Notre Dame Journal of Formal Logic
Volume XIII, Number 4, October 1972
NDJFAM

SOLUTION TO THE PROBLEM CONCERNING THE BOOLEAN BASES FOR CYLINDRIC ALGEBRAS

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In [1],[1] p. 162, Definition 1.1.1, a cylindric algebra of dimension α is defined, as follows:

(**A**) *A cylindric algebra of dimension* α, *where* α *is any ordinal number, is an algebraic structure*

$$\mathfrak{A} = \langle A, +, \times, -, 0, 1, \mathbf{c}_\kappa, \mathbf{d}_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$$

such that 0, 1, *and* $\mathbf{d}_{\kappa\lambda}$, *are distinguished* (constant)[2] *elements of the carrier set* A (*for all* $\kappa, \lambda < \alpha$), - *and* $\mathbf{c}_\kappa$ *are unary operations on* A (*for all* $\kappa < \alpha$), + *and* $\times$ *are binary operations on* A, *and such that the following postulates are satisfied for any* $x, y \epsilon A$ *and any* $\kappa, \lambda, \mu < \alpha$:

C0 The structure $\langle A, +, \times, -, 0, 1\rangle$ is a **BA**;
C1 $[\kappa] : \kappa < \alpha . \supset . \mathbf{c}_\kappa 0 = 0$;
C2 $[x\kappa] : x \epsilon A . \kappa < \alpha . \supset . x \leq \mathbf{c}_\kappa x$ (*i.e.*, $x + \mathbf{c}_\kappa x = \mathbf{c}_\kappa x$);
C3 $[xy\kappa] : x, y \epsilon A . \kappa < \alpha . \supset . \mathbf{c}_\kappa(x \times \mathbf{c}_\kappa y) = \mathbf{c}_\kappa x \times \mathbf{c}_\kappa y$;
C4 $[x\kappa\lambda] : x \epsilon A . \kappa, \lambda < \alpha . \supset . \mathbf{c}_\kappa \mathbf{c}_\lambda x = \mathbf{c}_\lambda \mathbf{c}_\kappa x$;
C5 $[\kappa] : \kappa < \alpha . \supset . \mathbf{d}_{\kappa\kappa} = 1$;
C6 $[\kappa\lambda\mu] : \kappa, \lambda, \mu < \alpha, \kappa \neq \lambda, \mu . \supset . \mathbf{d}_{\lambda\mu} = \mathbf{c}_\kappa(\mathbf{d}_{\lambda\kappa} \times \mathbf{d}_{\kappa\mu})$;
C7 $[x\kappa\lambda] : x \epsilon A . \kappa, \lambda < \alpha . \kappa \neq \lambda . \supset . \mathbf{c}_\kappa(\mathbf{d}_{\kappa\lambda} \times x) \times \mathbf{c}_\kappa(\mathbf{d}_{\kappa\lambda} \times -x) = 0$.

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1. An elementary familiarity with the theory of cylindric algebras and an acquaintance with the papers [2], [3] and [4] is presupposed. Concerning the symbols used in this paper it should be remarked that instead of "$a \cdot b$" which is used in [1] I am using "$a \times b$", and that instead of "$\overline{a}$" used in [3] I am using here "$-a$". An enumeration of the algebraic tables, *cf.* section **3** below, is a continuation of the enumeration of such tables given in [3], [5] and [6].

2. In this paper the difference between the distinguished elements and the constant elements will be disregarded because it is unessential for our present research.

Received November 1, 1971

The elements $\mathbf{d}_{\kappa\lambda}$ which are occurring in the definition given above are called diagonal elements, and the operations $\mathbf{c}_\kappa$ are called cylindrifications. The structure $\langle A, +, \times, -, 0, 1\rangle$ is the Boolean part of the considered system, and the abbreviation **BA** means Boolean Algebra. *Cf.* [1], p. 163 and p. 161. In particular, in [1], p. 161, the following system of Boolean algebra:

(**B**) *An algebraic structure*

$$\mathfrak{B} = \langle A, +, \times, -, 0, 1\rangle$$

in which A is the carrier set, + *and* $\times$ *are binary operations and* - *is a unary operation on* A, *and* 0 *and* 1 *are the constant elements belonging to* A, *is a Boolean algebra, if it satisfies the following postulates:*

B1 $[xy]:x,y \in A\,.\supset.\,x + y = y + x$
B2 $[xy]:x,y \in A\,.\supset.\,x \times y = y \times x$
B3 $[xyz]:x,y,z \in A\,.\supset.\,x + (y \times z) = (x + y) \times (x + z)$
B4 $[xyz]:x,y,z \in A\,.\supset.\,x \times (y + z) = (x \times y) + (x \times z)$
B5 $[x]:x \in A\,.\supset.\,x + 0 = x$
B6 $[x]:x \in A\,.\supset.\,x \times 1 = x$
B7 $[x]:x \in A\,.\supset.\,x + -x = 1$
B8 $[x]:x \in A\,.\supset.\,x \times -x = 0$

is accepted as the Boolean part of the system of cylindric algebra under consideration.

It should be noted that in [1] it is assumed for any investigated algebraic structure that

(a) the carrier set of such system is non-empty,
(b) the so-called closure postulates with respect to the primitive operations are assumed tacitly,

and that

(c) the logical relation = is extensional in regard to every argument of any operation under consideration.

