392
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CONCERNING THE PROPER AXIOM FOR S4.04 AND SOME RELATED SYSTEMS

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This paper examines the group of modal axioms covered by the general schema

(X) $$Xp \rightarrow (p \rightarrow Lp)$$

where X is an affirmative modality of S4. Familiarity is assumed with the properties of maximal-consistent sets of wff, and with the post-Henkin method of completeness proofs. Soundness proofs are left to the reader throughout.

(X) yields seven cases:

Case 1. Zeman's S4.04 axiom

L1 $LMLp \rightarrow (p \rightarrow Lp)$ *cf.* [5], p. 250

In the field of S4, **L1** is equivalent to

L2 $p \rightarrow L(MLp \rightarrow p)$

That **L1** is a consequence of **L2** is easy to see. For the converse we have[1]

(1)	$MLp \rightarrow ML(MLp \rightarrow p)$	C2
(2)	$\sim MLp \rightarrow (MLp \rightarrow p)$	**PC**
(3)	$\sim LMMLp \rightarrow ML(MLp \rightarrow p)$	(2), C2
(4)	$\sim MLp \rightarrow ML(MLp \rightarrow p)$	(3), S4, **PC**
(5)	$ML(MLp \rightarrow p)$	(1), (4), **PC**
(6)	$LML(MLp \rightarrow p)$	(5), T^0
(7)	$LML(MLp \rightarrow p) \rightarrow ((MLp \rightarrow p) \rightarrow L(MLp \rightarrow p))$	**L1**, $p/MLp \rightarrow p$
(8)	$(MLp \rightarrow p) \rightarrow L(MLp \rightarrow p)$	(6), (7), **PC**
(9)	$p \rightarrow L(MLp \rightarrow p)$	(8), **PC**

We now present a semantic analysis that distinguishes **L1** and **L2** in

1. This proof is due to Professor G. E. Hughes.

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systems weaker than S4. Let $\mathcal{K}$ be the modal logic whose rule of inference is

$$\text{Necessitation: } \frac{A}{LA}$$

and whose sole axiom is $L(p \to q) \to (Lp \to Lq)$.

For the definition of a (normal) $\mathcal{K}$-model, *cf.* [1] pp. 56-60 (the system is called $T(C)$ in that paper). If S is any normal extension of $\mathcal{K}$ we define $\mathcal{P}_S = (W_S, R, V)$ to be the *canonical model* for S, where

$$W_S = \{x \mid x \textit{ is an } S\textit{-maximal set of wff}\}$$
$$\forall x, y \in W_S, xRy \textit{ iff } \{A \mid LA \in x\} \subseteq y$$
$$V(p, x) = 1 \textit{ iff } p \in x, \textit{ for all propositional variables } p.$$

$\mathcal{P}_S$ falsifies every non-theorem of S, so to show that S is complete with respect to a class of models satisfying a certain condition, it suffices to show that $\mathcal{P}_S$ satisfies that condition. It is well known that S4 is characterized by the class of $\mathcal{K}$-models for which R is reflexive and transitive.

Proposition 1: *If S is a normal extension of $\mathcal{K}$L2, then $\mathcal{P}_S$ satisfies*

$$(xRy \,.\, x \neq y) \to \forall w(yRw \to wRy) \tag{a}$$

Proof. Suppose $x, y \in W_S$, xRy and $x \neq y$. Then there is some wff A such that $A \in x$ and $A \notin y$. Let w be such that yRw. If $LB \in w$ then $L(A \vee B) \in w$, by the $\mathcal{K}$-theorem $LB \to L(A \vee B)$. Now yRw, so $ML(A \vee B) \in y$. But $A \vee B \in x$ so by L2, $L(ML(A \vee B) \to A \vee B) \in x$ and hence $ML(A \vee B) \to A \vee B \in y$. Thus $A \vee B \in y$ and since $A \notin y$, $B \in y$. We have therefore shown that $\{B \mid LB \in w\} \subseteq y$, i.e., wRy.

