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DEGREES OF ISOLIC THEORIES

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1. *Introduction.* In this paper* we show that most of the commonly studied isolic structures fall into two categories as far as their first order theories are concerned. They are those whose theory is recursively isomorphic to that of second order arithmetic ($\langle\Lambda, +, \cdot\rangle$ for example) and those whose theory is recursively isomorphic to first order arithmetic ($\langle\Lambda_z, +, \cdot\rangle$ for example). These results are not remarkable, for after all the isols are obtained from $P(\omega)$ by a fairly simple construction. However they do suggest why so many first order questions about Λ reduce to first order questions about ω. And that is because it is hard to find algebraically interesting properties which distinguish Λ from Λ_z. For this reason we believe that it would be quite worth while to continue searching for algebraic distinctions between these structures. Basic concepts concerning Λ and Λ_z are to be found in [5]. The universal theories of Λ and Λ_z have received complete treatments in [11] and [7] respectively, and at least one kind of first order distinction between Λ and Λ_z is contained in [8]. Indeed [8] was the author's chief motivation in undertaking the present study.

We start by defining the isolic structures relevant to our discussion. First of all there are our basic Ω = the recursive equivalence types (commonly called RETs), Λ = the isols, and ω = the non-negative integers. It is also natural to consider $\Lambda(R)$ = the regressive isols (cf. [1]), $\Lambda(H)$ = hyperimmune isols and even $\Lambda(RH) = \Lambda(R) \cap \Lambda(H)$. On the more effective hand we have Ω_f = the RETs of sets with recursively enumerable complement, $\Lambda_z = \Omega_f \cap \Lambda$ = the cosimple isols, $\Lambda_z(R) = \Omega_f \cap \Lambda(R)$ = the cosimple regressive isols, $\Lambda_z(H) = \Omega_f \cap \Lambda(H)$ = the cohypersimple isols, and $\Lambda_z(RH) = \Omega_f \cap \Lambda(RH)$. The latter class has been added for the sake of symmetry only, for by T4 of [3] we have $\Lambda_z(R) = \Lambda_z(RH)$. In order to avoid a cumbersome repetition of names let us introduce a variable W which ranges over the symbols in $\{1, R, H, RH\}$ and use the notation $\Lambda(W)$ or $\Lambda_z(W)$ where

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$\Lambda(1) = \Lambda$, $\Lambda_z(1) = \Lambda_z$ and where $\Lambda(W)$, $\Lambda_z(W)$ have their obvious meanings for other values of W. Let F_n be the set of all n-ary recursive combinatorial functions and let finite $F \subseteq \bigcup_{n<\omega} F_n$ at least contain $\cdot$ and $+$. We should think of these functions as already extended to Ω and will use the same symbol $f : \mathsf{X}^n\omega \to \omega$ as for its extension $f : \mathsf{X}^n\Omega \to \Omega$. Throughout this paper we shall be concerned with algebraic structures of the form $\langle A, +, \cdot\rangle$ or $\langle A, F\rangle$. For our convenience these structures are divided into the following groups

(I) $\langle \Omega, +, \cdot\rangle$, $\langle \Omega, F\rangle$, $\langle \Lambda(W), +, \cdot\rangle$, $\langle \Lambda(W), F\rangle$
(II) $\langle \Omega_f, +, \cdot\rangle$, $\langle \Omega_f, F\rangle$, $\langle \Lambda_z(W), +, \cdot\rangle$, $\langle \Lambda_z(W), F\rangle$

where $W \in \{1, R, H, RH\}$. In each of these cases we are to think of the functions in F and $+$, $\cdot$ as restricted to the structure in question. It should be noted that $\Lambda(R)$ and $\Lambda_z(R)$ are not closed with respect to $+$, $\cdot$ (cf. T2 of [2]) nor to many of the functions in F_n where $n > 1$. Thus systems like $\langle \Lambda(R), F\rangle$ are really generalized algebras and would be best formulated as relational systems. We use the present notation to stress the uniformity of our approach.

For any algebraic system A let $\mathrm{L}^n(\mathsf{A})$ be a language which is appropriate for the n-th order theory of A and let $\mathrm{Th}^n(\mathsf{A})$ be the set of all sentences in $\mathrm{L}^n(\mathsf{A})$ which are true in A. Here we are only interested in the case where $n = 1$ or $n = 2$, and $n = 1$ will usually be omitted in this notation. We use $v_0, v_1, \ldots$ for individual variables and $\sigma_0, \sigma_1, \ldots$ as set variables. In most of this paper we shall be concerned with defining an isomorph of one structure in another. For this purpose an iota theory is much more convenient than the usual formulation of first order logic. The iota language ι - $\mathrm{L}^n(\mathsf{A})$ is obtained from $\mathrm{L}^n(\mathsf{A})$ by including all expressions $(\iota v)\ \varphi(v)$ as terms, where v is an individual variable and φ is a formula which itself may involve other iota terms. If $\mathsf{A} = \langle A, \ldots\rangle$ is an algebraic system, $a \in A$ is a fixed element and f is an assignment we interpret $\mathrm{val}_a((\iota v)\ \varphi(v), f)$ as the unique $x \in A$ such that $\mathsf{A} \models_a \varphi(v)\ [f(v/x)]$ if such an x exists and as a otherwise (here $f(v/x)$ is that assignment which agrees with f except for v and assigns x to v). The precise definition of val_a and $\models_a$ is by induction (cf. p. 223 of [9] for details). Finally let ι - $\mathrm{Th}^n_a(\mathsf{A})$ = all sentences φ in ι - $\mathrm{L}^n(\mathsf{A})$ such that $\mathsf{A} \models_a \varphi$. For many structures A there is a formula $\psi(v)$ in $\mathrm{L}^n(\mathsf{A})$ with just one free variable v such that $a \in A$ is the unique element of A satisfying $\psi(v)$ in A. In this case we say that ψ is *admissible* for A and define val_ψ, $\models_\psi$, and ι - $\mathrm{Th}^n_\psi(\mathsf{A})$ as val_a, $\models_a$, and ι - $\mathrm{Th}^n_a(A)$ respectively. From the folklore of model theory it is known that (cf. lemma 1.2 of [9])