Furthermore, concerning the system of cylindric algebra which is defined above, we have to remark that

(d) the postulate *C1* is independent of the remaining axioms if $\alpha = 1$, but it can be derived from them if $\alpha > 1$, *cf.* [1], p. 179,

and that

(e) a diagonal-free cylindric algebra of dimension α is an algebraic structure $\langle A, +, \times, -, 0, 1, \mathbf{c}_\kappa\rangle_{\kappa<\alpha}$ where A, +, $\times$, 0, 1, and $\mathbf{c}_\kappa$ are as in the definition of cylindric algebra given above and in which the postulates *C0*-*C4* are satisfied, *cf.* [1], p. 164, definition 1.1.2.

In this paper an affirmative solution will be given to the following problem:

Problem 1.2 *Is there a finite set of identities* (B_0)-(B_μ) *with the following properties:* (i) *the identities* (B_0)-(B_μ) *contain only variables and symbols for fundamental operations and distinguished elements of Boolean algebras;* (ii) *the identities* (B_0)-(B_μ) *hold in all Boolean algebras but do not form an adequate postulate system for Boolean algebras;* (iii) *the identities* (B_0)-(B_μ) *and C2-C7 jointly form an adequate postulate system for cylindric algebras of dimension* $\alpha \geq 2$?

which in [1], p. 245, is announced as open. In fact, since in the field of arbitrary Boolean algebra the postulate *C2* of $\mathfrak{A}$ is obviously inferentially equivalent to the formula

$C2^*$ $[x\kappa] : x \in A \,.\, \kappa < \alpha \,.\supset .\, x \times \mathbf{c}_\kappa x = x,$

we can distinguish two versions of Problem 1.2. Namely, versions A and B of Problem 1.2 such that each of them is formulated exactly as this problem is presented above, but with an exception that in clause (iii) of Version B instead of *C2* formula *C2** is accepted as an axiom.

It will be shown below that there are two algebraic systems, $\mathfrak{M}$ and $\mathfrak{N}$, such that:

1) each of them is finitely and equationally axiomatizable,
2) each of them is a proper subsystem of Boolean algebra, but neither of them is a lattice or even a semi-lattice,
3) that the use of system $\mathfrak{M}$ gives an affirmative solution to Version A, and similarly the use of system $\mathfrak{N}$ solves Version B of Problem 1.2.

It should be remarked that I was unable to find a proper subsystem of Boolean algebra such that in its field both versions could be solved simultaneously, and that although the systems $\mathfrak{M}$ and $\mathfrak{N}$ are akin and the axiom-systems *C2-C7* and *C2**, *C3-C7* are very similar the deductions which we are using in order to solve each of these two cases are different in some respect.

1 System $\mathfrak{M}$ and a solution of Version A. In [2] M. H. A. Newman constructed and investigated an algebraic system which he called the fully complemented non-associative double algebra, but which we shall call here simply (non-associative) Newman algebras. In [3][3] it has been established that there exists such formalization of these structures that any system of Newman algebras can be finitely and equationally axiomatized. Namely, in accordance with the points (a), (b) and (c) which are given above, we can define such an algebraic structure, *cf.* [3], p. 256, as follows:

(C) *Any algebraic structure*

$$\mathfrak{M} = \langle A, +, \times, - \rangle$$

3. It should be noted that I did not know of Problem 1.2 when I worked on the papers [3], [4], [5] and [6].

where + and × are two binary operations, and - is a unary operation defined on the carrier set A, is a non-associative Newman algebra, if it satisfies the following postulates:

M1 $[abc]:a,b,c \epsilon A . \supset . a \times (b + c) = (a \times b) + (a \times c)$
M2 $[abc]:a,b,c \epsilon A . \supset . (a + b) \times c = (a \times c) + (b \times c)$
M3 $[ab]:a,b \epsilon A . \supset . a = a + (b \times - b)$
M4 $[ab]:a,b \epsilon A . \supset . a = a \times (b + - b)$
M5 $[ab]:a,b \epsilon A . \supset . a = (b + - b) \times a$

It is self-evident that if the symbols +, × and - are understood as the symbols of the lattice theoretical "join", "meet" and "complement" respectively, then the postulates *M1-M5* are valid formulas in the field of any Boolean algebra. On the other hand, in [2] Newman has proved, *cf.* section **3** below, that the formulas

P1 $[abc]:a,b,c \epsilon A . \supset . a \times (b \times c) = (a \times b) \times c$

and

R1 $[a]:a \epsilon A . \supset . a = a + a$

are independent from the axioms *M1-M5*. Moreover, *cf.* [5], p. 266, formula *R1* is not a consequence of the axioms *M1-M5* and *P1*. Therefore, system $\mathfrak{M}$ (and even a system of associative Newman algebras)[4] is a proper subsystem of a corresponding Boolean algebra.

