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# A COMPLETENESS PROOF FOR *C*-CALCULUS

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*To Alfred Tarski who first axiomatized C-calculus*

*Introduction.* Every true formula of the classical implicational logic, the $C$-calculus, is provable, by means of substitution and detachment, from the following three axioms:

1. $CCCpqrCqr$
2. $CCCpqrCCprr$
3. $CCqrCCCprrCCpqr$[1]

In effect 1, 2 and 3 jointly assert the inferential equivalence of a formula of the form $CC\alpha\beta\gamma$ with the set of two formulas of the forms $C\beta\gamma$ and $CC\alpha\gamma\gamma$.[2] The completeness proof which follows is of elementary nature.[3] First, the deduction of useful theorems is given. Then, it is shown that a formula in the implicational normal form is true if and only if it satisfies the chain condition, and that every formula in the implicational normal form which satisfies the chain condition is deducible from 1, 2 and 3. Finally, it is shown that every formula of the $C$-calculus is inferentially equivalent to a finite set of formulas in the implicational normal form.

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1. This axiomatization was discovered in 1961.

2. Equivalence asserting axiomatizations, besides being pedagogically transparent, may be of interest in connection with systematization of metalogic by means of inferential equivalence; see [1].

3. The first completeness proof of an axiomatization of $C$-calculus was given by Tarski, but never published. See footnote to p. 145 of [3]. Formula 2 was used by Tarski in his first axiomatization of $C$-calculus. Another completeness proof of $C$-calculus was given by Kurt Schütte, *cf.* [5] and [4], pp. 214-217. Schütte's proof presupposes completeness of the logic of implication and negation (the $C$-$N$-calculus).

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*Derivation of useful theorems.*

1. *CCCpqrCqr*
2. *CCCpqrCCprr*
3. *CCqrCCCprrCCpqr*

 1 *p/Cpq, q/r, r/Cqr* × *C*1-4

4. *CrCqr*

 2 *q/r, r/CqCpr* × *C*4 *r/Cpr*-5

5. *CCpCqCprCqCpr*

 3 *q/Cpr, r/CqCpr* × *C*4 *r/Cpr*-*C*5-6

6. *CCpCprCqCpr*

 6 *r/p, q/CrCqr* × *C*4 *r/p, q/p*-*C*4-7

7. *Cpp*

 2 *r/Cpq* × *C*7 *p/Cpq*-8

8. *CCpCpqCpq*

 1 *p/r, r/CpCrq* × *C*4 *r/Crq, q/p*-9

9. *CqCpCrq*

 3 *r/CpCrq* × *C*9-*C*8 *q/Crq*-10

10. *CCpqCpCrq*

 3 *q/Cqr, r/CqCpr* × *C*10 *p/q, q/r, r/p*-*C*5-11

11. *CCpCqrCqCpr*

 11 *p/Cqr, r/Cpr* × *C*10 *p/q, q/r, r/p*-12

12. *CqCCqrCpr*

 3 *r/CCqrCpr* × *C*12-*C*5 *q/Cqr*-13

13. *CCpqCCqrCpr*

 11 *p/CCppp, q/Cpp, r/p* × *C*7 *p/CCppp*-*C*7-14

14. *CCCpppp*

 13 *p/CCpqp, q/CCppp, r/p* × *C*2 *r/p*-*C*14-15

15. *CCCpqpp*[4]

 13 *p/Cpq, q/CCqrCpr, r/CCCprsCCqrs* × *C*13-*C*13 *p/Cqr, q/Cpr, r/s*-16

16. *CCpqCCCprsCCqrs*

 11 *p/Cpq, q/CCprs, r/CCqrs* × *C*16-17

17. *CCCprsCCpqCCqrs*

 17 *p/Cpq, r/p, s/p, q/r* × *C*15-18

18. *CCCpqrCCrpp*

 18 *r/q* × 19

19. *CCCpqqCCqpp*[5]

 11 *p/Cpq, q/Cqr, r/Cpr* × *C*13-20

20. *CCqrCCpqCpr*

Metatheorem I. *If* **W**$\beta$ *is a theorem where* **W** *is an n-term series*

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4. 4, 13 and 15 form the Tarski-Bernays axiomatization of the *C*-calculus; see [3], p. 145 and p. 296. In the original Tarski axiomatization, 2 was used instead of 15.