(1) *If ψ is a formula, then for every sentence φ in ι - $\mathrm{L}^n(\mathsf{A})$ we can effectively find a unique sentence φ^* in $\mathrm{L}^n(\mathsf{A})$ such that if ψ is admissible for A then $\varphi \in \iota$ - $\mathrm{Th}^n_\psi(\mathsf{A})$ if and only if $\varphi^* \in \mathrm{Th}^n(\mathsf{A})$.*

Another problem that we will be concerned with is that of identity. Let $\mathsf{A} = \langle A, \ldots\rangle$ and $\mathsf{B} = \langle B, \ldots\rangle$ be algebraic structures, and suppose that in our attempt to define an isomorph of A in B we are only able to define a structure A^* in B for which there is a congruence relation $E \subseteq \mathsf{X}^2B$ such that the natural quotient A^*/E is isomorphic to A. Sometimes we can find a formula $\psi(v_0, v_1)$ whose free variables are v_0 and v_1 such that $E = \{\langle x_0, x_1\rangle : B \models \psi[x_0, x_1]\}$. In this case we say that ψ is *acceptable* for A and B with respect to the interpretation D of A^* in B. Again from the folklore of model theory (lemma 4.4 of [6])

(2) *If ψ is a formula and D is an interpretation then for every sentence φ we can effectively find a unique sentence φ^* such that if ψ is acceptable for A and B with respect to the interpretation D then $\varphi \epsilon \mathrm{Th}^n(A)$ if and only if $\varphi^* \epsilon \mathrm{Th}^n(B)$.*

We will also use certain notations from recursion theory. Let us use $\leqslant_1$ for one-one reducibility and $\cong$ for recursive isomorphism. The fundamental result for these notions is (cf. theorem 18 of [10])

(3) *If $\alpha \leqslant_1 \beta$ and $\beta \leqslant_1 \alpha$ then $\alpha \cong \beta$ (where $\alpha, \beta \subseteq \omega$).*

By applying (3) to the preceding paragraph it is clear that if we identify sentences with their Gödel numbers then under the hypotheses of (1) we have $\mathrm{Th}^n(A) \cong \iota - \mathrm{Th}^n_\psi(A)$. Since all of the structures under discussion satisfy $(\exists ! v)(\forall u)(uv = v)$ and since our strongest condition concerns recursive isomorphism of theories we shall identify $\mathrm{Th}^n(A)$ with $\iota - \mathrm{Th}^n_\psi(A)$ where ψ is the formula $(\forall u)(uv = v)$. Also note that under the hypotheses of (2) we have $\mathrm{Th}^n(A) \leqslant_1 \mathrm{Th}^n(B)$. This fact together with (3) will be used in the next section to obtain recursive isomorphism. All other notions from recursion theory that we use can be found in the standard isolic literature.

Our paper is organized as follows. Section 2 is devoted to a computation of the recursive functions necessary to imply $\leqslant_1$ between relevant theories. In general a formula will be exhibited, and in case the author feels that the formal expression is difficult to read an explanation will follow. Except in cases of mathematical interest no attempt will be made to prove that the formula does the required job. That it does so will be clear from the obvious absoluteness. Except when we wish to be precise informal variables will be used throughout. The letters $x, y, z, \ldots$ will denote individual variables and $\alpha, \beta, \gamma, \ldots$ will denote set variables. In section 3 we prove two lemmas which are needed in order to justify the claims of section 2. Both appear to be independently interesting and the author hopes that (in other contexts) they will be useful to isol theorists.

2. *Reducibilities.*

Theorem 1. *Each of the structures in group* I *has a first order theory which is recursively isomorphic to* $\mathrm{Th}^2(\omega, +, \cdot)$.

Theorem 2. *Each of the structures in group* II *has a first order theory which is recursively isomorphic to* $\mathrm{Th}(\omega, +, \cdot)$.

First, let us list the reducibilities which are necessary to obtain these results. Their grouping is meant to indicate similarity in techniques of our proofs.

$$(4)\quad \begin{cases} \mathrm{Th}(\Omega, +, \cdot) \leqslant_1 \mathrm{Th}(\Omega, F),\ \mathrm{Th}(\Omega_f, +, \cdot) \leqslant_1 \mathrm{Th}(\Omega_f, F) \\ \mathrm{Th}(\Lambda(W), +, \cdot) \leqslant_1 \mathrm{Th}(\Lambda(W), F),\ \mathrm{Th}(\Lambda_z(W), +, \cdot) \leqslant_1 \mathrm{Th}(\Lambda_z(W), F) \end{cases}$$