1.1 In [3], pp. 258-262, it has been proved that the formulas which are given below and which we shall need for our end are the consequences of the axioms *M1-M5*. For this reason they are presented here without the proofs:

M6 $[ab]:a,b \epsilon A . \supset . a + - a = b + - b$
[*M4*; *M5*; *cf.* a proof of *F4* in [3], p. 258]

Hence, having *M6* we can introduce into the system the following definition:

D1 $[a]:a \epsilon A . \supset . a + - a = 1$ [*M6*; *cf. D1* in [3], p. 258]
M7 $[a]:a \epsilon A . \supset . a = a \times 1$ [*M4*; *D1*; *cf. F5* in [3], p. 258]
M8 $[a]:a \epsilon A . \supset . a = 1 \times a$ [*M5*; *D1*; *cf. F6* in [3], p. 258]
M9 $[a]:a \epsilon A . \supset . a = a \times a$ [*M4*; *M1*; *M3*; *cf. F7* in [3], p. 258]
M10 $[a]:a \epsilon A . \supset . a + 1 = 1 + a$
[*M4*; *M1*; *M2*; *M9*; *M8*; *D1*; *M3*; *cf. F8* in [3], pp. 258-9]
M11 $[ab]:a,b \epsilon A . \supset . a \times - a = (b \times - b) \times (a \times - a)$
[*M9*; *M3*; *M2*; *M8*; *M10*; *cf. F9* in [3], p. 259]
M12 $[ab]:a,b \epsilon A . \supset . a \times - a = (a \times - a) \times (b \times - b)$
[*M9*; *M3*; *M1*; *M7*; *M10*; *cf. F10* in [3], p. 259]
M13 $[ab]:a,b \epsilon A . \supset . a \times - a = b \times - b$ [*M11*; *M12*; *cf. F11* in [3], p. 259]

4. Concerning the associative Newman algebras see [2], p. 265, [4] and [5].

Hence, having *M13* we can introduce into the system the following definition:

D2 $[a]:a\epsilon A.\supset.a\times -a=0$ [*M13*, *cf.* *D2* in [3], p. 259]
M14 $[a]:a\epsilon A.\supset.a=a+0$ [*M3*; *D2*; *cf.* *F12* in [3], p. 259]
M15 $[a]:a\epsilon A.\supset.0+(0+-(-a))=a$
[*D2*; *M9*; *M2*; *M6*; *M3*; *M1*; *M4*; *cf.* *F13* in [3], p. 259]
M16 $[a]:a\epsilon A.\supset.a\times 0=0$
[*M15*; *M2*; *M9*; *D2*; *M1*; *M14*; *M3*; *cf.* *F14* in [3], p. 259]
M17 $[a]:a\epsilon A.\supset.0\times a=0$
[*M15*; *M1*; *M9*; *D2*; *M2*; *M14*; *M3*; *cf.* *F15* in [3], p. 259]
M18 $0+-(-1)=1$
[*M9*; *M1*; *M2*; *M16*; *M17*; *M9*; *M15*; *cf.* *F16* in [3], p. 259]
M19 $[a]:a\epsilon A.\supset.a=0+a$
[*M5*; *M15*; *M18*; *M1*; *M16*; *M7*; *cf.* *F17* in [3], p. 259]
M20 $[a]:a\epsilon A.\supset.a=-(-a)$ [*M15*; *M19*; *cf.* *F19* in [3], p. 260]

Moreover, the points (a) and (b) which are given above together with the definitions *D1* and *D2* imply at once that the following formulas

M21 $1\epsilon A$ [*Cf.* *F24* in [3], p. 260]
M22 $0\epsilon A$ [*Cf.* *F25* in [3], p. 260]

are valid in the field of this system. And, since it is established in [3], p. 260, section **2.3**, that every formula which is provable in the field of Newman's postulates of his system is also a consequence of the axioms *M1*-*M5*, we can accept without the proofs the formulas:

M23 $[ab]:a,b\epsilon A.a+b=0.\supset.a=b$ [*Cf.* **P16** in [2], p. 259]
M24 $[ab]:a,b\epsilon A.\supset.a+b=b+a$ [*Cf.* **P17** in [2], p. 260]
M25 $[abc]:a,b,c\epsilon A.\supset.a+(b+c)=(a+b)+c$ [*Cf.* **P18** in [2], p. 261]

1.2 It is well-known, *cf.* [2], although not presented in a formal way, that an adequate set of postulates for Newman algebra together with the low of idempotency with respect to the binary operation + forms a sufficient axiom-system for Boolean algebra. It will be proved here that it holds for the postulates *M1*, *M2*, *M3*, *M4*, *M5* and *R1*. Namely:

R2 $[a]:a\epsilon A.\supset.a+1=1$
PR $[a]:\mathrm{Hp}(1).\supset.$
$a+1=a+(a+-a)=(a+a)+-a=a+-a=1$ [1; *D1*; *M25*; *R1*; *D1*]
R3 $[ab]:a,b\epsilon A.\supset.-a\times -b=-(a+b)$
PR $[ab]:\mathrm{Hp}(1).\supset.$
2. $0=b\times -b=(b\times(a+-a))\times -b$ [1; *D2*; *M4*]
$=((b\times a)\times -b)+((b\times -a)\times -b).$ [*M1*; *M2*]
3. $((b\times a)\times -b)=((b\times -a)\times -b).$ [1; 2; *M23*]
4. $0=((b\times -a)\times -b)+((b\times -a)\times -b)$ [1; 2; 3]
$=(b\times -a)\times -b.$ [*R1*]
5. $0=-(a+b)\times(a+b)=(-(a+b)\times a)+(-(a+b)\times b).$
[1; *D2*; *M13*; *M20*; *M1*]
6. $(-(a+b)\times a)=(-(a+b)\times b).$ [1; 5; *M23*]