Proposition 2: *If S is a normal extension of $\mathcal{K}$L1, then $\mathcal{P}_S$ satisfies*

$$(xRy \,.\, x \neq y) \to \exists z(xRz \,.\, \forall w(zRw \to wRy)) \tag{b}$$

Proof. Suppose xRy and $x \neq y$. Let

$$z_0 = \{A \mid LA \in x\} \cup \{LMB \mid B \in y\}$$

If z_0 is not S-consistent then there are wff A_i such that $LA_i \in x$ $(1 \leqq i \leqq n)$ and LMB_i such that $B_i \in y$ $(1 \leqq i \leqq m)$ for which

$$\vdash_S \sim(A_1 \ldots A_n \,.\, LMB_1 \ldots LMB_m)$$

and hence

$$\vdash_S A \to \sim(LMB_1 \ldots LMB_m) \text{ where } A = A_1 \ldots A_n$$

By the $\mathcal{K}$-theorem $\sim(LMp \,.\, LMq) \to \sim LM(p \,.\, q)$ we then deduce

$$\vdash_S A \to \sim LMB \text{ where } B = B_1 \ldots B_m$$

Since $x \neq y$ there is some wff C such that $C \in x$ and $\sim C \in y$. Using the $\mathcal{K}$-theorem $\sim LMp \to \sim LM(p \,.\, q)$ we now deduce

$$\vdash_S A \to \sim LM(B \,.\, \sim C)$$

and so

$\vdash_S LA \rightarrow L \sim LM(B . \sim C)$

i.e.,

$\vdash_S LA \rightarrow LML \sim (B . \sim C)$

But

$\vdash_K LA \longleftrightarrow (LA_1 \ldots LA_n)$

so $LA \epsilon x$, which gives $LML \sim (B . \sim C) \epsilon x$ (1)

Now $\sim C \notin x$, so $(B . \sim C) \notin x$, hence $\sim (B . \sim C) \epsilon y$. (2)

In the presence of **L1**, (1) and (2) together yield $L \sim (B . \sim C) \epsilon x$ and so $\sim (B . \sim C) \epsilon y$ (since xRy).

But $B \epsilon y$ and $\sim C \epsilon y$, so $(B . \sim C) \epsilon y$, which contradicts the **PC**-consistency of y. The upshot of all this is that z_0 must be S-consistent and so, by Lindenbaum's Lemma, may be extended to an S-maximal set z. Since $\{A \mid LA \epsilon x\} \subseteq z_0 \subseteq z$, we have xRz. Finally, suppose zRw. We want to show wRy. But if $B \notin y$ then $\sim B \epsilon y$ and so $LM \sim B \epsilon z$. But zRw so $M \sim B \epsilon w$, hence $\sim M \sim B \notin w$, i.e., $LB \notin w$. This shows that wRy, and the proof is complete.

Proposition 3: *If $\mathcal{P}$ is an* S4-*model, then $\mathcal{P}$ satisfies condition* (a) *iff $\mathcal{P}$ satisfies condition* (b).

Proof. If $\mathcal{P}$ satisfies (a) then putting $y = z$ it is immediate that (b) is satisfied. Conversely, suppose xRy and $x \neq y$. We want to show that if yRt then tRy. From (b) we deduce

$$\exists z(xRz . \forall w(zRw \rightarrow wRy)) \quad (1)$$

Since R is reflexive, zRz, and so by (1) zRy. Then if yRt, we have zRt by the transitivity of R. Using (1) again we deduce tRy.

From Proposition 1 it follows that S4.04 is complete with respect to the class of S4-models that satisfy condition (a). The axiom corresponding to (a) is **L2**. **L1** corresponds (Proposition 2) to a rather more complex condition that reduces (Proposition 3) to (a) in S4-models. Our discussion would seem to indicate then that from a model-theoretic stand-point **L2** is the "right" axiom for S4.04.

Case 2. Sobociński's K4 axiom

P1 $MLMp \rightarrow (p \rightarrow Lp)$ *cf.* [4], p. 349

Proposition 4: *If S is a normal extension of K_{P1} then $\mathcal{P}_S$ satisfies*

$$(xRy . x \neq y) \rightarrow \forall z(xRz \rightarrow \exists w(zRw . \forall t(wRt \rightarrow t = y))) \quad \text{(c)}$$

Proof. If xRy and $x \neq y$ then there is some wff C such that $C \epsilon x$ and $\sim C \epsilon y$. Let xRz and put

$$w_0 = \{A \mid LA \epsilon z\} \cup \{LB \mid B \epsilon y\}$$

If w_0 is not S-consistent then reasoning as in Proposition 2 we find there are wff A, B such that $LA \in z$, $B \in y$ and

$$\vdash_S A \rightarrow \sim L(B \,.\, \sim C).$$

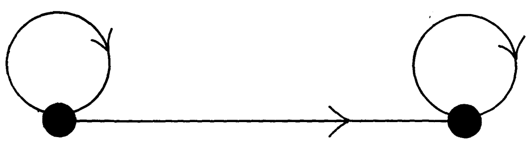
Thus

$$\vdash_S LA \rightarrow L \sim L(B \,.\, \sim C)$$

and so $LM \sim (B \,.\, \sim C) \in z$. But xRz and so $MLM \sim (B \,.\, \sim C) \in x$.