(5) $\mathrm{Th}(\Lambda, +, \cdot) \leqslant_1 \mathrm{Th}(\Omega, +, \cdot)$, $\mathrm{Th}(\Lambda_z, +, \cdot) \leqslant_1 \mathrm{Th}(\Omega_f, +, \cdot)$
(6) $\mathrm{Th}(\Omega, F) \leqslant_1 \mathrm{Th}^2(\omega, +, \cdot)$
(7) $\mathrm{Th}(\Lambda(R), F) \leqslant_1 \mathrm{Th}^2(\omega, +, \cdot)$
(8) $\mathrm{Th}(\Lambda(H), F) \leqslant_1 \mathrm{Th}^2(\omega, +, \cdot)$
(9) $\mathrm{Th}(\Lambda(RH), F) \leqslant_1 \mathrm{Th}^2(\omega, +, \cdot)$
(10) $\mathrm{Th}(\Omega_f, F) \leqslant_1 \mathrm{Th}(\omega, +, \cdot)$
(11) $\mathrm{Th}(\Lambda_z(R), F) \leqslant_1 \mathrm{Th}(\omega, +, \cdot)$
(12) $\mathrm{Th}(\Lambda_z(H), F) \leqslant_1 \mathrm{Th}(\omega, +, \cdot)$
(13) $\mathrm{Th}^2(\omega, +, \cdot) \leqslant_1 \mathrm{Th}(\Lambda(W), +, \cdot)$
(14) $\mathrm{Th}(\omega, +, \cdot) \leqslant_1 \mathrm{Th}(\Lambda_z(W), +, \cdot)$

The identity map suffices for (4). For (5) define in $\mathrm{L}(\Omega, +, \cdot)$

(15) $\mathbf{0} = (\iota x)(\forall y)(xy = x)$, $\mathbf{1} = (\iota x)(\forall y)(xy = y)$, $\mathbf{2} = \mathbf{1} + \mathbf{1}$
(16) $\mathrm{isol}(x) \equiv x \neq x + \mathbf{1}$

It is clear that (16) defines Λ in Ω and Λ_z in Ω_f giving both parts of (5). To prove (10)-(12) we include (15) and define in $\mathrm{L}(\omega, +, \cdot)$

(17) $x \leqslant y \equiv (\exists z)(x + z = y)$, $x < y \equiv x \leqslant y \cap x \neq y$
(18) $\mathrm{j}(x, y) = (\iota z)(\mathbf{2}z = \mathbf{2}x + (x+y)(x+y+\mathbf{1}))$
(19) $\mathrm{k}(z) = (\iota x)(\exists y)(\mathrm{j}(x, y) = z)$, $\mathrm{l}(z) = (\iota y)(\exists x)(\mathrm{j}(x, y) = z)$
(20) $\mathrm{rm}(x, y) = (\iota z)(\exists u)(x = yu + z \wedge z < y)$
(21) $\mathrm{gd}(x, y) = \mathrm{rm}(\mathrm{k}(x), \mathbf{1} + \mathrm{l}(x)(\mathbf{1} + y))$
(22) $\mathrm{lh}(x) = \mathrm{gd}(x, \mathbf{0})$, $(x)_y = \mathrm{gd}(x, \mathbf{1} + y)$
(23) $\mathrm{seq}(x) \equiv (\forall y < x)(\mathrm{lh}(y) \neq \mathrm{lh}(x) \vee (\exists z < \mathrm{lh}(x))((y)_z \neq (x)_z))$
(24) $\mathrm{set}(x) \equiv \mathrm{seq}(x) \wedge (\forall y, z)(y < z < \mathrm{lh}(x) \to (x)_y < (x)_z)$
(25) $x \mathrel{\dot\epsilon} y \equiv \mathrm{set}(y) \wedge (\exists z < \mathrm{lh}(y))(x = (y)_z)$
(26) $x \mathrel{\dot\subseteq} y \equiv \mathrm{set}(x) \wedge \mathrm{set}(y) \wedge (\forall z)(z \mathrel{\dot\epsilon} x \to z \mathrel{\dot\epsilon} y)$
(27) $x \mathrel{\dot\cap} y = (\iota z)(\mathrm{set}(x, y, z) \wedge (\forall u)(u \mathrel{\dot\epsilon} z \equiv u \mathrel{\dot\epsilon} x \wedge u \mathrel{\dot\epsilon} y))$
(28) $\mathrm{card}(x) = (\iota y)(\mathrm{set}(x) \wedge y = \mathrm{lh}(x))$
(29) $k\text{-}\mathrm{set}(x) \equiv \mathrm{seq}(x) \wedge \mathrm{lh}(x) = k \wedge (\forall y < k)(\mathrm{set}((x)_y))$
(30) $x \mathrel{\dot\subseteq^k} y \equiv k\text{-}\mathrm{set}(x) \wedge k\text{-}\mathrm{set}(y) \wedge (\forall z < k)((x)_z \mathrel{\dot\subseteq} (y)_z)$
(31) $x \mathrel{\dot\cap^k} y = (\iota z)(k\text{-}\mathrm{set}(x, y, z) \wedge (\forall u < k)((z)_u = (x)_u \mathrel{\dot\cap} (y)_u))$
(32) $\mathrm{card}^k(x) = (\iota y)(k\text{-}\mathrm{set}(x) \wedge \mathrm{seq}(y) \wedge \mathrm{lh}(y) = k \wedge (\forall u < k)(\mathrm{card}((x)_u) = (y)_u))$

These definitions arithmetize the finite subsets of ω and part of their theory. In (18) and (19) j is the usual pairing function of recursion theory and k, l it's inverses. (20) is the remainder function and (21) is Gödel's β-function. (23) defines the sequence numbers, and we code sets as ascending sequences. In (25)-(28) we define elementary set concepts and in (29)-(32) extend them pointwise to k-tuples (k here is a variable). Let $\mathrm{T}_2(n, x, y, z)$ be the Kleene T-predicate which as we know is definable in arithmetic. We continue with