7. $0 = (-(a+b) \times a) + (-(a+b) \times a)$ [1; 5; 6]
$= -(a+b) \times a.$ [*R1*]
8. $0 = -(a+b) \times b.$ [1; 7; 6]
9. $-(a+b) \times -b = 0 + (-(a+b) \times -b)$ [1; *M19*]
$= (-(a+b) \times (a+b)) + (-(a+b) \times -b)$ [*D2*; *M13*; *M20*]
$= -(a+b) \times ((a+b) + -b)$ [*M1*]
$= -(a+b) \times (a + (b + -b))$ [*M25*]
$= -(a+b) \times (a+1) = -(a+b) \times 1$ [*D1*; *R2*]
$= -(a+b).$ [*M7*]
10. $-(a+b) \times -a = (-(a+b) \times -a) + (-(a+b) \times (a+b))$ [1; *M3*; *M13*; *M20*]
$= -(a+b) \times (-a + (a+b))$ [*M1*]
$= -(a+b) \times (b+1) = -(a+b) \times 1$ [*M24*; *M25*; *D1*; *R2*]
$= -(a+b).$ [*M7*]
11. $b \times -a = 0 + (b \times -a) = (a \times -a) + (b \times -a)$ [1; *M19*; *D2*]
$= (a+b) \times -a.$ [*M2*]
12. $-(a+b) + (b \times -a) = (-(a+b) \times -a) + ((a+b) \times -a)$ [1; 10; 11]
$= (-(a+b) + (a+b)) \times -a$ [*M2*]
$= 1 \times -a = -a.$ [*M24*; *D1*; *M8*]
$-a \times -b = (-(a+b) + (b \times -a)) \times -b$ [1; 12]
$= (-(a+b) \times -b) + ((b \times -a) \times -b)$ [*M2*]
$= -(a+b) + 0 = -(a+b)$[5] [9; 4; *M14*]

R4 $[ab]: a, b \epsilon A . \supset . a + b = -(-a \times -b)$ [*M20*; *R3*]
R5 $[ab]: a, b \epsilon A . \supset . a \times b = -(-a + -b)$ [*M20*; *R3*]
R6 $[ab]: a, b \epsilon A . \supset . -(a \times b) = -a + -b$ [*R5*; *M20*]
R7 $[abc]: a, b, c \epsilon A . \supset . a + (b \times c) = (a+b) \times (a+c)$
PR $[abc]: \mathrm{Hp}(1) . \supset .$
$a + (b \times c) = -(-(a + (b \times c))) = -(-a \times -(b \times c))$ [1; *M20*; *R3*]
$= -(-a \times (-b + -c))$ [*R6*]
$= -((-a \times -b) + (-a \times -b))$ [*M1*]
$= -(-a \times -b) \times -(-a \times -c)$ [*R3*]
$= (a+b) \times (a+c)$ [*R4*]
R8 $[ab]: a, b \epsilon A . \supset . a \times b = b \times a$[6] [*R5*; *M24*]

Since the formulas *M24*, *R8*, *R7*, *M1*, *M12*, *M7*, *D1* and *D2* correspond respectively to the postulates *B1*, *B2*, *B3*, *B4*, *B5*, *B6*, *B7* and *B8* which are given in (**B**), the proof that the axioms *M1*-*M5* and *R1* constitute an adequate postulate system for Boolean algebras is complete.

1.3 It is self-evident that in the field of the fixed carrier set *A* the formulas *M1*-*M5* are inferentially equivalent to the formulas *M1*, *M2*, *M14*, *M7*, *M8*, *D1* and *D2*. Hence, we can reformulate definition (**C**) of the non-associative Newman algebras, as follows:

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5. Concerning the proof of *R3*, *cf.* [2], p. 269, formula **P30**, and [7], Theorem 12_1.

6. We can deduce *R8* from *M1* -*M5* without the use of *R1*, *cf.* [2], pp. 269-270.

(**D**) *Any algebraic structure*

$$\mathfrak{M} = \langle A, +, \times, -, 0, 1\rangle$$

where + and × are two binary operations, and - is a unary operation defined on the carrier set A, and 0 *and* 1 *are the constant elements belonging to A, is a non-associative Newman algebra, if it satisfies the postulates M1, M2, M14, M7, M8, D1 and D2.*

Since in the postulates *C1-C7*, *cf.* definition (**A**), the constants 0 and 1 occur as undefined notions, the formulation (**D**) in which *D1* and *D2* are not the definitions but the postulates is more convenient for our final purpose than definition (**C**) of the Newman algebras.

1.4 Now, let us assume the axioms *M1*, *M2*, *M14*, *M7*, *M8*, *D1*, *D2* and, as an additional postulate, a formula:

C2 $[x\kappa]:x \epsilon A \,.\, \kappa < \alpha \,.\supset.\, x + \mathbf{c}_\kappa x = \mathbf{c}_\kappa x$, for any $\alpha \geq 1$.

