations corresponding with the intersected lines, and right braces are conveniently written following the relevant relation symbols.

*Syntax of FF+* From the purely syntactical point of view it is simplest just to add to the rules (FS) the rule

$$\text{relation} : = \text{relation}\} \quad (\text{FS+})$$

This introduces some unwanted formulations but they are all interpretable. We shall also now require right brackets round logical connections between relations (though not predicates) and must amend the fourth line of the rewriting rule for "relation" in (FS) to

$$(\text{relation connective relation}).$$

*FF+ into English* English is unreliable in these matters. It seems that the brace, as introduced, is often represented by a pause; or we discriminate  $fAg\}Ea$  from  $fAgEa$  by reading E in the first case as "some" and in the second as "any." One may arbitrarily suggest reading } simply as "brace." Some similar arbitrary solution, say "bracket," will be necessary to the case of ).

*FF+ into LPC* Consider first LPC into **FF+**. The brace may be regarded as introduced into **FF+** by EG from formulae such as

$$\psi\xi\eta \cdot \phi\xi\xi$$

yielding

$$\xi\phi E\psi\}\eta$$

More generally, however, there may be some (irreducible) conjunction, on

right or left of the quantifier, of relations with possibly different individual left or right arguments: thus

$$\alpha\xi \cdot \psi_1\xi\eta_1 \cdot \psi_2\xi\eta_2 \cdot \phi_1\xi_1\xi \cdot \phi_2\xi_2\xi$$

yields by EG

$$\xi_1\phi_1 \cdot \xi_2\phi_2 \mathbf{E} (\alpha \cdot (\psi_1\} \eta_1 \cdot \psi_2\} \eta_2$$

(If either  $\psi_1$  or  $\psi_2$  is in turn of this branching form the brace needs to be inserted in all its branches.) Subsequent translation must replace  $\xi_1, \xi_2, \eta_1, \eta_2$  by expressions containing quantifiers and may associate braces with  $\phi_1, \phi_2$ . In reversing the procedure and translating from **FF+** to **LPC** we must assume the reverse of these procedures already performed. The general rule, where  $\pi$  denotes repeated conjunction, is: If the  $\eta_j$  and possibly  $\xi_i$  are nominals containing quantifiers, and if none of the  $\phi_i$  end with braces,

$$\begin{aligned} \pi_i \xi_i [\phi_i \xi \eta] \mathbf{E} ([\alpha \xi] \cdot \pi_j [\psi_j \xi \eta]) \eta_j \\ \rightarrow [\exists \xi (\alpha \xi \cdot \pi_j \psi_j \xi \eta_j \cdot \pi_i \phi_i \xi_i \xi)] \end{aligned}$$

Similarly if  $\sigma$  denotes repeated disjunction

$$\begin{aligned} \sigma_i \xi_i [\phi_i \xi \eta] \mathbf{A} ([\alpha \xi] \cdot \pi_j [\psi_j \xi \eta]) \eta_j \\ \rightarrow [\forall \xi ((\alpha \xi \cdot \pi_j \psi_j \xi \eta_j) \supset \sigma_i \phi_i \xi_i \xi)] \end{aligned}$$

*Logic of FF+* J can be defined in the new context

$$fJg\} = f\mathbf{E}(g\} \cdot =Ag\}$$

Identity can be introduced into **FF+** as into **LPC** by prefixing of axioms to formulae containing it, but only if  $s$ , which is needed in formulating the axiom of reflexivity, is in turn primitive. The brace is non-significant in an individual formula with a single quantifier; thus

$$(xfQg\}y \equiv xfQgy$$

Hence given a conjunction

$$(ya \cdot xfQgy$$

we may insert a brace in the right member:  $fQg\}$  is now a relation and we may use EG2 to infer

$$xfQg\}\mathbf{E}a$$

Similarly for comparable applications of UI2, etc. Now given

$$((az \cdot \pi_j z g_j y_j \cdot \pi_i x_i f_i z \rightarrow \pi_i x_i f_i \mathbf{E}(a \cdot \pi_j g_j\} y_j) \quad (\mathbf{EG3})$$

and similar variants of UI, etc. we have

**Theorem 6.** *Natural deduction is a complete proof procedure within FF+.*

Proof as before.

*Limitations of FF+*

**Theorem 7.** *Every three-quantifier formula has an equivalent expression in FF+.*

Let the prenex quantifiers be  $Q_1 \xi Q_2 \eta \exists \zeta$  (or proceed dually if the last is  $\forall \zeta$ ). Drive the last quantifier inwards until expressions of the form

$$\exists \zeta (\gamma \zeta \cdot \phi \xi \zeta \cdot \psi \zeta \eta)$$

are reached: translate into FF+ as

$$\xi \phi E(\gamma \cdot \psi) \eta$$

The resulting formula may be translated as in Theorem 2.