(33) $\mathrm{K}_2(n, x, y) \equiv (\exists z)\mathrm{T}_2(n, x, y, z)$
(34) $x \ \mathrm{dom}\ n \equiv (\exists y)\mathrm{K}_2(n, x, y)$
(35) $y \ \mathrm{rng}\ n \equiv (\exists x)\mathrm{K}_2(n, x, y)$

(36) $\mathrm{fnc}(n) \equiv (\forall x)\,(x \text{ dom } n \to (\exists !y)\mathrm{K}_2(n, x, y))$
(37) $\mathrm{fnc}^{-1}(n) \equiv (\forall y)\,(y \text{ rng } n \to (\exists !x)\mathrm{K}_2(n, x, y))$
(38) $\mathrm{map}(n) \equiv \mathrm{fnc}(n) \wedge \mathrm{fnc}^{-1}(n)$
(39) $\{n\}(x) = (\iota y)\mathrm{K}_2(n, x, y)$
(40) $k\text{-op}(n) \equiv \mathrm{fnc}(n) \wedge (\forall x)\,(x \text{ dom } n \equiv k\text{-set}(x)) \wedge (\forall y)\,(y \text{ rng } n \to \mathrm{set}(y))$
(41) $k\text{-coop}(n) \equiv k\text{-op}(n) \wedge (\forall x, y)\,(x \text{ dom } n \wedge y \text{ dom } n \to \{n\}(x \mathbin{\dot\cap}^k y)$
$= \{n\}(x) \mathbin{\dot\cap} \{n\}(y))$
(42) $k\text{-myop}(n) \equiv k\text{-coop}(n) \wedge (\forall x, y)\,(x \text{ dom } n \wedge y \text{ dom } n \wedge \mathrm{card}^k(x)$
$= \mathrm{card}^k(y) \to \mathrm{card}(\{n\}(x)) = \mathrm{card}(\{n\}(y)))$
(43) $x\eta a \equiv \sim (\exists y)\mathrm{K}_2(a, x, y)$
(44) $x \mathrel{\ddot\subseteq} a \equiv \mathrm{set}(x) \wedge (\forall y)\,(y \in x \to y\eta a)$
(45) $x \mathrel{\ddot\subseteq}^k a \equiv k\text{-set}(x) \wedge \mathrm{seq}(a) \wedge \mathrm{lh}(a) = k \wedge (\forall y < k)\,((x)_y \mathrel{\ddot\subseteq} (a)_y)$
(46) $a \mathrel{\dot\simeq} b \equiv (\exists n)\,(\mathrm{map}(n) \wedge (\forall x)\,(x\eta a \to (x \text{ dom } n \wedge \{n\}(x)\eta b))$
$\wedge (\forall y)\,(y\eta b \to (\exists x)\,(x\eta a \wedge \{n\}(x) = y)))$
(47) $\Phi_n(a) = (\iota z)\,(\mathrm{k}(n)\text{-myop}(\mathrm{l}(n)) \wedge \mathrm{seq}(a) \wedge \mathrm{lh}(a) = \mathrm{k}(n)) \wedge$
$(\forall y)\,(y\eta z \equiv (\exists x)\,(\mathrm{k}(n)\text{-set}(x) \wedge x \mathrel{\ddot\subseteq}^{k(n)} a \wedge y \mathrel{\dot\in} \{\mathrm{l}(n)\}(x)))$

and for each integer $k < \omega$ (with the associated numeral $\mathbf{k}$)

(48) $\langle a_0, \ldots, a_{k-1}\rangle = (\iota a)\,(\mathrm{seq}(a) \wedge \mathrm{lh}(a) = \mathbf{k} \wedge a_0 = (a)_{\mathbf{0}} \ldots a_{k-1} = (a)_{\mathbf{k-1}})$

We interpret (46)-(47) in the following way. Let $\omega(n)$ be the standard enumeration of r.e. sets and let $\xi(n) = \omega - \omega(n)$. If we let φ_n enumerate recursive combinatorial operators (via the code indicated in (47)) then

(49) $\langle \omega, +, \cdot\rangle \models v_0 \mathrel{\dot\simeq} v_1[x_0, x_1]$ if and only if $\xi(x_0) \simeq \xi(x_1)$
(50) $\langle \omega, +, \cdot\rangle \models \Phi_{\mathrm{j(k,n)}}(\langle v_0, \ldots, v_{k-1}\rangle) = v_k[x_0, \ldots, x_k]$
if and only if $\varphi_{j(k,n)}(\xi(x_0), \ldots, \xi(x_{k-1})) = \xi(x_k)$.