Then, *cf.* sections **1.3** and **1.1**, we have at our disposal the formulas *M1-M25*, and, moreover:

R1 $[a]:a \epsilon A \,.\supset.\, a = a + a$

PR $[a]:\mathrm{Hp}(1)\,.\supset.$

$$a = a \times 1 = a \times (\mathbf{c}_0 1 + - \mathbf{c}_0 1) = a \times ((1 + \mathbf{c}_0 1) + - \mathbf{c}_0 1) \quad [1; M7; D1; C2]$$
$$= a \times (1 + (\mathbf{c}_0 1 + - \mathbf{c}_0 1)) = a \times (1 + 1) = (a \times 1) + (a \times 1) \quad [M25; D1; M1]$$
$$= a + a \quad [M7]$$

Since, *cf.* sections **1.3** and **1.2**, the axioms *M1*, *M2*, *M14*, *M7*, *M8*, *D1*, *D2* and *R1* form an adequate postulate system for the Boolean algebras, the deduction presented above allows us to establish the following theorem:

Theorem I. *Let an algebraic structure*

$$\mathfrak{A} = \langle A, +, \times, -, 0, 1, \mathbf{c}_\kappa, \mathbf{d}_{\kappa\lambda}\rangle$$

and the postulates C1-C7 be formulated exactly as they are given in (**A**). *And, let us modify C0, as follows:*

C0 the structure* $\langle A, +, \times, -, 0, 1\rangle$ *is a non-associative Newman algebra* (e.g. defined as in (**D**)).

Then:

(i) *For any ordinal number* $\alpha \geq 1$, *and every* $\kappa, \lambda, \mu < \alpha$, *system* $\mathfrak{A}$ *is a cylindric algebra of dimension* α, *if it satisfies the postulates C0*, C1-C7 for any* $\kappa, \lambda, \mu < \alpha$.

(ii) *For any ordinal number* $\alpha \geq 1$, *and every* $\kappa < \alpha$, *the substructure* $\langle A, +, \times, -, 0, 1, \mathbf{c}_\kappa\rangle$ *of* $\mathfrak{A}$ *is a diagonal-free cylindric algebra of dimension* α, *if it satisfies the postulates C0*, C1-C4 for any* $\kappa < \alpha$.

(iii) *For any ordinal number* $\alpha \geq 2$, *and every* $\kappa, \lambda, \mu < \alpha$, *system* $\mathfrak{A}$ *is a cylindric algebra of dimension* α, *if it satisfies the postulates C0*, C2-C7 for any* $\kappa, \lambda, \mu < \alpha$.

Proof: The Theorem follows at once from the considerations given above in this section and the fact that, if $\alpha \geq 2$, then *C1* is a consequence of *C0*, i.e. *C0** and *C2*, and the postulates *C2*-*C7*, *cf.* [1], p. 179. A restriction that $\alpha \geq 1$ given in (i) and (ii) is necessary, since, if $\alpha = 0$, then the systems considered there would be automatically reduced to *C0**, i.e. to a Newman algebra, while the classical definitions of these systems, *cf.* (**A**), requires that in such a case they be reduced to a Boolean algebra.

Thus, Version A of the problem 1.2 is solved affirmatively.

2 System $\mathfrak{N}$ and a solution of Version B. In order to obtain a solution of Version B of the Problem 1.2 we have to use an algebraic system other than $\mathfrak{M}$, because I was unable to resolve this case in the field of the latter algebras. An algebraic system which will be called here system $\mathfrak{N}$ and which we shall use for our present purpose is a dual non-associative Newman algebra. As far as I know such algebras were never previously discussed in the literature. Obviously, we can define system $\mathfrak{N}$, as follows:

(**E**) *Any algebraic structure*

$$\mathfrak{N} = \langle A, +, \times, - \rangle$$

where + *and* $\times$ *are two binary operations, and* - *is a unary operation defined on the carrier set A is a dual non-associative Newman algebra, if it satisfies the following postulates*:

N1 $[abc]: a,b,c \in A . \supset . a + (b \times c) = (a+b) \times (a+c)$
N2 $[abc]: a,b,c \in A . \supset . (a \times b) + c = (a+c) \times (b+c)$
N3 $[ab]: a,b \in A . \supset . a = a \times (b + -b)$
N4 $[ab]: a,b \in A . \supset . a = a + (b \times -b)$
N5 $[ab]: a,b \in A . \supset . a = (b \times -b) + a$

It is self-evident that if the symbols +, $\times$ and - are understood as the symbols of lattice theoretical "join", "meet" and "complement" respectively, then the postulates *N1*-*N5* are valid formulas in the field of any Boolean algebra. On the other hand, it will be proved in section **3** below, that the formulas

S1 $[abc]: a,b,c \in A . \supset . a + (b + c) = (a+b) + c$

and

T1 $[a]: a \in A . \supset . a = a \times a$

are independent from the axioms *N1*-*N5*. Moreover, it will be shown in section **3** that *T1* is not a consequence of the axioms *N1*-*N5* and formula *S1*. Hence, system $\mathfrak{N}$ (and even a system of a dual associative Newman algebra) is a proper subsystem of a corresponding Boolean algebra.

