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ON THE EXTENSIONS OF S5

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1 The purpose of this paper is to investigate the extensions of the Lewis system S5. To some extent this is meant to be a complement to what is shown in Scroggs [7]. We will show that any formula containing only one variable, if added to S5, will give an inconsistency or make the system collapse into classical propositional calculus (**PC**). We then examine the proper extensions of S5 obtained by adding formulas containing more than one variable. We describe Kripke-type semantics for these systems and prove their completeness.

2 A *normal* extension of S5 is an extension which is closed under the rules of substitution and (material) detachment. A *proper* extension of S5 is a normal extension where some formula not valid in S5 is derivable, but the formula $p \rightarrow Lp$ is not derivable.

Theorem 1: *If any wff, containing only one propositional variable, is added as a new axiom to* S5, *then the system thus obtained is not a proper extension of* S5.

Proof: Any formula is, in S5, equivalent to a formula in modal conjunctive normal form (**MNCF**).[1] We want to show that no formula of the form

(i) $$a \vee Lb_1 \vee Lb_2 \vee \ldots \vee Lb_n \vee Mc$$

where a, b_1, . . . b_n and c all are **PC**-formulas (possibly empty), can be used to give a proper extension of S5. Since any **MNCF**-formula is a conjunction of disjunctions of this form the theorem will follow. Our assumption is that a, b_1, . . . b_n and c contain only one propositional variable, say p. In [3] it is shown[2] that a formula of the form (i) is S5-valid (and hence derivable) iff at least one of the formulas $a \vee c$, $b_1 \vee c$, . . ., $b_n \vee c$ is **PC**-valid. So in the added formula, none of these

1. See Hughes and Cresswell [3], pp. 54-56.

2. See Hughes and Cresswell [3], pp. 118-120.

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formulas can be **PC**-valid. Hence none of $a, b_1, \ldots, b_n$ or c can be a tautology. Since they contain only one propositional variable, any of them must be **PC**-equivalent to p, $-p$ or $p \,\&\, -p$. Since $-(p \,\&\, -p)$, $-L(p \,\&\, -p)$ and $-M(p \,\&\, -p)$ are theorems in S5, any formula of the form (i) containing any of the disjuncts $p \,\&\, -p$, $L(p \,\&\, -p)$ or $M(p \,\&\, -p)$ is equivalent to the formula where the disjunct is dropped (if the formula contains only disjuncts among these three, the addition of the formula will yield an inconsistency). Hence, by the rule of substitution, we only have to regard formulas of the form (i) where $a, b_1, \ldots, b_n$ and c are p, $-p$ or empty. If c is $-p$ and any of $a, b_1, \ldots, b_n$ is p, some of the above mentioned formulas will be $p \vee -p$ and the formula will be derivable in S5, which is against our premises. If c is p and any of $a, b_1, \ldots, b_n$ is $-p$, we have a similar case. By this exclusion we only have the following possibilities left for the added formula: p, Lp, Mp, $p \vee Lp$, $p \vee Mp$, $Lp \vee Mp$, $p \vee Lp \vee Mp$, $p \vee L - p$, $Lp \vee L - p$ or $p \vee Lp \vee L - p$. From any of these we can derive either $p \vee Lp \vee Mp$ or $p \vee Lp \vee L - p$, so one of these formulas would be a thesis of the enlarged system. Now if $p \vee Lp \vee Mp$ were a theorem $(p \,\&\, -p) \vee L(p \,\&\, -p) \vee M(p \,\&\, -p)$ would also be one, and so we would have an inconsistency. The formula $p \vee Lp \vee L - p$ is equivalent in S5 to $p \rightarrow Lp$, so if this formula is added the system will collapse into **PC**.

3 Definition: The finite matrix $\mathfrak{H} = \langle K, \{1\}, \cap, -, *\rangle$ is an *extension Henle matrix* iff:

i) K is a Boolean algebra with respect to $-$ and $\cap$.
ii) If $A \in K$ and $A \neq 0$, then $A^* = 1$ (1 is the unit element and the designated value).
iii) If $A \in K$ and $A = 0$, then $A^* = 0$.

We will use the notation $\mathfrak{H}_n$ for the extension Henle matrix containing 2^n elements.[3] If we try to find a formula containing more than one variable, which, if added to S5, would give a proper extension of S5, we can delimit ourselves to formulas in **MNCF** of the form (i) which fulfill the following conditions:

I) None of $a \vee c$, $b_1 \vee c, \ldots, b_n \vee c$ is a tautology.
II) In any substitution-instance of the formula containing only one variable, at least one of the disjunctions above is a tautology.

