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TOLERANCE GEOMETRY

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1. *Introduction** In his paper on visual perception [7], Zeeman points out that any model of perception must take account of the fact that we cannot distinguish between points that are sufficiently close. A similar observation has been made for choice behavior by Luce [1]. Zeeman's observation leads him directly to a notion of a "tolerance" within which "we allow an object to move before we notice any difference." Other authors use the terms "threshold" and "just noticeable difference," for the same notion.

Zeeman defines a *tolerance* I on a set A as a binary relation on A which is reflexive and symmetric, and he calls the pair (A, I) a *tolerance space*. We shall use the more common term *graph* for this concept and prefer to think of tolerance spaces or tolerance relations as graphs with more specialized properties, motivated by the notion of "closeness." Zeeman studies various properties of and relations between tolerance spaces (graphs), using topological techniques.

In studying visual perception, it is convenient to distinguish between physical space and (subjective) visual space, the space from which we draw our "conscious" perceptions. It has been observed in the literature that visual space has a non-Euclidean geometry (see Roberts and Suppes [4]). To determine what this geometry is, observed relations such as betweenness, alignment, perpendicularity (of two aligned sets of points), parallelism, etc. are studied, and their properties are determined.

In order to study visual geometry, to take account of the tolerance effect, it seems desirable to replace classical primitives, such as betweenness, straightness, perpendicularity, and parallelism, with more general notions, obtained from the classical ones by substituting closeness for identity. We shall use the term *tolerance geometry* for any geometry whose

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primitives are obtained by such a perturbation. In this paper, we develop an axiomatization for tolerance geometry on the line.

There is one important difference between the axiomatizations for tolerance geometries and those for classical ones. For the latter we are usually interested in categorical axiomatizations. For example, we are interested in listing axioms necessary and sufficient for a given geometry to be isomorphic onto the real plane. In the case of perception, we usually deal only with finite point sets—or at least we can only observe finitely many points to test our axioms. So it is more interesting to study finite sets and to study axioms necessary and sufficient for isomorphism (or homomorphism) *into* certain kinds of spaces. This will be our approach.

2. *Indifference Graphs* Perhaps the simplest example of a tolerance geometry arises from the theory of indifference graphs, motivated by observing judgments of indifference, similarity, etc. Here we take as the only primitive a binary relation I on a finite set A, interpreted as "closeness." In the linear case, we ask for axioms on (A,I) necessary and sufficient for an isomorphism (homomorphism) into the real line. In particular, we shall consider the simplest case where closeness on the line is defined by "being within ϵ." Then, given $\epsilon > 0$, we seek conditions on (A,I) necessary and sufficient for the existence of a function $f:A \to \mathcal{R}$ so that for all $x, y \epsilon A$,

(1) $$xIy \longleftrightarrow |f(x) - f(y)| < \epsilon.$$

The conditions will, of course, be independent of the particular value of ϵ.

The theory of indifference graphs is developed in Roberts [2, 3]. For later use, we shall need to state below at least one of the axiom systems for indifference graphs. Using standard terminology, we shall call the relation I in a graph (A,I) the *adjacency* relation. A *subgraph* (B,J) of (A,I) is a graph (B,J), where $B \subseteq A$ and J is the restriction of I to B. (A,I) is *connected* if for every x and y there is a *path* between x and y, i.e., a sequence $x, x_1, x_2, \ldots, x_n, y$ from A so that $xIx_1Ix_2I\ldots Ix_nIy$. A *component* of a graph is a maximal connected subgraph. If x and y are in the same component, the *distance* $d(x, y)$ is the length of the shortest path between x and y. $d(x, y) = 0$ if and only if $x = y$.