5. The derivation from 16 to 19 follows essentially that of Łukasiewicz given in [2].

($n \geq 2$) *of* $C\alpha_k$ ($1 \leq k \leq n$) *and each* $\alpha_k$ *and* $\beta$ *is a well-formed formula, then* $\mathbf{U}\beta$ *is a theorem where* $\mathbf{U}$ *is a series like* $\mathbf{W}$ *except for the order of elements.*

As any permutation is obtainable by successive permutations of neighboring elements, it suffices to show that in $\mathbf{W}\beta$ any successive $\alpha_h$ and $\alpha_{h+1}$ can be interchanged preserving theoremhood. Let

a. $C\alpha_1 C\alpha_2 \ldots C\alpha_{h-1} C\alpha_h C\alpha_{h+1}\gamma$

be a theorem. By 11 $p/\alpha_h$, $q/\alpha_{h+1}$, $r/\gamma$,

b. $CC\alpha_h C\alpha_{h+1}\gamma C\alpha_{h+1} C\alpha_h\gamma$

By 20 and b,

c. $CC\alpha_{h-1} C\alpha_h C\alpha_{h+1}\gamma C\alpha_{h-1} C\alpha_{h+1} C\alpha_h\gamma$

Continuing in the same way,

d. $CC\alpha_1 C\alpha_2 \ldots C\alpha_{h-1} C\alpha_h C\alpha_{h+1}\gamma C\alpha_1 C\alpha_2 \ldots C\alpha_{h-1} C\alpha_{h+1} C\alpha_h\gamma$

Detaching a from d,

$C\alpha_1 C\alpha_2 \ldots C\alpha_{h-1} C\alpha_{h+1} C\alpha_h\gamma$

4 $r/CCpqCCqrCpr$, $q/CrCCpqr \times C13$-21

21. $CCrCCpqrCCpqCCqrCpr$
21 I $\times$ 22

22. $CpCCqrCCrCCpqrCCpqr$
20 $p/Cqr$, $q/CCrCCpqrCCpqr$, $r/CCCCpqrrr \times C19$ $p/r$, $q/CCpqr$-23

23. $CCCqrCCrCCpqrCCpqrCCqrCCCCpqrrr$
20 $q/CCqrCCrCCpqrCCpqr$, $r/CCqrCCCCpqrrr \times C23$-$C22$-24

24. $CpCCqrCCCCpqrrr$
24 I $\times$ 25

25. $CCCCpqrrCpCCqrr$
20 $q/CCqrr$, $r/CCrqq \times C19$ $p/q$, $q/r$-26

26. $CCpCCqrrCpCCrqq$
13 $p/CpCCrqq$, $q/CCrqCpq$, $r/CCrCpqCpq \times C11$ $q/Crq$, $r/q$-$C2$ $p/r$, $r/Cpq$-27

27. $CCpCCrqqCCrCpqCpq$
13 $p/CpCCqrr$, $q/CpCCrqq$, $r/CCrCpqCpq \times C26$-$C27$-28

28. $CCpCCqrrCCrCpqCpq$
20 $q/CCrCpqCpq$, $r/CCCpqrr$, $p/CpCCqrr \times C19$ $p/r$, $q/Cpq$-$C28$-29

29. $CCpCCqrrCCCpqrr$
4 $r/CCCprpp$, $q/Cpq \times C15$ $q/r$-30

30. $CCpqCCCprpp$
29 $p/Cpq$, $q/Cpr$, $r/p \times C30$-31

31. $CCCCpqCprpp$
19 $p/CCpqCpr$, $q/p \times C31$-32

32. *CCpCCpqCprCCpqCpr*

3 $q/Cqr$, $r/CCpqCpr \times C20$-$C32$-33

33. *CCpCqrCCpqCpr*

*Implicational normal form.* We write 'δ' with a numerical subscript as a metalinguistic variable ranging over variables. A well-formed formula $\alpha$ is in the implicational normal form if and only if $\alpha$ is