The simplest formula satisfying these conditions is $L - p \vee L(p \vee -q) \vee L(p \vee q)$. As is easily shown this formula, which we will call L_2, is not valid and thus not derivable in S5. That the addition of L_2 would not make S5 collapse into **PC** can be shown by using the extension Henle matrix $\mathfrak{H}_2$ (this is group III in Lewis and Langford [5]).[4] For this matrix L_2 is satisfied, but not the formula $p \rightarrow Lp$.

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3. Scroggs, [7], p. 118, has $\mathfrak{H}_n$ as a special case of his more general notion 'Henle matrix.'

4. See Lewis and Langford [5], p. 493.

Following Dugundji [1], let F_n represent the formula

$$\sum_{1 \leqslant i < k \leqslant n} (p_i = p_k)$$

where $\sum$ stands for a chain of disjunctions and '=' is the strict equivalence sign. Scroggs [7] has shown[5] that these formulas can be used to axiomatize all possible normal extensions of S5, and specially if we have $n > 3$, we get a proper extension. Scroggs also shows that the characteristic matrix for the system S5 + F_n is the extension Henle matrix $\mathfrak{H}_k$, where k is the greatest integer such that 2^k is less than n, and furthermore any proper extension of S5 has as its characteristic matrix some extension Henle matrix. Since L_2 is satisfied by $\mathfrak{H}_2$, but not by any $\mathfrak{H}_m$ where $m > 2$, it follows that the characteristic matrix for the system S5 + L_2[6] is $\mathfrak{H}_2$.

Consider the following sequence of formulas:

L_1: $L - p_1 \vee Lp_1$
L_2: $L - p_1 \vee L(p_1 \vee -p_2) \vee L(p_1 \vee p_2)$
L_3: $L - p_1 \vee L(p_1 \vee -p_2) \vee L(p_1 \vee p_2 \vee -p_3) \vee L(p_1 \vee p_2 \vee p_3)$
⋮

All these formulas except L_1 satisfy I) and II) as stated above and they yield proper extensions of S5. From the system S5 + L_n any of the formulas L_m with $m \geq n$ can be derived but none where $m < n$. An argument similar to that for S5 + L_2 shows that the characteristic matrix for S5 + L_n is $\mathfrak{H}_n$. The sequence $L_1, L_2, L_3, \ldots$ provides a somewhat simpler basis for the extensions of S5 than the Dugundji formulas.

4 Following the ideas of Kripke, we now define the appropriate semantic notion for the systems S5 + L_n. An S5 + L_n-model is an ordered triple $\langle W, R, V\rangle$, where W is a set of 'worlds' containing *at most n* elements, R is the universal relation, defined over the members of W, and V is a value assignment satisfying the usual S5-conditions. Given this semantics, it is easily verified that all theorems of S5 + L_n are valid and all formulas L_m, where $m < n$, are invalid.

Following Lemmon [4],[7] we now define an algebra $\mathfrak{N}_n$ on the model structure $\langle W, R\rangle$ as $\langle M, \{W\}, \cap, -, *\rangle$ where:

i) M is the power set of W (we can assume that W contains exactly n elements).
ii) $\cap$, - are the set-theoretic operations intersection and complementation restricted to M.
iii) If $A \in M$, then $A^* = \{x: \exists y(y \in A \;\&\; Rxy)\}$.

Theorem 2: *All the systems* S5 + L_n *are complete.*[8]

5. Scroggs [7], pp. 119-120.

6. In [9], Sobociński calls this system V2.

7. Lemmon [4], p. 57.

8. This result is also in Segerberg [8].

Proof: From a very general result of Lemmon[9] it follows that for any wff A, A is satisfied by the algebra $\mathfrak{R}_n$ iff it is valid in every S5 + L_n-model. Since R is the universal relation, for any $A \epsilon M$ if $A \neq \emptyset$, then $A^* = W$, it can now be shown that the algebra $\mathfrak{R}_n$ is isomorphic to the extension Henle matrix $\mathfrak{H}_n$. This matrix is the characteristic for S5 + L_n and therefore A is derivable in S5 + L_n iff A is valid in every S5 + L_n-model.

5 A result of Parry [6] can be formulated:[10]

Any wff A containing at most k variables is a theorem in S5 *iff A is satisfied by* $\mathfrak{H}_{2k}$.

From this together with Scroggs results we can give a new proof of Theorem 1. Suppose we add to S5 a wff, not provable in S5, containing only one variable. Since the formula, say A, is not a theorem in S5, S5 + A cannot have $\mathfrak{H}_2$ as its characteristic matrix. A is not satisfied by $\mathfrak{H}_2$ and therefore not satisfied by any $\mathfrak{H}_n$ where $n > 2$. So if S5 + A is consistent it must have $\mathfrak{H}_1$ as its characteristic matrix, and then S5 + A is equivalent to **PC**.

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9. Lemmon [4], Theorem 21, p. 61.

10. See S. Halldén [2], p. 28.