2.1 Now, let us assume the axioms *N1*-*N5*. Then:

N6 $[ab]: a,b \in A . \supset . a \times -a = b \times -b$
[*N4*; *N5*; *cf.* an analogous proof of *M6* in section **1.1**]

Hence, having *N6* we can introduce into the system the following definition:

Df1 $[a]:a \epsilon A . \supset . a \times - a = 0$ [*N6*; *cf. D1* in **1.1**]
N7 $[a]:a \epsilon A . \supset . a = a + 0$ [*N4*; *Df1*; *cf. M7*]
N8 $[a]:a \epsilon A . \supset . a = 0 + a$ [*N5*; *Df1*; *cf. M8*]
N9 $[a]:a \epsilon A . \supset . a = a + a$ [*N4*; *N1*; *N3*; *cf. M9*]
N10 $[a]:a \epsilon A . \supset . a \times 0 = 0 \times a$ [*N4*; *N1*; *N2*; *N9*; *N8*; *Df1*; *N3*; *cf. M10*]
N11 $[ab]:a, b \epsilon A . \supset . a + - a = (b + - b) + (a + - a)$
[*N9*; *N3*; *N2*; *N8*; *N10*; *cf. M11*]
N12 $[ab]:a, b \epsilon A . \supset . a + - a = (a + - a) + (b + - b)$
[*N9*; *N3*; *N1*; *N7*; *N10*; *cf. M12*]
N13 $[ab]:a, b \epsilon A . \supset . a + - a = b + - b$ [*N11*; *N12*; *cf. M13*]

Hence, having *N13* we can introduce into the system the following definition:

Df2 $[a]:a \epsilon A . \supset . a + - a = 1$ [*N13*; *cf. D2*]
N14 $[a]:a \epsilon A . \supset . a = a \times 1$ [*N3*; *Df2*; *cf. M14*]
N15 $[a]:a \epsilon A . \supset . 1 \times (1 \times - (- a)) = a$
[*Df2*; *M9*; *N2*; *N6*; *N3*; *N1*; *N4*; *cf. M15*]
N16 $[a]:a \epsilon A . \supset . a + 1 = 1$ [*N15*; *N2*; *N9*; *Df2*; *N1*; *N14*; *N3*; *cf. M16*]
N17 $[a]:a \epsilon A . \supset . 1 + a = 1$ [*N15*; *N1*; *N9*; *Df2*; *N2*; *N14*; *N3*; *cf. M17*]
N18 $1 \times - (- 0) = 0$ [*N9*; *N1*; *N2*; *N16*; *N17*; *N9*; *N15*; *cf. M18*]
N19 $[a]:a \epsilon A . \supset . a = 1 \times a$ [*N5*; *N15*; *N18*; *N1*; *N16*; *N7*; *cf. M19*]
N20 $[a]:a \epsilon A . \supset . a = - (- a)$ [*N15*; *N19*; *cf. M20*]
N21 $0 \epsilon A$ [*Df1*; *cf. M21*]
N22 $1 \epsilon A$ [*Df2*; *cf. M22*]

The proofs of the formulas which are established above are given in the abbreviated forms, since these theses are the duals of the formulas *M6*-*M22* which are already proven in section **1.1**. On the other hand, we have to present the complete proofs of the next formulas *N23*, *N24* and *N25*, because in this paper their duals, i.e. the formulas *M23*, *M24* and *M25*, are accepted as proven, due to a certain metatheorem which cannot be applied directly to the system $\mathfrak{N}$, *cf.* section **1.1**.

N23 $[ab]:a, b \epsilon A . a \times b = 1 . \supset . a = b$
PR $[ab]:\mathrm{Hp}(2) . \supset .$
$a = a + (b \times - b) = (a + b) \times ((a + - b) \times (b + - b))$ [1; *N4*; *N1*; *N3*]
$= (a + b) \times ((a \times b) + - b) = (a + b) \times (1 + - b)$ [*N2*; 2]
$= (a + b) \times 1 = 1 \times (a + b) = (- a + 1) \times (a + b)$ [*N17*; *N14*; *N19*; *N16*]
$= (- a + (-(- a) \times b)) \times (a + b)$ [2; *N20*]
$= ((- a + - (- a)) \times (- a + b)) \times (a + b)$ [*N1*]
$= (1 \times (- a + b)) \times (a + b) = (- a + b) \times (-(- a) + b)$ [*Df2*; *N19*; *N20*]
$= (- a \times - (- a)) + b = b$ [*N2*; *N5*]
N24 $[ab]:a, b \epsilon A . \supset . a \times b = b \times a$
PR $[ab]:\mathrm{Hp}(1) . \supset .$
$a \times b = (a \times b) + (a \times - a)$ [1; *N4*]

$= ((a + a) \times (b + a)) \times ((a + - a) \times (b + - a))$ [*N1*; *N2*]
$= ((0 + a) \times (b + a)) \times ((b + - a) \times (a + - a))$
[*N9*; *N8*; *Df2*; *N19*; *N3*]
$= ((b \times 0) + a) \times ((b \times a) + - a)$ [*N2*; *N10*; *N2*]
$= ((b + a) \times (a + a)) \times ((b \times a) + - a)$ [*N2*; *N8*; *N9*]
$= ((b \times a) + a) \times ((b \times a) + - a)$ [*N2*]
$= (b \times a) + (a \times - a) = b \times a.$ [*N1*; *N4*]