Following Scott and Suppes [5], an equivalence relation E on A is defined by

$$xEy \longleftrightarrow (\forall z)\ [xIz \longleftrightarrow yIz].$$

x^* will denote the equivalence class containing x, A^* the collection of equivalence classes. I^* is defined on A^* by

$$x^*I^*y^* \longleftrightarrow xIy.$$

$G^* = (A^*, I^*)$ will be called the *reduction* of the graph $G = (A, I)$. If $G \cong G^*$, we shall say G is *reduced.*

Roberts [2] defines a point a in (A,I) as an *extreme point* if whenever x and y are adjacent to but not equivalent to a, then xIy and $(\exists z)$ $[xIz$ &

yIz & $\sim aIz$]. (This definition is motivated by considering extreme points of finite subsets of $\mathcal{R}$ and letting I mean "within ϵ.") It is easy to see that

Lemma 1. *If a and b are extreme points in a graph (A, I), then*
(a) $(aIx$ *and* $aIy) \rightarrow xIy$.
(b) $aIb \rightarrow aEb$.

Definition. A graph $G = (A, I)$ is an *indifference graph* if for every (finite)[1] connected subgraph H of G, either H^* has exactly one point or H^* has precisely two extreme points.

Remark. It is easy to write this out in terms of first order, universal sentences. It can be shown however that one first order universal sentence is not sufficient.

Theorem 1 (Roberts [2]). *A finite graph (A, I) is an indifference graph if and only if there is a function $f: A \rightarrow \mathcal{R}$ satisfying* (1).[2]

Indifference graphs are linear tolerance spaces in a somewhat more general sense than Theorem 1. The points of an indifference graph can be simply (linearly) ordered so that points which are "close" (adjacent) can never surround points which are not close (nonadjacent). In particular, it is proved in Roberts [3] that a graph (A, I) is an indifference graph if and only if there is a simple order R on A so that for all $x, y, u, v \in A$,

$$xRuRvRy \ \& \ xIy \rightarrow uIv. \tag{2}$$

If R satisfies (2), we shall say that R is *compatible* with I. It turns out that extreme points of the graph correspond to extreme points (maxima and minima) of the compatible simple order. Summarizing, we have

Theorem 2 (Roberts [3]).
(a) *A graph (A, I) is an indifference graph if and only if there is a simple order R on A compatible with I.*
(b) *If (A, I) is a reduced, connected indifference graph, then a is an extreme point of (A, I) if and only if a is an extreme point (maximum or minimum) of every compatible simple order.*
(c) *If $G = (A, I)$ is a connected indifference graph and a, b are nonadjacent extreme points of G, then there is a compatible simple order R on A so that for all $x \in A$, $aRxRb$.*

Note. Part (c) of this theorem follows from part (b), given the observation that a is an extreme point of G if and only if a^* is an extreme point of G^* and the observation that we may pass from a compatible simple order on G^* to one on G by arbitrarily ordering points within equivalence classes.

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1. All the graphs in this paper are finite, so finiteness of H will be understood below.

2. The function may be taken 1-1 without loss of generality.

3. *ϵ-Betweenness on the Line* In Tarski's [6] fundamental axiomatization of Euclidean geometry, he uses as primitives a ternary relation of betweenness B on a set A and a quaternary relation of equidistance on A. The main point of this paper is to state a Tarski-type axiomatization for tolerance geometry for the simplest case, namely, the one-dimensional one.

It turns out that if we limit ourselves to the line, the classical Tarski axioms can be replaced by a simple set of axioms stated only in terms of betweenness. We list these below. For ease of comparison, we shall number them starting with Axiom C2.

Classical Axioms for Betweenness on the Line For all x, y, z, u, v in A:

C2. $B(x, y, z) \to B(z, y, x)$.
C3. $B(x, y, z)$ *or* $B(x, z, y)$ *or* $B(y, x, z)$.
C4. $B(x, y, u)$ *and* $B(y, z, u) \to B(x, y, z)$.
C5. *If* $u \neq v$, *then* $B(x, u, v)$ *and* $B(u, v, y) \to B(x, u, y)$.
C6. $B(x, y, z)$ *and* $B(y, x, z) \to x = y$.
C7. $x = y \to B(x, y, z)$.