$$C\beta_n C\beta_{n-1} \ldots C\beta_1 \delta_0$$

with $n \geq 0$, with each $\beta_k$ $(1 \leq k \leq n)$ either a variable or an implication of the form $C\delta_x\delta_y$. A formula in the implicational normal form is true if and only if there is a chain $\beta_{s_1}, \beta_{s_2}, \ldots, \beta_{s_m}$ $(1 \leq s_1, s_2, \ldots, s_m \leq n)$ such that either

$$\beta_{s_1} = \delta_0$$

or else

$$\beta_{s_m} = \delta_m,\ \beta_{s_{m-1}} = C\delta_m\delta_{m-1},\ \ldots,\ \beta_{s_1} = C\delta_1\delta_0.$$

Metatheorem II. *Every true formula in the implicational normal form is a theorem.*

Let $\alpha$ be a true formula in the implicational normal form and let $\beta_{s_1}$, $\beta_{s_2}, \ldots, \beta_{s_m}$ be the chain of conditions as described. The proof of Metatheorem II is by induction on the length of the chain. If $\beta_{s_1} = \delta_0$, then $C\beta_{s_1}\delta_0$ is 7 or a substitution instance of 7. Suppose that $C\delta_m CC\delta_m\delta_{m-1} \ldots CC\delta_1\delta_0\delta_0$ is a theorem, then by detaching it from 20 $q/\delta_m, r/CC\delta_m\delta_{m-1} \ldots CC\delta_1\delta_0\delta_0$, $r/\delta_{m+1}$ we obtain as a theorem $CC\delta_{m+1}\delta_m C\delta_{m+1}CC\delta_m\delta_{m-1} \ldots CC\delta_1\delta_0\delta_0$ and, by Metatheorem I, $C\delta_{m+1}CC\delta_{m+1}\delta_m CC\delta_m\delta_{m-1} \ldots CC\delta_1\delta_0\delta_0$. Thus, by induction $C\beta_{s_m}C\beta_{s_{m-1}}C\beta_{s_{m-2}} \ldots C\beta_{s_1}\delta_0$. To this theorem we can add by 4 any condition which is in $\alpha$ and not in the chain and by Metatheorem I we can place it in the required position.

*Reduction to sets of formulas in the implicational normal form.* For the purpose at hand, we say that a formula $\alpha$ is inferentially equivalent to a finite set of formulas $\{\beta_1, \beta_2, \ldots, \beta_n\}$ if and only if $C\alpha\beta_1, C\alpha\beta_2, \ldots, C\alpha\beta_n$ and $C\beta_1C\beta_2 \ldots C\beta_n\alpha$ are theorems. Then, also, $\alpha$ is a theorem if and only if $\beta_1, \beta_2, \ldots, \beta_n$ are theorems. The completeness proof will be provided by showing that every formula is inferentially equivalent to a finite set of formulas in the implicational normal form. As every true formula will be provable from a set of true formulas inferentially equivalent to it and of the implicational normal form, and since all true formulas in the implicational normal form are provable, every true formula is a theorem.

Metatheorem III. $CC\alpha\beta\gamma \longleftrightarrow \{C\beta\gamma,\ CC\alpha\gamma\gamma\}$[6] [1, 2, 3]

Metatheorem IV. $CC\alpha\beta\beta \longleftrightarrow CC\beta\alpha\alpha$ [19]

Metatheorem V. $CCC\alpha\beta\gamma\gamma \longleftrightarrow C\alpha CC\beta\gamma\gamma$ [25, 29]

Metatheorem VI. *If* $\beta \longleftrightarrow \{\gamma, \varepsilon\}$, *then* $C\alpha\beta \longleftrightarrow \{C\alpha\gamma,\ C\alpha\varepsilon\}$

6. '$\longleftrightarrow$' stands for 'is inferentially equivalent to.'