These results are obtained by interpreting (15)-(48) in $\langle \omega, +, \cdot\rangle$. Then (10) is an immediate consequence of (2), (49), and (50) via the theory of a structure consisting of the co-r.e. sets, the combinatorial operators corresponding to functions in F, with the $\simeq$ relation replacing $=$. To prove (11)-(12) it will suffice to find r.e. codes for the finite, immune, hyperimmune and regressive sets, all with r.e. complements. Let

(51) $\mathrm{ft}(a) \equiv (\exists x)\,(\mathrm{set}(x) \wedge (\forall y)\,(y \mathrel{\dot\in} x \equiv y\eta a))$
(52) $\mathrm{coft}(a) \equiv (\exists x)\,(\mathrm{set}(x) \wedge (\forall y)\,(y \mathrel{\dot\in} x \equiv \sim y\eta a))$
(53) $\mathrm{im}(a) \equiv (\forall b)\,((\forall x)\,(x\eta a \vee x\eta b) \to \mathrm{coft}(b))$
(54) $\mathrm{ar}(n) \equiv \mathrm{fnc}(n) \wedge (\forall x)\,(x \text{ dom } n \wedge \mathrm{set}(\{n\}(x)) \wedge (\exists y)$
$(y \mathrel{\dot\in} \{n\}(x))) \wedge (\forall x, y)\,(\exists z)\,((z \mathrel{\dot\in} \{n\}(x) \mathbin{\dot\cap} \{n\}(y)) \to x = y)$
(55) $\mathrm{hyp}(a) \equiv (\forall n)\,(\mathrm{ar}(n) \to (\exists x)\,(\forall y)\,(y \mathrel{\dot\in} \{n\}(x) \to \sim y\eta a))$
(56) $\mathrm{pre}(x, y, a) \equiv x\eta a \wedge y\eta a \wedge x < y \wedge (\forall z)\,(x < z < y \to \sim z\eta a)$
(57) $\mathrm{retr}(a) \equiv (\exists n)\,(\mathrm{fnc}(n) \wedge (\forall x)\,(x\eta a \to x \text{ dom } n) \wedge (\forall x, y)\,(\mathrm{pre}(x, y, a)$
$\to x = \{n\}\,(y)) \wedge (\forall y)\,(y\eta a \wedge (\forall x)\,(\sim\mathrm{pre}(x, y, a)) \to y = \{n\}(y)))$
(58) $\mathrm{regr}(a) \equiv (\exists b)\,(a \mathrel{\dot\simeq} b \wedge \mathrm{retr}(b))$

Thus in (51) we have defined the finite sets, in (52) the cofinite sets, in (53) the immune sets, in (55) the hyperimmune sets by means of discrete array defined in (54), in (57) the retraceable sets and finally the regressive sets in (58). By the previous remarks (11)-(12) follow immediately. We continue our definitions in $L^2(\omega, +, .)$ (remember we use Greek letters for set variables and interpret $\alpha(x)$ for set membership and $(\iota\alpha)\varphi$ as the empty set when φ is not uniquely satisfied in our model).

(59) $(\alpha)_x = (\iota\beta)(\forall y)(\beta(y) \equiv \alpha(\mathsf{j}(x, y)))$

(60) $x \ddot{\subseteq} \alpha \equiv \mathsf{set}(x) \wedge (\forall y)(y \dot{\epsilon}\, x \rightarrow \alpha(y))$

(61) $x \ddot{\subseteq}^k \alpha \equiv k\text{-}\mathsf{set}(x) \wedge (\forall y < k)((x)_y \ddot{\subseteq} (\alpha)_y)$

(62) $\alpha \doteq \beta \equiv (\exists n)(\mathsf{map}(n) \wedge (\forall x)(\alpha(x) \rightarrow x \text{ dom } n \wedge \beta(\{n\}(x)))$
$\wedge (\forall y)(\beta(y) \rightarrow (\exists x)(\alpha(x) \wedge \{n\}(x) = y)))$

(63) $\Phi_n(\alpha) = (\iota\beta)(\mathsf{k}(n) - \mathsf{myop}(\mathsf{l}(n)) \wedge (\forall y)(\beta(y) \equiv (\exists x)(\mathsf{k}(n) -$
$\mathsf{set}(x) \wedge x \ddot{\subseteq}^{k(n)} \alpha \wedge y \dot{\epsilon} \{\mathsf{l}(n)\}(x))))$

and for each $k < \omega$ (with the associated numeral $\mathbf{k}$)

(64) $\langle\alpha_0, \ldots, \alpha_{k-1}\rangle = (\iota\alpha)(\alpha_0 = (\alpha)_0 \wedge \ldots \wedge \alpha_{k-1} = (\alpha)_{\mathsf{k}-1} \wedge$
$(\forall x)(\mathbf{k} \leqslant x \rightarrow (\alpha)_x = (\iota\beta)(\beta = \beta)))$

Then exactly as in the previous case it can be shown that

(65) $\langle\omega, +, \cdot\rangle \models \sigma_0 \doteq \sigma_1[\alpha_0, \alpha_1]$ if and only if $\alpha_0 \simeq \alpha_1$

(66) $\langle\omega, +, \cdot\rangle \models \Phi_{\mathsf{j}(\mathsf{k},\mathsf{n})}(\langle\sigma_0, \ldots, \sigma_{k-1}\rangle) = \sigma_k[\alpha_0, \ldots, \alpha_k]$ if and only if
$\varphi_{j(k,n)}(\alpha_0, \ldots, \alpha_{k-1}) = \alpha_k$

Then (6) is an immediate consequence of (2), (65), and (66) via the theory of a structure consisting of all sets of integers, the combinatorial operators corresponding to functions in F, with the $\simeq$ relation replacing =. As in the previous case we obtain (7)-(9) be defining

(67) $\mathsf{ft}(\alpha) \equiv (\exists x)(\mathsf{set}(x) \wedge (\forall y)(y \dot{\epsilon}\, x \equiv \alpha(y)))$

(68) $\mathsf{coft}(\alpha) \equiv (\exists x)(\mathsf{set}(x) \wedge (\forall y)(y \dot{\epsilon}\, x \equiv \sim \alpha(y)))$