**Theorem 8.** *There exist four-quantifier LPC formulae that have no equivalent expression in FF+.*

Such a formula is

$$\exists \xi \exists \eta \exists \zeta \exists \omega (f_1 \xi \eta \cdot f_2 \xi \zeta \cdot f_3 \xi \omega \cdot f_4 \eta \zeta \cdot f_5 \eta \omega \cdot f_6 \zeta \omega)$$

Thus eliminate  $\exists \omega$  getting

$$\exists \xi \exists \eta \exists \zeta (f_1 \xi \eta \cdot f_2 \xi \zeta \cdot f_4 \eta \zeta \cdot [\xi f_3 E(\zeta f_5) \eta \cdot \zeta f_6])$$

and eliminate  $\exists \zeta$ , say in the form

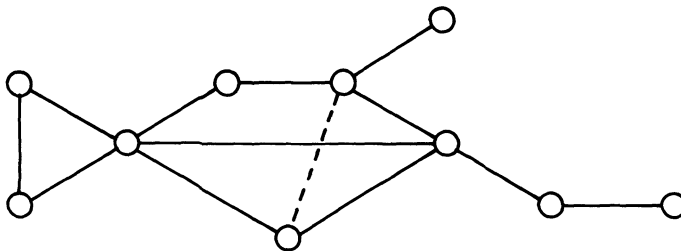
$$\exists \xi \exists \eta (f_1 \xi \eta \cdot [\xi f_3 E(\zeta f_5) \eta \cdot \zeta f_6] E(\zeta f_2) \xi \cdot \zeta f_4 \eta)$$

The formula is not now in a form in which elimination of  $\exists \eta$  or of  $\exists \xi$  is possible: the same difficulty attends the alternative forms of elimination of  $\exists \zeta$ .

A consequence of Theorems 7 and 8 is that although the numerical quantifier  $\exists_2$  can be defined in FF+,  $\exists_3$  cannot. The number system of FF+ is "one, two, many," as is reputedly that of certain primitive natural languages.

**Theorem 9.** *Every LPC formula whose relation graph does not contain four points pairwise linked by independent paths has an equivalent expression in FF+.*

For example, an LPC formula with relation graph as in (C) is expressible provided the dotted line is not included. The result is



(C)

independent of the order of the quantifiers as a consequence of the genesis of the brace-notation, but it is necessary to prove that a suitable path can be traced through the graph from any starting-point. Note that by conjunction of forms  $\phi_1 Q \phi_2 \dots \phi_n$ , possibly with predicates conjoined with the  $\phi_i$ , we can represent branches that later recombine; also that using the form  $\phi \dots \psi \dots$  we can represent a path that loops back to its starting-point.

In view of the example of Theorem 8 the graph can certainly be drawn without crossing of lines. Let a starting-point be given and suppose an arrow placed on each line in such a way that any point on the graph can be reached from the starting-point by travelling along arrows. We show that this can be done in such a way that when two paths diverge and later converge again no other paths except open-ended ones diverge from them along their length.

Thus let  $\lambda$  be the starting-point and let  $\xi - \eta - \zeta$  and  $\xi - \theta - \zeta$  be parallel branches of a section of the graph such that either  $\lambda = \xi$  or there is a path  $\lambda - \xi$  separately from the branches: let  $\xi$  be the first branch point with two or more non-open branches on the path  $\lambda \rightarrow \xi$ . We assume first that there is no independent path between  $\xi$  and  $\zeta$ : it follows that  $\zeta$  does not lead independently to  $\lambda$ . Under these circumstances if there is a path  $\eta - \theta$  we can put in arrows  $\xi \rightarrow \eta \rightarrow \theta \rightarrow \zeta$  and  $\eta \rightarrow \zeta \rightarrow \theta$ ; if there is a path  $\eta - \kappa$  to a point  $\kappa$  on a branch beyond  $\zeta$  we can put in arrows  $\eta \rightarrow \kappa \rightarrow \zeta$ .

Alternatively assume that there is an independent path between  $\xi$  and  $\zeta$ . Now a path  $\eta - \theta$  would contravene the assumptions and so would a path  $\eta - \kappa$ . But if there is no such path arrows can be put in  $\xi \rightarrow \eta \rightarrow \zeta$ ,  $\xi \rightarrow \theta \rightarrow \zeta$ ; and if  $\lambda$  is on the independent path,  $\zeta \rightarrow \lambda \rightarrow \xi$ . If  $\lambda$  is not on the path it is merely another parallel path  $\xi \rightarrow \zeta$ . Any separate system of parallel branches can be similarly dealt with.

Now we can return to the theorem. Given  $F$ , draw its graph and, starting from the point corresponding with the first quantifier, trace a path through the graph placing arrows on the lines; invert relations as necessary so that the order of variables corresponds. Then absorb quantifiers in turn, starting with the last. Alternatively absorb quantifiers in any convenient order disregarding braces, then insert braces as indicated by enclosure circles.

**Theorem 10.** *FF<sub>+</sub> is undecidable.*

This is a corollary of Theorem 7 and of the known undecidability of LPC, since by a result of Surányi [10, 11] every formula of LPC is equivalent in respect of validity to one of the form

$$\exists x \exists y \exists z B \cdot \exists x \exists y \forall z C$$

or according to Kahr, Moore and Wang ([5]; see also [2]), to one of the form

$$\exists x \forall y \exists z B$$

where  $B$  and  $C$  contain at most one- and two-place predicates and no quantifiers.

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