Using the axiom **P1**, a contradiction then follows exactly as in Proposition 2. Thus w_0 is S-consistent and may be extended to an S-maximal w such that zRw. Now let wRt. Then if $B \in y$, $LB \in w$, and since wRt, $B \in t$. Hence $y \subseteq t$ and so by the maximality of y, $y = t$.

K4 is known to be characterized by Lewis and Langford's Group II matrix (*cf.* [6] p. 349) and so it has two-element models that may be represented graphically as follows: (*cf.* [1] p. 63).



If $\mathcal{P}$ is a reflexive model that satisfies (c) then it may be seen by elementary reasoning that if y and z are distinct from x, then $(xRz \,.\, xRy) \rightarrow (x = y)$. If $\mathcal{P}$ is also transitive and connected in the sense of [1] p. 193, then it has the form indicated above.

Case 3. When X is the improper modality, the resulting axiom is (equivalent to) $p \rightarrow Lp$ corresponding to the model condition $\forall x \forall y (xRy \rightarrow x = y)$, *cf.* [1] p. 214.

Case 4. $X = L$ gives a substitution-instance of a **PC**-tautology.

Case 5. $X = M$ gives

X1 $Mp \rightarrow (p \rightarrow Lp)$

Proposition 5: *If S is a normal extension of* K**X1** *then* $\mathcal{P}_S$ *satisfies*

$$(xRy \,.\, x \neq y) \rightarrow \forall z (xRz \rightarrow z = y) \qquad \text{(d)}$$

Proof. There is some wff A such that $A \in x$ and $A \notin y$. If $B \in z$, $A \vee B \in z$ and so $M(A \vee B) \in x$. But $A \vee B \in x$ so by **X1**, $L(A \vee B) \in x$. Thus $A \vee B \in y$, and since $A \notin y$, $B \in y$. This shows $z \subseteq y$, which is enough to prove $z = y$.

Given R reflexive, corresponding to $Lp \rightarrow p$, then condition (d) above reduces to

$$\forall x \forall y (xRy \rightarrow x = y)$$

which is Case 3 above. A syntactic proof is straightforward.

Case 6. $X = ML$ gives the S4.4 axiom

R1 $MLp \rightarrow (p \rightarrow Lp)$

whose corresponding model condition is

$$(xRy \,.\, x \neq y) \to \forall z(xRz \to zRy).$$

A proof is given in [3].

Case 7. $X = LM$ yields

H2 $LMp \to (p \to Lp)$

Proposition 6: *If S is a normal extension of* KH2 *then* $\bar{\mathcal{P}}_S$ *satisfies*

$$(xRy \,.\, x \neq y) \to \exists z(xRz \,.\, \forall w(zRw \to w = y)) \qquad \text{(e)}$$

Proof. By a similar method to Proposition 4 we may show

$$z_0 = \{A \mid LA \in x\} \cup \{LB \mid B \in y\}$$

is S-consistent and may be extended to an S-maximal z with the required properties. When R is reflexive, condition (e) reduces to

$$(xRy \,.\, x \neq y) \to \forall z(yRz \to y = z)$$

which was shown in [3] to be the characteristic model condition for the K1.2 axiom

H1 $p \to L(Mp \to p)$

In [4] a proof is given that H1 and H2 are equivalent in the field of S4. The following proof shows that the equivalence holds in the field of S2:

(1)	$LM(Mp \to p) \to ((Mp \to p) \to L(Mp \to p))$	H2 $p/Mp \to p$
(2)	$LM(Mp \to p)$	S2, *cf.* [2], p. 140, 22.8
(3)	$(Mp \to p) \to L(Mp \to p)$	(1), (2), PC
(4)	$p \to L(Mp \to p)$	(3), PC

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