(69) $\mathsf{im}(\alpha) \equiv (\forall a)((\forall x)(\alpha(x) \vee x\eta a) \rightarrow \mathsf{coft}(a))$

(70) $\mathsf{hyp}(\alpha) \equiv (\forall n)(\mathsf{ar}(n) \rightarrow (\exists x)(\forall y)(y \dot{\epsilon} \{n\}(x) \rightarrow \sim \alpha(y)))$

(71) $\mathsf{pre}(x, y, \alpha) \equiv \alpha(x) \wedge \alpha(y) \wedge x < y \wedge (\forall z)(x < z < y \rightarrow \sim \alpha(z))$

(72) $\mathsf{retr}(\alpha) \equiv (\exists n)(\mathsf{fnc}(n) \wedge (\forall x)(\alpha(x) \rightarrow x \text{ dom } n) \wedge (\forall x, y)(\mathsf{pre}(x, y, \alpha) \rightarrow$
$x = \{n\}(y)) \wedge (\forall y)(\alpha(y) \wedge (\forall x)(\sim\mathsf{pre}(x, y, \alpha)) \rightarrow y = \{n\}(y)))$

(73) $\mathsf{regr}(\alpha) \equiv (\exists\beta)(\alpha \doteq \beta \wedge \mathsf{retr}(\beta))$

Thus we have defined finite, cofinite, immune, hyperimmune, retraceable, and regressive, just as before except in a second order theory and without requiring that their complements be r.e. This proves (7)-(9). In the next collection of results we show that arithmetic can be defined in first order isolic theories. In $L(\Lambda, +, \cdot)$ (= $L(\omega, +, \cdot)$) take (15), (17) as given and then define

(74) $\mathsf{fin}(x) \equiv (\forall y)(x \leqslant y \vee y \leqslant x)$

(75) $\mathsf{fin}^*(x) \equiv (\forall y)(y \leqslant x \rightarrow \mathsf{fin}(y))$

By lemma 1 of the following section fin* defines ω in each of the structures in groups I or II and thus has the intended interpretation of 'the finite.' But this (with the obvious reducibility) immediately implies (14). Let K_2^* be obtained from K_2 of (33) by relativization of variables to fin* and let n be the Gödel number of the function $(p_{k(x)})^{l(x)}$ where p_x is the x-th prime integer starting with $p_0 = 2$. We continue our list of definitions with

(76) $\theta(x, y) = (\iota z)(\mathsf{fin}^*(x, y, z) \quad \mathsf{K}_2^*(\mathbf{n}, \mathsf{j}(x, y), z))$

(77) $y \mid z \equiv (\exists x)(xy = z),\ y \nmid z \equiv \sim(y \mid z)$

(78) $\mathsf{nd}(x, y) = (\iota z)(\mathsf{fin}^*(y, z) \wedge \theta(y, z) \mid x \wedge (\theta(y, z+1) \nmid x))$

(79) $\mathsf{cls}(x) \equiv (\forall y)(\mathsf{fin}^*(y) \rightarrow \mathsf{nd}(x, y) = 1 \vee \mathsf{nd}(x, y) = 2)$

(80) $x(y) \equiv \mathsf{cls}(x) \wedge \mathsf{fin}^*(y) \wedge \mathsf{nd}(x, y) = 1$

(81) $x \doteq y \equiv \mathsf{cls}(x) \wedge \mathsf{cls}(y) \wedge (\forall z)(\mathsf{fin}^*(z) \rightarrow (x(z) \equiv y(z)))$

Now θ defines the prime power function mentioned above, and (78) like (21) uses an element to code a sequence (possibly eventually = 0) of integers. However by lemma 2 for each $a \in \{1, 2\}^\omega$ there is an $x_0 \in \Lambda(W)$ such that

(82) $\langle \Lambda(W), +, \cdot \rangle \models \mathsf{nd}(v_0, v_1) = v_2[x_0, x_1, x_2]$ if and only if $(x_1 \notin \omega$ and $x_2 = 0)$ or $(x_1 \in \omega$ and $x_2 = a(x_1))$

Thus (79)-(81) adequately formalize the class concept, membership, and class equality. With the obvious reducibility (13) follows immediately and thus subject to the results of the next section concludes our proof of theorems 1 and 2.

3. *Definability.*

Lemma 1. *The formula* fin* *of* (75) *defines* ω *in each of the structures* (i) $\langle \Lambda(W), \leqslant \rangle$, *and* (ii) $\langle \Lambda_z(W), \leqslant \rangle$, *where* $W \in \{1, R, H, RH\}$.

Proof of (i). Let $\alpha \in A \in \Lambda(W)$. If $A \in \omega$ then A clearly satisfies fin* in each $\Lambda(W)$. Thus suppose $A \in \Lambda(W) - \omega$ so that α is an immune set. Now (a) if α is immune then every infinite subset of α is immune, (b) if α is hyperimmune then every infinite subset of α is hyperimmune, and (c) if α is an infinite regressive set then α has $2^{\aleph_0}$ infinite regressive subsets. (a) and (b) are immediate and (c) holds because an infinite regressive α contains an infinite retraceable subset β (cf. proposition 7 of [1]), and an infinite retraceable β contains at least $2^{\aleph_0}$ infinite retraceable subsets (cf. proposition 4 of [3]). Since every isol has at most $\aleph_0$ predecessors, and contains exactly $\aleph_0$ sets, we can use (a)-(c) to choose an infinite $\beta \subseteq \alpha$ such that if $B = \mathrm{Req}(\beta)$ then $B \nleqslant A$ and $A \in \Lambda(W)$ implies $B \in \Lambda(W)$ for each choice of W ($\mathrm{Req}(\beta)$ denotes the RET to which β belongs). Also $A \nleqslant B$, for otherwise there would exist a partial recursive isomorphism mapping α onto a proper subset of itself contradicting the fact that $A \in \Lambda$. Thus A and B are incomparables in Λ, hence in $\Lambda(W)$, each of which is closed under $\leqslant$. But this violates the formula fin*. QED.