*Proof*: a. $\beta \longleftrightarrow \{\gamma, \varepsilon\}$

b. $C\beta\gamma$

c. $C\beta\varepsilon$

d. $C\gamma C\varepsilon\beta$ [a]

e. $CC\alpha\beta C\alpha\gamma$ [20, b]

f. $CC\alpha\beta C\alpha\varepsilon$ [20, c]

g. $CC\alpha\gamma C\alpha C\varepsilon\beta$ [20, d]

h. $CC\alpha\gamma CC\alpha\varepsilon C\alpha\beta$ [13, 33, g]

$C\alpha\beta \longleftrightarrow \{C\alpha\gamma, C\alpha\varepsilon\}$ [e, f, h]

Metatheorem VII. *If* $\alpha \longleftrightarrow \{\beta_1, \beta_2\}$ *and* $\beta_1 \longleftrightarrow \{\gamma_1, \gamma_2\}$ *then* $\alpha \longleftrightarrow \{\gamma_1, \gamma_2, \beta_2\}$ [13, 33, I]

Metatheorems VI and VII also cover the cases where $\gamma = \varepsilon$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$. Every well-formed formula $\alpha$ is of the form

$$\alpha = C\alpha_1 C\alpha_2 \ldots C\alpha_n \delta_0$$

If $n = 0$, $\alpha$ is a variable and $\alpha$ is not true. Let $\mathbf{g}\beta$ be the number of occurrences of '$C$' in $\beta$ and $\mathbf{g}[\beta] = \max \mathbf{g}\beta_i$ $(1 \leqslant i \leqslant m)$ where $\beta = C\beta_1 \ldots C\beta_m\delta_1$. Let $\mathbf{g}\alpha_j = \mathbf{g}[\alpha]$ and, in addition, let $\alpha_j$ be the last occurring $\alpha_i$ such that $\mathbf{g}\alpha_i = \mathbf{g}[\alpha]$. If $\mathbf{g}\alpha_j \leqslant 1$, then $\alpha$ is in the implicational normal form. Suppose now that $\mathbf{g}\alpha_j > 1$. One of the cases holds: A. $\alpha_j = CC\beta\gamma\varepsilon$, B. $\alpha_j = C\beta C\gamma\varepsilon$.

In case A, $C\alpha_j C\alpha_{j+1} \ldots C\alpha_n\delta_0 = CCC\beta\gamma\varepsilon C\alpha_{j+1} \ldots C\alpha_n\delta_0$. By III this formula is inferentially equivalent to the set $\{\theta_j, \theta_j'\}$,

$\theta_j = C\varepsilon C\alpha_{j+1} \ldots C\alpha_n\delta_0$

$\theta_j' = CCC\beta\gamma C\alpha_{j+1} \ldots C\alpha_n\delta_0 C\alpha_{j+1} \ldots C\alpha_n\delta_0$

$\theta_j' \longleftrightarrow CCC\alpha_{j+1} \ldots C\alpha_n\delta_0 C\beta\gamma C\beta\gamma$, by IV,

$\theta_j' \longleftrightarrow C\alpha_{j+1} CCC\alpha_{j+2} \ldots C\alpha_n\delta_0 C\beta\gamma C\beta\gamma$, by V.

Continuing in the same way, by V and VI,

$\theta_j' \longleftrightarrow C\alpha_{j+1} \ldots C\alpha_n CC\delta_0 C\beta\gamma C\beta\gamma$

$\theta_j' \longleftrightarrow C\alpha_{j+1} \ldots C\alpha_n CCC\beta\gamma\delta_0\delta_0$, by IV and $n-j$ times VI.

$\theta_j' \longleftrightarrow C\alpha_{j+1} \ldots C\alpha_n C\beta CC\gamma\delta_0\delta_0 = \theta_j''$, by V and $n-j$ times VI.

$\mathbf{g}[\theta_j] < \mathbf{g}[\alpha]$ and $\mathbf{g}[\theta_j''] < \mathbf{g}[\alpha]$.