Theorem 3. *Suppose B is a ternary relation on a finite set A. Then Axioms* C2-C7 *are necessary and sufficient for the existence of a* 1-1 *function $f: A \to \mathcal{R}$ so that for all $x, y, z \in A$,*

$$B(x, y, z) \leftrightarrow [f(x) \leqq f(y) \leqq f(z) \text{ or } f(z) \leqq f(y) \leqq f(x)]. \tag{3}$$

Proof. Necessity is straightforward and sufficiency can be verified by induction on $|A|$.

The tolerance axioms for betweenness on the line characterize the relation of *ϵ-betweenness*. If B is the relation of classical betweenness on the set A, then there is a function $f: A \to \mathcal{R}$ so that for all $x, y, z \in A$,

$$B(x, y, z) \leftrightarrow |f(x) - f(y)| + |f(y) - f(z)| = |f(x) - f(z)|. \tag{4}$$

For ϵ-betweenness, we would like conditions on (A, B) necessary and sufficient for the existence of a function $f: A \to \mathcal{R}$ satisfying, for all $x, y, z \in A$,

$$B(x, y, z) \leftrightarrow |f(x) - f(y)| + |f(y) - f(z)| < |f(x) - f(z)| + \epsilon. \tag{5}$$

The tolerance axioms are stated in terms of B and a binary relation I on A *defined* from B by

$$xIy \leftrightarrow B(x, y, x). \tag{6}$$

Axioms for ϵ-Betweenness on the Line For all x, y, z, u, v in A:

T1. *(A, I) is an indifference graph.*
T2. $B(x, y, z) \to B(z, y, x)$.
T3. $B(x, y, z)$ *or* $B(x, z, y)$ *or* $B(y, x, z)$.
T4. $B(x, y, u)$ *and* $B(y, z, u)$ *and* $\sim B(x, y, z) \to uIy$ *and* uIz.
T5. *If* $\sim uIv$, *then* $B(x, u, v)$ *and* $B(u, v, y) \to B(x, u, y)$.
T6. $B(x, y, z)$ *and* $B(y, x, z) \to xIy$ *or* $(zIx$ *and* $zIy)$.
T7. $xIy \to B(x, y, z)$.

Our main objective is to prove

Theorem 4. *Suppose B is a ternary relation on a finite set A and $\epsilon > 0$ is given. Then Axioms* T1-T7 *are necessary and sufficient for the existence of a function $f: A \to \mathcal{R}$ satisfying* (5).[3]

To prove necessity of the axioms, use the observation that $xIy \longleftrightarrow |f(x) - f(y)| < \epsilon_2$. The sufficiency proof starts with the observation that if (A, I) is reduced, then B restricted to the collection of extreme points of (A,I) satisfies the classical axioms C2-C7 (Lemma 3). It follows from Theorem 3 that in general B on the set of extreme points comes from a *weak order*, i.e., a binary relation R which is reflexive, transitive, and complete (i.e., for all x, y, xRy, or yRx). We choose an extreme point x_0 minimal in this order.

Define a binary relation P on A by

$$xPy \longleftrightarrow B(x_0, y, x) \text{ and } \sim xIy. \tag{7}$$

The basic result is that P is a semiorder, a concept due to Luce [1] and to Scott and Suppes [5]. To prove this, we use the theorem (Roberts [2]) that a binary relation is a semiorder if and only if it is asymmetric and transitive and its symmetric complement is an indifference graph. (The *symmetric complement* of a binary relation (A, P) is a binary relation (A, I) so that $xIy \longleftrightarrow \sim xPy$ & $\sim yPx$.) The key to the proof hinges on showing that in fact, I is the symmetric complement of P. This is finally established in Lemma 7, after proving Lemmas 4-6 as preliminary lemmas.