N25 $[abc] : a, b, c \epsilon A . \supset . a \times (b \times c) = (a \times b) \times c$

PR $[abc] : \mathrm{Hp}(1) . \supset .$

2. $0 \times (0 \times c) = (0 \times (0 \times c)) + (c \times - c)$ [1; *N4*]
$= ((0 + c) \times ((0 + c) \times (c + c))) \times ((0 + - c)$
$\times ((0 + - c) \times (c + - c)))$ [*N1*; *N2*]
$= ((0 + c) \times ((0 \times 0) + c)) \times (((0 \times 0) + - c) \times (c + - c))$
[*N9*; *N8*; *N3*; *N2*; *N3*]
$= (((0 \times 0) \times c) + c) \times (((0 \times 0) \times c) + - c)$ [*N24*; *N8*; *N9*; *N2*]
$= ((0 \times 0) \times c) + (c \times - c) = (0 \times 0) \times c.$ [*N1*; *N4*]

3. $0 \times (b \times c) = (0 \times (b \times c)) + (b \times - b)$ [1; *N4*]
$= ((0 \times (0 \times c)) + b) \times (((0 \times b) \times c) + - b)$
[*N1*; *N2*; *N9*; *N8*; *Df2*; *N19*; *N3*; *N2*]
$= (((0 \times 0) \times c) + b) \times (((0 \times b) \times c) + - b)$ [2]
$= (((0 \times b) \times c) + b) \times (((0 \times b) \times c) + - b)$ [*N2*; *N8*; *N9*; *N2*]
$= ((0 \times b) \times c) + (b \times - b) = (0 \times b) \times c.$ [*N1*; *N4*]

$a \times (b \times c) = (a \times (b \times c)) + (a \times - a)$ [1; *N4*]
$= ((0 + a) \times ((b + a) \times (c + a))) \times ((b + - a) \times (c + - a))$
[*N1*; *N2*; *N9*; *N8*; *Df2*; *N19*]
$= ((0 \times (b \times c)) + a) \times (((a + - a) \times (b + - a)) \times (c + - a))$
[*N2*; *N19*; *Df2*]
$= (((0 \times b) \times c) + a) \times (((a \times b) \times c) + - a)$ [3; *N2*]
$= (((a \times b) \times c) + a) \times (((a \times b) \times c) + - a)$ [*N2*; *N8*; *N9*; *N2*]
$= ((a \times b) \times c) + (a \times - a) = (a \times b) \times c$ [*N1*; *N4*]

It should be remarked that the proofs which are given above for *N23*-*N25* are patterned after the deductions which Newman used in order to obtain the theses *M23*-*M25*, *cf.* [2], pp. 259-261.

2.2 It will be shown here that system $\mathfrak{N}$ possesses a property analogous to that which was established for system $\mathfrak{M}$ in section **1.2** above. Namely, that its postulates together with the law of idempotency with respect to the binary operation $\times$ form a sufficient axiom system for Boolean algebras. For this end let us assume the axioms *N1*-*N5* and the formula *T1*. Then:

T2 $[a] : a \epsilon A . \supset . a \times 0 = 0$
[*Df1*; *N25*; *T1*; *cf.* an analogous proof of *R2* in section **1.2**]

T3 $[ab] : a, b \epsilon A . \supset . - a + - b = - (a \times b)$
[*Df2*; *N4*; *N1*; *N2*; *N23*; *T1*; *N13*; *N20*; *N19*; *N25*; *Df1*; *T2*; *N7*; *N3*; *N24*; *N8*; *N14*; *cf.* *R3*]

T4 $[ab] : a, b \epsilon A . \supset . a \times b = - (- a + - b)$ [*N20*; *T3*; *cf.* *R4*]

T5 $[ab] : a, b \epsilon A . \supset . a + b = - (- a \times - b)$ [*N20*; *T3*, *cf.* *R5*]

$T6$ $[ab]:a,b \epsilon A . \supset . - (a + b) = - a \times - b$ [$N20$; $T5$; *cf.* $R6$]
$T7$ $[abc]:a,b,c \epsilon A . \supset . a \times (b + c) = (a \times b) + (a \times c)$
[$N20$; $T3$; $T6$; $N1$; $T4$; *cf.* $R7$]
$T8$ $[ab]:a,b \epsilon A . \supset . a + b = b + a$[7] [$T5$; $N24$; $R8$]

Since the formulas $T8, N24, N1, T7, N7, N16, Df2$ and $Df1$ correspond respectively to the postulates $B1, B2, B3, B4, B5, B6, B7$ and $B8$ which are given in (**B**), the proof that the axioms $N1$-$N5$ and $T1$ constitute an adequate postulate system for Boolean algebras is complete.

2.3 Obviously, in the field of the fixed carrier set A the formulas $N1$-$N5$ are inferentially equivalent to the formulas $N1, N2, N14, N7, N8$, $Df1$ and $Df2$. Hence, we can reformulate definition (**E**) of the dual non-associative Newman algebras, as follows:

(**F**) *Any algebraic structure*

$$\mathfrak{N} = \langle A, +, \times, -, 0, 1 \rangle$$

where + *and* $\times$ *are two binary operations, and* - *is a unary operation defined on the carrier set* A, *and* 0 *and* 1 *are the constant elements belonging to* A, *is a dual non-associative Newman algebra, if it satisfies the postulates* $N1$, $N2, N14, N7, N8$, $Df1$ *and* $Df2$.