Proof of (ii). In [2] it is shown how a Turing degree $\Delta(A)$ can be associated with each $A \in \Lambda_z$ or $A \in \Lambda(R)$. In the first case it is the common degree of the $\alpha \in A$ with r.e. complement and in the second case it is the common degree of the $\alpha \in A$ which are retraceable. The ingredients of our proof consist of (a) for every r.e. degree $\mathbf{a} > \mathbf{0}$ there is an $A \in \Lambda_z(RH)$ such that $\mathbf{a} = \Delta(A)$, (b) if $A, B \in \Lambda(R) - \omega$ and $A \leqslant B$ then $\Delta(A) = \Delta(B)$, and (c) there are at least three r.e. degrees. (a) and (b) are respectively propositions 16 and

17 of [2]. Now for our proof. Again we need only take care of the case $A \in \Lambda_z(W) - \omega$. Case (1) $A \in \Lambda(R)$. Choose $B \in \Lambda_z(RH)$ with non-recursive degree $\Delta(B) \neq \Delta(A)$. Then A is incomparable with B in Λ, by (b), hence in $\Lambda_z(W)$. Thus A violates fin*. Case (2) $A \notin \Lambda(R)$. Take any $B \in \Lambda_z(RH)$ with non-recursive degree $\Delta(B)$. $A \nleqslant B$, for otherwise by proposition 9 of [1], $\Lambda(R)$ is closed under $\leqslant$, giving $A \in \Lambda(R)$. If $B \nleqslant A$ then A violates fin*. On the other hand if $B \leqslant A$ then by (a) we can choose a $C \in \Delta_z(RH)$ with non-recursive degree $\Delta(C) \neq \Delta(B)$. Then B is incomparable with C by (b), and $B \leqslant A$ by cases. Thus A violates fin*. Since $\Lambda_z(W)$ is closed under $\leqslant$ these violations occur in $\Lambda_z(W)$. QED.

If we examine our proof we see that ω is actually defined in $\Lambda(W)$, $\Lambda_z(R)$ by the formula fin. This was observed by S. Tennenbaum for the case of Λ (cf. the footnote on page 103 of [5]). In [8] a complicated priority argument is used to show that fin defines ω in Λ_z. Whether it also works for $\Lambda_z(H)$ is an open question. Our point is that if one is willing to replace fin by a slightly more complicated formula then most of the technical details are avoided.

Let $a : \omega \to \omega$ be one one and let $b : \omega \to \omega$. In [2] $a \leqslant^* b$ is defined to mean that there exists a partial recursive function q such that $\rho a \subseteq \delta q$ and $q(a_n) = b_n$ for each $n \in \omega$ (we use the notation of [2] in the rest of this paper, interchangeably writing a_n for $a(n)$ whenever it is more convenient, except that we still use ω for the integers).

Sublemma. *If $f : \omega \to \omega$ is strictly increasing and $a : \omega \to \omega - \{0\}$ then there exists a strictly increasing regressive function t such that $t \leqslant^* a$ and $f(t_n) < t_{n+1}$ for every $n \in \omega$.*

Proof. Let p enumerate the primes in increasing order starting out with $p(0) = 2$. Define $t_0 = p(0)^{a(0)}$ and $t_{n+1} = t_n \cdot p(f(t_n))^{a(n+1)}$ for $n \in \omega$. First note that $t_{n+1} > p(f(t_n)) > f(t_n) \geqslant t_n$ giving $f(t_n) < t_{n+1}$ for $n \in \omega$. Also $p(f(t_n))$ is greater than every prime factor of t_n and this implies that the highest prime factor of t_n occurs with exponent $a(n)$. Let $q(0) = 0$ and otherwise $q(x) =$ the exponent to which the highest prime of x occurs in x. q is recursive and $q(t_n) = a_n$. Thus $t \leqslant^* a$. Let $r(x) = t_0$ for $x \in \{0, t_0\}$ and otherwise let $r(x) = x$ divided by its highest prime factor raised to the $q(x)$-th power. r is recursive and $r(t_0) = t_0$, $r(t_{n+1}) = t_n$ for $n \in \omega$. Thus t is a regressive function. QED.

In the construction of [4] a precursor infinite product of isols is used. Let $\nu(n) = \{x : x < n\}$ and $j(x \times \alpha) = \{j(x, y) : y \in \alpha\}$. We code eventually vanishing sequences $x = \{x_0, \ldots, x_k, 0, \ldots\}$ and $x_k \neq 0$ with the number $\{x_n\}^* = (\Pi_{i \leqslant k} p(i)^{x(i)}) - 1$. Note that for the sequence consisting entirely of 0's (denoted by $\{\theta\}$) $\{\theta\}^* = 0$. There are recursive functions c, d such that $c(0) = c(\{\theta\}^*) = 0$, and $c(\{x_0, \ldots, x_k, 0, \ldots\}^*) = k + 1$ for $x_k \neq 0$, and $d(k, \{x_n\}^*) = x_k$. These notions are all explained in [4]. For any $a : \omega \to \omega - \{0\}$ and regressive function t let $\gamma_n = \{0\} \cup j(t_n \times (\nu(a_n) - \{0\}))$ and then define

(83) $\xi = \{\{x_n\}^* : (\forall n)\,(x_n \in \gamma_n)\}$.