Once it is established that P is a semiorder, the Scott-Suppes [5] representation theorem for semiorders gives an f so that

$$xPy \longleftrightarrow f(x) \geq f(y) + \epsilon/2.$$

This f satisfies (5) as well.

We now turn to the details of the proof. Note first that I is symmetric and reflexive by Axiom T1, since all graphs are symmetric and reflexive by definition.

Lemma 2 (Symmetric versions of Axioms T4 and T5.)

(a) $B(x, y, z)$ & $B(x, z, u)$ & $\sim B(y, z, u) \to xIy$ & xIz.
(b) *If* $\sim uIv$, *then* $B(x, u, v)$ & $B(u, v, y) \to B(x, v, y)$.

Proof. (a) Ax. T2 and Ax. T4. (b) Ax. T2 and Ax. T5.

Lemma 3. *If (A, I) is reduced, then B restricted to the collection of extreme points of (A, I) satisfies the classical axioms,* C2-C7.

Proof. Note that Axioms C2 and C3 go through unchanged from Axioms T2 and T3. Axioms C5 and C7 follow from our corresponding T-axioms if we note that by Lemma 1b, two extreme points which are adjacent are also

3. It follows from the proof that the function f may be taken 1-1 without loss of generality.

equivalent and hence, since (A, I) is reduced, are equal. Axiom C6 follows from Ax. T6, using both parts of Lemma 1. Finally, to verify Ax. C4, suppose $B(x, y, u)$ and $B(y, z, u)$ and $\sim B(x, y, z)$. Then by Ax. T4, uIy and uIz. By the extremality of u and Lemma 1a, yIz. Thus, by Ax. T7, $B(x, y, z)$, which is a contradiction. Q.E.D.

To handle the case where (A, I) may not be reduced, we define B^* on A^* by $B^*(x^*, y^*, z^*) \longleftrightarrow B(x, y, z)$. Proof that B^* is well-defined is tedious but relatively straight-forward. If (A, B) satisfies Axioms T1-T7, then it is easy to verify that (A^*, B^*) does also. One technical point: the I defined from B^* by $B^*(x^*, y^*, x^*)$ is in fact I^*.

Note that (A^*, I^*) is reduced. Thus, by Lemma 3, there is a simple order R^* on the collection of extreme points of (A^*, I^*) such that for all extreme points α, β, γ in (A^*, I^*), $B^*(\alpha, \beta, \gamma)$ if and only if β is between α and γ in R^*. Note that x is an extreme point of (A, I) if and only if x^* is an extreme point of (A^*, I^*). Define a weak order R on the collection of extreme points in (A, I) by $xRy \longleftrightarrow x^*R^*y^*$. Then for all extreme points x, y, z in (A, I), $B(x, y, z)$ if and only if y is between x and z in R.

Let x_0 be a minimum point in the weak order R, i.e., an extreme point so that whenever x and y are extreme points then $B(x_0, x, y)$ or $B(x_0, y, x)$. Define P on A by (7). We shall prove (Lemma 8) that P is a semiorder. To establish this, we show first that I is the symmetric complement of P, by a series of lemmas, culminating in Lemma 7.

Lemma 4. *Suppose x and y are in the same component K of (A, I) and $a \notin K$. Then $\sim B(x, a, y)$.*

Proof. It is sufficient to prove either $B(x, y, a)$ or $B(y, x, a)$. For suppose, say, $B(x, y, a)$ holds. Then, $B(x, a, y)$ cannot hold, otherwise by Ax. T6, either aIy or (xIa and xIy). Both contradict $a \notin K$.

The proof proceeds by induction on the distance $d(x, y)$. If $d(x, y) = 0$, then $x = y$ and so xIy. $B(x, y, a)$ follows by Ax. T7. By way of induction, let $x \ldots zy$ be a minimum length path from x to y. Then $d(x, z) < d(x, y)$, so by inductive assumption, $B(x, z, a)$ or $B(z, x, a)$. Suppose $\sim B(x, y, a)$ and $\sim B(y, x, a)$. By Ax. T3, $B(x, a, y)$. If $B(x, z, a)$ is the case, then by Lemma 2a, we conclude $B(z, a, y)$, since xIa implies $a \in K$, contrary to assumption. Now yIz, so $B(a, z, y)$. By Ax. T6 we conclude $a \in K$, again a contradiction.

