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A STRONGER DEFINITION OF A RECURSIVELY INFINITE SET

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1 Introduction. The purpose of this announcement is to strengthen the definition of a recursively infinite set as defined by Dekker and Myhill in [2]. This can be done after we have proved that any function that maps an immune set, α, one-to-one into itself and has a partial recursive extension must be an ω-permutation of α.

2 Preliminaries. Let ε stand for the set of nonnegative integers (*numbers*), V for the class of all subcollections of ε (*sets*), and $\mathcal{F}$ for the set of all mappings from a subset of ε into ε (*functions*). If f is a function, we write δf and ρf for its domain and range respectively. The relation of inclusion is denoted by $\subset$ and that of proper inclusion by $\subsetneq$. Certain families of functions are denoted by special symbols.

$$\mathcal{F}_{1-1} = \{f \in \mathcal{F} \mid f \text{ is one-to-one}\},$$
$$\mathcal{A} = \{f \in \mathcal{F} \mid f \text{ has a partial recursive extension}\},$$
$$\mathcal{A}_{1-1} = \{f \in \mathcal{A} \mid f \text{ has a one-to-one partial recursive extension}\}.$$

The sets α and β are *recursively equivalent* [written: $\alpha \simeq \beta$], if $\delta f = \alpha$ and $\rho f = \beta$, for some $f \in \mathcal{A}_{1-1}$.

We recall from [1], Proposition 1 that

$$(*) \qquad f \in \mathcal{A}_{1-1} \Longleftrightarrow f, f^{-1} \in \mathcal{A}, \text{ for } f \in \mathcal{F}_{1-1}.$$

A permutation of a set α is an ω-permutation, if $f \in \mathcal{A}_{1-1}$. The reader is assumed to be familiar with the contents of [2].

3 Main Results.

Notation. For $f \in \mathcal{F}$, f^n is defined for $n \in \varepsilon$, as follows: $f^0 = i$, where i is the identity function, and $f^{n+1} = f \circ f^n$, where $\circ$ is function composition, and f^{n+1} has the appropriate domain.

Theorem 1. *Let α be an immune set and $f \in \mathcal{F}_{1-1} \cap \mathcal{A}$ such that $\delta f = \alpha$ and $\rho f \subset \alpha$, then f is an ω-permutation of α.*

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Proof: Let $y \epsilon \rho f$. Put $\beta = \{f^i(y) | i \epsilon \varepsilon\}$. Thus $\beta \subset \alpha$ and β is r.e. Hence β must be finite. It follows that there exist numbers $i < j$ such that $f^j(y) = f^i(y)$. But $f \epsilon \mathcal{I}_{1-1}$, hence

$$(f^{-1})^i \circ f^j(y) = (f^{-1})^i \circ f^i(y).$$

Thus $f^{j-i}(y) = y$. It follows that $f(f^{j-i-1}(y)) = y$. So $f^{-1}(y) = f^{j-i-1}(y)$, where $j - i - 1 \geq 0$. Hence by putting

$$f^{-1}(y) = f^k(y), \text{ where } k = (\mu n > 0)(f^n(y) = y) - 1,$$

it is clear that $f^{-1} \epsilon \mathcal{A}$. Thus by (*), $f \epsilon \mathcal{A}_{1-1}$. But since α is immune, it follows that $\rho f = \alpha$. Hence f is an ω-permutation of α.

Remark. We recall from [2] that a set α is recursively infinite (r.i.) if there is an $f \epsilon \mathcal{A}_{1-1}$ such that $\delta f = \alpha$ and $\rho f \subsetneq \alpha$, i.e., $\alpha \simeq \beta$, where $\beta \subsetneq \alpha$. It is also known that α is r.i. if and only if α has an infinite r.e. subset. By using Theorem 1, we can now strengthen the definition of r.i.

Theorem 2. *A set α is* r.i. *if and only if there exists an $f \epsilon \mathcal{I}_{1-1} \cap \mathcal{A}$ such that $\delta f = \alpha$ and $\rho f \subsetneq \alpha$.*

Proof: The only if part is immediate. Thus let there exist an $f \epsilon \mathcal{I}_{1-1} \cap \mathcal{A}$ such that $\delta f = \alpha$ and $\rho f \subsetneq \alpha$. It suffices to show that α has an infinite r.e. subset. But if α has no infinite r.e. subset, then α is immune and by Theorem 1, $\rho f = \alpha$. Since $\rho f \subsetneq \alpha$, we are done.

Remark. Theorem 1 is useful in the study of automorphisms of algebraic structures and Theorem 2 makes it easier to prove a set is not immune.

REFERENCES

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