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MANY-VALUED LOGICS AND THE LEWIS PARADOXES

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W. C. Wilcox has recently suggested that certain features of C. I. Lewis's paradoxes of material implication can be used to "lend some justification" to Łukasiewicz's well known claim that there is no interesting n-valued logic between 3-valued and infinitely many-valued systems.[1] The purpose of this note is to show just the contrary, that Lewis's paradoxes can in fact be used to lend some doubt to Łukasiewicz's claim.

One way in which Lewis tries to establish the paradoxical character of material implication and equivalence is by listing, and then formulating in language, some peculiar sounding theorems of *Principia Mathematica*:[2]

$p \supset (q \supset p)$	$\sim(p \supset q) \supset p$	$pq \supset (p \equiv q)$
$\sim p \supset (p \supset q)$	$\sim(p \supset q) \supset \sim q$	$\sim p \sim q \supset (p \equiv q)$
$pq \supset (p \supset q)$	$\sim(p \supset q) \supset p \sim q$	$p \sim q \supset \sim(p \equiv q)$
$pq \supset (q \supset p)$	$\sim(p \supset q) \supset (p \supset \sim q)$	etc.
$\sim p \sim q \supset (p \supset q)$	$\sim(p \supset q) \supset (\sim p \supset q)$	
$\sim p \sim q \supset (q \supset p)$	$\sim(p \supset q) \supset (\sim p \supset \sim q)$	

But a more dramatic, if not more effective, technique used by Lewis against the horseshoe requires that we imagine a large number of true statements and an equal number of false statements—the more variegated the better—written on slips of paper and randomized in a large hat. Then let two of these statements be drawn at random. The probabilities of the

1. "On infinite matrices and the paradoxes of material implication," *Notre Dame Journal of Formal Logic*, vol. XI (1970), p. 254. It should be remarked that Łukasiewicz claims only that no "philosophical significance" attaches to n-valued ($3 < n < \infty$) logics: "Philosophical remarks on many-valued systems of propositional logic," *Polish Logic*, Storrs McCall, Ed., Oxford (1967), p. 60.

2. Lewis and Langford, *Symbolic Logic*, Century Co., New York (1932), esp. pp. 85-89. It is interesting that most, if not all, the ten "illogical" inferences rejected by William S. Cooper ("Propositional Logic of Ordinary Discourse," *Inquiry* (1968), pp. 295-320) appear to be variations on Lewis's list.

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implication-relationship holding between the first statement (p) and the second (q) are as follows:[3]

(i) $P(p \supset q) = 3/4$
(ii) $P(q \supset p) = 3/4$
(iii) $P(p \equiv q) = 1/2$
(iv) $P(p \supset q . \vee . q \supset p) = 1$
(v) $P(\sim(p \supset q) \cdot \sim(q \supset p)) = 0$

Lewis appears not to have recognized that his paradoxes are at least partially resolved in many-valued systems of logic.[4] In the classical 3-valued logic of Łukasiewicz, for example, the probability that the first proposition drawn will imply the second (number (i) above) is reduced to 2/3, while the probability that the two propositions are equivalent (number (iii) above) decreases to 1/3. And in general the probability ratios in classical n-valued logics (as may easily be proved by mathematical induction on the number of values for appropriate matrices) may be computed by using the following formulas:

$$P(p \supset q) = \frac{n+1}{2n} \qquad P(p \equiv q) = \frac{1}{n}$$

Nevertheless the probabilities asserted in (iv) and (v) above remain the same in classical many-valued logics. If the first proposition drawn does not imply the second, then the reverse implication *will* hold. Moreover every one of the paradoxical theorems listed by Lewis persists in n-valued logics generated by Łukasiewicz's formulas.

A wider and more interesting resolution of the paradoxes takes place, however, if we adopt a "regular," rather than a classical, many-valued extension of basic propositional logic.[5] First, although the tautologies of regular many-valued logics mainly duplicate those of classical many-valued logics, and appear intuitively to be as plausible when interpreted as laws of logic, yet *none* of the paradoxical sounding theorems cited by Lewis continue to hold in regular many-valued systems.[6] More gratifying still is the fissure in the Russellian "block-universe" introduced when we calculate new probability ratios in a 3-valued regular system:

3. *Ibid.*, p. 145.

4. *Cf.* especially Chap. VIII of *Symbolic Logic* where Lewis suggests in several different contexts that all truth-functional systems are guilty of the same paradoxicality.

5. S. C. Kleene, *Introduction to Metamathematics*, Amsterdam (1952), p. 334 f. "Regular" n-valued logics can be generated from the 2-valued calculus by substituting for Łukasiewicz's definition: $p \vee q = \text{df}. (p \supset q) \supset q$, the "regular" definition: $p \vee q = \text{df}. \sim p \supset q$.

6. Only four of Cooper's objectionable inferences validated in the 2-valued propositional calculus (Footnote 2 above) are rejected in classical many-valued logic, whereas all ten are rejected in regular many-valued logic.

(i)′ $P(p \supset q) = 5/9$
(ii)′ $P(q \supset p) = 5/9$
(iii)′ $P(p \equiv q) = 2/9$
(iv)′ $P(p \supset q .\text{v}. q \supset p) = 8/9$ (!)
(v)′ $P(\sim(p \supset q) \cdot \sim(q \supset p)) = 1/9$

Or, more generally, in an n-valued regular logic:

$P(p \supset q) = \frac{2n - 1}{n^2}$ (*E.g.*, in a 10-valued system: $P(p \supset q) = 19/100$)

$P(p \equiv q) = \frac{2}{n^2}$ (*E.g.*, in a 10-valued system: $P(p \equiv q) = 1/50$)

$P(p \supset q .\text{v}. q \supset p) = \frac{4n - 4}{n^2}$ (*E.g.* in a 10-valued system: $P(p \supset q .\text{v}. q \supset p) = 9/25$)

I am not sure what "philosophical significance," to use Łukasiewicz's phrase, we should place upon such applications of n-valued ($3 < n < \infty$) logics. But I think it premature, at least, to dismiss these systems as uninteresting.[7]

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7. *Cf.* in this context, Nicholas Rescher, "An intuitive interpretation of systems of four-valued logic," *Notre Dame Journal of Formal Logic,* vol. VI (1965), pp. 154-156.