If $B(z, x, a)$, then using $B(x, a, y)$, Lemma 2b and the fact that $a \notin K$, we conclude $B(z, a, y)$. Since yIz, we also have $B(a, z, y)$. By Ax. T6, we again conclude $a \in K$, contrary to assumption. Q.E.D.

Lemma 5. *Suppose R is a compatible simple order on (A, I) and (A, I) is connected. Then $xRyRz \rightarrow B(x, y, z)$.*

Proof. Suppose $xRyRz$ holds. We may assume that x, y, and z are distinct. We may also assume $\sim xIy$ and $\sim yIz$, for otherwise $B(x, y, z)$ follows by Ax. T7. Let $A' = A - \{u \neq x, y\colon xRuRy\}$, let $A'' = A - \{v \neq y, z\colon yRvRz\}$, and let I' and I'' be the restrictions of I to A' and A'', respectively. Using the definition of compatibility, one proves that y and z are in one component of

(A', I') while x is in a different one. Thus, by Lemma 4, $\sim B(y, x, z)$. Similarly, using (A'', I''), we find $\sim B(x, z, y)$. Thus by Ax. T3, $B(x, y, z)$, as desired. Q.E.D.

Lemma 6. *If a is an extreme point of a component K of (A, I) and $x, y \in K$, then $B(a, x, y)$ or $B(a, y, x)$.*

Proof. If K^* has only one point, then aEx, so aIx and we conclude $B(a, x, y)$ from Ax. T7. If K^* has more than one point, then, since (A, I) is an indifference graph, by definition K^* has exactly two extreme points. It follows by Lemma 1b that there must be an extreme point b of (A, I) nonadjacent to a. It follows from Theorem 2c that there is a compatible simple order R on A such that $aRyRb$. We conclude $B(a, y, b)$, by Lemma 5. Similarly, $B(a, x, b)$. Suppose now Lemma 6 is false, i.e., suppose $B(x, a, y)$. Now $B(x, a, y)$ and $B(a, y, b)$ implies either aIy or $B(x, a, b)$, by Ax. T5. If aIy, then $B(a, y, x)$, and Lemma 6 is true. If $B(x, a, b)$, then, since $B(a, x, b)$, we conclude by Ax. T6 that aIx or (bIa and bIx). But $\sim bIa$, by choice of b. Thus, aIx; and so from Ax. T7, we conclude $B(a, x, y)$, as desired. Q.E.D.

Lemma 7. *I is the symmetric complement of P.*

Proof. By definition of P, if xIy, then $\sim xPy$ and $\sim yPx$. To complete the proof, it is sufficient to verify the claim that for all x, y, $B(x_0, x, y)$ or $B(x_0, y, x)$. For then $\sim xIy \rightarrow (xPy$ or $yPx)$. We verify the claim by cases. K_0 will denote the component containing x_0.

Case 1. $x, y \in$ the same component K and $K = K_0$.
Case 2. $x, y \in$ the same component K and $K \neq K_0$.
Case 3. $x, y \in$ different components K_1 and K_2 and K_1, say, is the same as K_0.
Case 4. $x, y \in$ different components K_1 and K_2 and neither is K_0.

Case 1. The claim follows by Lemma 6.
Case 2. The claim follows by Lemma 4 and Ax. T3.
Case 3. By Lemma 4, $B(x, x_0, y)$ or $B(x_0, x, y)$. The latter is as desired. Therefore, assume the former. If K_0^* has just one point, then xEx_0 and $B(x_0, x, y)$ follows by Ax. T7. If K_0^* has two extreme points, then, since (A, I) is an indifference graph, there is an extreme point y_0 of K_0 nonadjacent to x_0. Let a be an extreme point in K_2.