In case B, $C\alpha_j C\alpha_{j+1} \ldots C\alpha_n\delta_0 = CC\beta C\gamma\varepsilon C\alpha_{j+1} \ldots C\alpha_n\delta_0$. By III this formula is inferentially equivalent to the set $\{\theta_j, \theta_j'\}$,

$\theta_j = CC\gamma\varepsilon C\alpha_{j+1} \ldots C\alpha_n\delta_0$

$\theta_j' = CC\beta C\alpha_{j+1} \ldots C\alpha_n\delta_0 C\alpha_{j+1} \ldots C\alpha_n\delta_0$

Proceeding as in case A,

$\theta_j' \longleftrightarrow CCC\alpha_{j+1} \ldots C\alpha_n\delta_0\beta\beta$

$\theta_j' \longleftrightarrow C\alpha_{j+1} C\alpha_{j+2} \ldots C\alpha_n CC\delta_0\beta\beta$

$\theta_j' \longleftrightarrow C\alpha_{j+1} C\alpha_{j+2} \ldots C\alpha_n CC\beta\delta_0\delta_0 = \theta_j''$

Again, $\mathbf{g}[\theta_j] < \mathbf{g}[\alpha]$ and $\mathbf{g}[\theta_j''] < \mathbf{g}[\alpha]$. In either case, by successive applications of VI and VII

$$\alpha \longleftrightarrow \{C\alpha_1 \ldots C\alpha_{j-1}\theta_j, C\alpha_1 \ldots C\alpha_{j-1}\theta_j''\}$$

If it is still the case that $\mathbf{g}[C\alpha_1 \ldots C\alpha_{j-1}\theta_j] = \mathbf{g}[\alpha]$, then by a similar reasoning there are formulas $\theta_k$ and $\theta_k''$ such that $\mathbf{g}[\theta_k] < \mathbf{g}[\alpha]$, $\mathbf{g}[\theta_k''] < \mathbf{g}[\alpha]$, $k < j$ and $C\alpha_1 \ldots C\alpha_{j-1}\theta_j \longleftrightarrow \{C\alpha_1 \ldots C\alpha_{k-1}\theta_k, C\alpha_1 \ldots C\alpha_{k-1}\theta_k''\}$. Similarly, there are formulas $\lambda_\kappa$ and $\lambda_\kappa''$ such that $\mathbf{g}[\lambda_\kappa] < \mathbf{g}[\alpha]$, $\mathbf{g}[\lambda_\kappa''] < \mathbf{g}[\alpha]$ and $C\alpha_1 \ldots C\alpha_{j-1}\theta_j'' \longleftrightarrow \{C\alpha_1 \ldots C\alpha_{k-1}\lambda_k, C\alpha_1 \ldots C\alpha_{k-1}\lambda_k''\}$. By VII $\alpha \longleftrightarrow \{C\alpha_1 \ldots C\alpha_{k-1}\theta_k, C\alpha_1 \ldots C\alpha_{k-1}\theta_k'', C\alpha_1 \ldots C\alpha_{k-1}\lambda_k, C\alpha_1 \ldots C\alpha_{k-1}\lambda_k''\}$. Repeating the reasoning as many times as there are $\alpha_i$ with $\mathbf{g}\alpha_i = \mathbf{g}[\alpha]$ we obtain a set of formulas which is inferentially equivalent to $\alpha$ and, for each formula $\beta$ in the set, $\mathbf{g}[\beta] < \mathbf{g}[\alpha]$.

Repeating the entire procedure $\mathbf{g}[\alpha] - 1$ times, we obtain a finite set of formulas which is inferentially equivalent to $\alpha$ and for each formula $\beta$ in the set, $\mathbf{g}[\beta] \leqslant 1$, i.e., $\beta$ is in the implicational normal form, which completes the completeness proof.

## REFERENCES


[1] Hiż, H., "Inferential equivalence and natural deduction," *The Journal of Symbolic Logic*, vol. 22 (1957), pp. 237-240.

[2] Łukasiewicz, J., *Elementy logiki matematycznej*, Second edition, Warsaw (1958).

[3] Łukasiewicz, J., *Selected works*, edited by L. Borkowski, Amsterdam (1970).

[4] Schmidt, H. A., *Mathematische Gesetze der Logik I*, Berlin-Göttingen-Heidelberg (1960).

[5] Schütte, K., "Über einen Teilbereich des Aussagenkalküls," *Comptes Rendus des Séances de la Société des Sciences et Lettres de Varsovie*, Cl. III, 26 (1933), pp. 30-32.



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