Lemma 2. *For any $a : \omega \to \omega - \{0\}$ there is a regressive function t such that ξ (as defined by (83)) is a regressive hyperimmune set.*

Proof. Let q and r be as in the sublemma and let $s(n) = \Pi_{i<n} a_i$. We define ξ (and consequently t) in stages. Let $\xi_0 = \{\{\theta\}^*\} = \{0\}$ and $y_0 = 0$. Given ξ_n let $y_{n+1} = \{0, \ldots, 0, j(t_n, 1), 0, \ldots\}^*$ where the sequence contains an initial string of n zeros and where t_n is chosen so as to satisfy (i) $q(t_n) = a_n$, (ii) $r(t_n) = t_{n-1}$ if $n > 0$ and $r(t_0) = t_0$ otherwise, (iii) $y_{n+1} > x$ for every $x \in \xi_n$, and (iv) $t_n > g(s(n))$ where g ranges over the first n recursive functions enumerated in some order. Thus construction is legitimatized by the sublemma (note that by the induction hypothesis, in (iii) y_n is a function of t_{n-1}). Now define

(84) $\xi_{n+1} = \xi_n \cup \{\{x_i\}^* : c(\{x_i\}^*) = n + 1 \wedge (\forall i \leqslant n)(x_i \in \gamma_i)\}$

where the γ_i have been constructed from the t_i as in (83). By the construction t is regressive and it is evident that $\xi = \mathbf{U}_{n<\omega}\xi_n$ where ξ is given by (83). We first show that ξ is regressive (in fact retraceable) by defining a partial recursive function u which retraces it. Let $u(0) = 0$ and for any $x \neq 0$ we start the following computation. Compute $c(x) = n + 1$ and $d(i, x)$ for $i \leqslant n$. This gives us complete knowledge of the sequence which $x = \{x_0, \ldots, x_n, 0, \ldots\}^*$, $x_n \neq 0$ represents. Compute $A = \{i \leqslant n : x_i \neq 0\}$ (note that $n \in A$) and $k(x_i)$, $l(x_i)$, $q(k(x_i))$, $r^{(m)}(k(x_i))$, and $r^*(k(x_i))$ for each $i \in A$ (recall that $r^{(m)}$ is the m-th iterate of r and that $r^*(a)$ is the least m such that $r^{(m)}(a) = r^{(m+1)}(a)$). If $l(x_i) > q(k(x_i)$ or $l(x_i) = 0$ for any $i \in A$ then $u(x)$ is undefined. If $r^*(k(x_n)) \neq n$ or $r^{(n-i)}(k(x_n)) \neq k(x_i)$ for any $i \in A$ then $u(x)$ is undefined. Otherwise

(85) $\eta_x = \{\{z_i\}^* : c(\{z_i\}^*) \leqslant n + 1 \wedge (\forall i \leqslant n)(z_i \in \{0\} \cup j(r^{(n-i)}(k(x_n)) \times (\nu(q(r^{(n-i)}(k(x_n)))) - \{0\})))\}$

can be effectively computed from x. By our eliminations it is clear that $x \in \eta_x$. Arrange the elements of η_x in size order and define $u(x)$ as the immediate predecessor of x in that order. $u(x)$ is clearly a partial recursive function and in order to see that it retraces ξ first note that if $x \in \xi$, say $x \in \xi_{n+1} - \xi_n$, then $x \in \delta u$. Moreover it is apparent by inspecting (84)-(85) that $\eta_x = \xi_{n+1}$. By (iii) if $m > n$ then every element in $\xi_{m+1} - \xi_m$ exceeds every element in η_x. Thus the location of x as the i-th element of η_x is its location in ξ. This proves that ξ is retraceable. Now we show that ξ is hyperimmune. Let x_n enumerate ξ in increasing order and suppose that $x_n < g(n)$ for all n, for some recursive function g. By (iii) $\nu(y_{n+1}) \cap \xi = \xi_n$ which contains $s(n)$ elements. Thus $y_{n+1} = x_{s(n)} < g(s(n))$. But $t_n < y_{n+1}$ so that $t_n < g(s(n))$ for all n which contradicts (iv). QED.

From this lemma and T1 of [4] we obtain the desired result

Corollary. *For every sequence* $a : \omega \to \omega - \{0\}$ *there is an* $x \in \Lambda(RH)$ *such that* $p_n^y | x$ *if and only if* $y \leqslant a_n$.

It should be remarked that T4.4 of [12] would provide a very short proof that $\Lambda(R)$ contains an x satisfying the corollary, however it would be necessary to modify that theorem to get the x in $\Lambda(RH)$. It is probably true that (83) defines a regressive hyperimmune ξ for any regressive hyperimmune t satisfying $t \leqslant^* a$. The difficulty is in regressing as complicated an object as ξ. Our idea was that if one were only in need of an example, a special and pathological t would do.

We draw two final conclusions. First if $f_1, \ldots, f_n$ and g are recursive combinatorial functions then we cannot show that g is undefinable in $\langle \Lambda, +, \cdot, f_1, \ldots, f_n \rangle$ by a computation of degrees. Similarly for all the other structures mentioned in this paper. Second, that the theory of Λ_z is a constructible set but it seems likely that the theory of Λ is not. That is indeed a strong difference.

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