We have by definition of x_0 either $B(x_0, y_0, a)$ or $B(x_0, a, y_0)$. The latter is impossible, by Lemma 4. Thus, $B(x_0, y_0, a)$. This gives us the initial step of an inductive proof that in fact $B(x_0, y_0, y)$ holds, where the induction is on $d(y, a)$. Using $B(x_0, y_0, y)$ and the assumption $B(x, x_0, y)$, we conclude by Ax. T4 that either $B(x, x_0, y_0)$ or $(yIx_0$ and $yIy_0)$. The latter is impossible, since $y \in K_2$, $y_0 \in K_0$ and $K_2 \neq K_0$. Thus $B(x, x_0, y_0)$. But by Lemma 5, $B(x_0, x, y_0)$, since by Theorem 2c there is a compatible simple order R on (A, I) such that x_0RxRy_0. Thus, $B(x, x_0, y_0)$ and $B(x_0, x, y_0)$; and so by Ax. T6, either xIx_0 or $(y_0Ix$ and $y_0Ix_0)$. The latter is impossible, since y_0 and x_0 are nonadjacent. Thus xIx_0. The conclusion $B(x_0, x, y)$ follows by Ax. T7.

Case 4. Suppose $\sim B(x_0, x, y)$ and $\sim B(x_0, y, x)$. Then by Ax. T3, $B(x, x_0, y)$. Let a and b be extreme points of K_1 and K_2, respectively. We show $B(a, x_0, b)$. This is a contradiction. For by definition of x_0, either $B(x_0, a, b)$ or $B(x_0, b, a)$ holds. In either case, an application of Ax. T6 shows that a, b, x_0 are not in different components, and this is a contradiction.

To establish $B(a, x_0, b)$, one first establishes $B(a, x_0, y)$ by induction on $d(a, x)$ and then argues by induction on $d(b, y)$. Q.E.D.

Lemma 8. *P as defined by* (7) *is a semiorder.*

Proof. It follows from Theorem 5, Corollary 2 of Roberts [2] that a binary relation P is a semiorder if it is irreflexive and transitive and its symmetric complement is an indifference graph. By Lemma 7 and Ax. T1, it is sufficient to prove that P is transitive. To do this, suppose xPy and yPz. Then $B(x_0, y, x)$ and $B(x_0, z, y)$. By Lemma 2a, either $B(z, y, x)$ or (x_0Iy and x_0Iz). If x_0Iy and x_0Iz, then by Lemma 1a, the extremality of x_0 implies yIz, violating yPz. We conclude $B(x, y, z)$. This conclusion implies $\sim xIz$, for if xIz, then by Ax. T7, $B(y, x, z)$. Thus, by Ax. T6, either xIy or (zIx and zIy), violating xPy or yPz.

To finish the proof of xPz, it is sufficient by Lemma 7 to show $\sim zPx$. But suppose zPx. Then $B(x_0, x, z)$. This plus $B(x_0, y, x)$ implies $B(y, x, z)$ or (x_0Iy and x_0Ix), by Lemma 2a. But by the extremality of x_0, x_0Iy and x_0Ix implies yIx, whence $\sim xPy$. Thus, $B(y, x, z)$. Also $B(x, y, z)$, as proved above. Axiom T6 now implies $\sim yPx$ or $\sim yPz$. These contradictions show that the assumption zPx is impossible. Q.E.D.

To complete the proof of sufficiency of the Axioms T1-T7, we note that by a theorem of Scott and Suppes [5], there is a function $f: A \rightarrow \mathcal{R}$ so that[4]

(8) $$xPy \leftrightarrow f(x) \geq f(y) + \epsilon/2.$$

Observing that $xIy \leftrightarrow |f(x) - f(y)| < \epsilon/2$, the reader can verify that f satisfies (5) of Theorem 4, as desired.

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