We introduced definition (**F**) exactly for the same reason which is given in regard to system $\mathfrak{M}$ in section **1.3** above.

2.4 Now, it will be proved that using system $\mathfrak{N}$ we are able to obtain an affirmative solution to Version B of the Problem 1.2. But, although system $\mathfrak{N}$ is a dual of the system $\mathfrak{M}$, formula $C2^*$ is obviously not a dual of the formula $C2$. For this reason the deductions which will be used in this section differ considerably from the proof presented in section **1.4.** Namely:

2.4.1 Let us assume the axioms of the system $\mathfrak{N}$, i.e. $N1, N2, N14, N7, N8$, $Df1$ and $Df2$, and, as the additional postulates, the cylindric formulas $C2^*$, $C3$, $C5$, $C6$ and $C7$ with a proviso that a dimension of each of these formulas is $\alpha \geq 2$. Then, *cf.* sections **2.1** and **2.3**, we have at our disposal the formulas $N1$-$N25$ and, moreover:

$V1$ $[\kappa]: \kappa < \alpha . \supset . 1 = \mathbf{c}_\kappa 1$
PR $[\kappa]:\text{Hp}(1) . \supset .$
$1 = 1 \times \mathbf{c}_\kappa 1 = \mathbf{c}_\kappa 1$ [1; $C2^*$; $N19$]
$V2$ $[x\kappa]:x \epsilon A . \kappa < \alpha . \supset . \mathbf{c}_\kappa x = \mathbf{c}_\kappa \mathbf{c}_\kappa x$
PR $[x\kappa]:\text{Hp}(2) . \supset .$
$\mathbf{c}_\kappa x = 1 \times \mathbf{c}_\kappa x = \mathbf{c}_\kappa 1 \times \mathbf{c}_\kappa x = \mathbf{c}_\kappa (1 \times \mathbf{c}_\kappa x) = \mathbf{c}_\kappa \mathbf{c}_\kappa x$
[1; 2; $N19$; $V1$; $C3$; $N19$]

Since it is assumed that $\alpha \geq 2$, there are at least two ordinal numbers, viz. 0 and 1, such that $0 < 1 < \alpha$. Hence:

7. We can obtain $T8$ from $N1$-$N5$ without the use of $T1$.

V3 $\mathbf{d}_{1,1} = 1$ [*C5*]
V4 $\mathbf{d}_{1,1} = \mathbf{c}_0(\mathbf{d}_{1,0} \times \mathbf{d}_{0,1})$ [*C6*]
V5 $[x]:x \epsilon A . \supset . \mathbf{c}_0(\mathbf{d}_{0,1} \times x) \times \mathbf{c}_0(\mathbf{d}_{0,1} \times - x) = 0$ [*C7*]

Therefore:

V6 $0 = \mathbf{c}_0(\mathbf{d}_{0,1} \times - \mathbf{d}_{1,0})$
PR $0 = \mathbf{c}_0(\mathbf{d}_{0,1} \times \mathbf{d}_{1,0}) \times \mathbf{c}_0(\mathbf{d}_{0,1} \times - \mathbf{d}_{1,0})$ [*V5*]
$= \mathbf{c}_0(\mathbf{d}_{1,0} \times \mathbf{d}_{0,1}) \times \mathbf{c}_0(\mathbf{d}_{0,1} \times - \mathbf{d}_{1,0})$ [*N24*]
$= \mathbf{d}_{1,1} \times \mathbf{c}_0(\mathbf{d}_{0,1} \times - \mathbf{d}_{1,0}) = 1 \times \mathbf{c}_0(\mathbf{d}_{0,1} \times - \mathbf{d}_{1,0})$ [*V4*; *V3*]
$= \mathbf{c}_0(\mathbf{d}_{0,1} \times - \mathbf{d}_{1,0})$ [*N19*]
V7 $\mathbf{c}_0 0 = 0$
PR $\mathbf{c}_0 0 = \mathbf{c}_0\mathbf{c}_0(\mathbf{d}_{0,1} \times - \mathbf{d}_{1,0}) = \mathbf{c}_0(\mathbf{d}_{0,1} \times - \mathbf{d}_{1,0}) = 0$ [*V6*; *V2*; *V6*]

2.4.2 Now, assume the axioms mentioned in **2.4.1** of $\mathfrak{N}$ and, as the additional postulates, the cylindric formulas *C1* and *C2** each of them of dimension $\alpha \geq 1$. Then, we have the formulas *N1*-*N25*, *C1* yields *V7* and *C2** implies

V8 $[x]:x \epsilon A . \supset . x \times \mathbf{c}_0 x = x$ [*C2**]

Hence:

T1 $[a]:a \epsilon A . \supset . a = a \times a$
PR $[a]:\mathrm{Hp}(1) . \supset .$
$a = a + 0 = a + (0 \times \mathbf{c}_0 0) = a + (0 \times 0)$ [1; *N7*; *V8*; *N21*; *V7*]
$= (a + 0) \times (a + 0) = a \times a$ [*N1*; *N7*]

2.4.3 It follows from **2.4.1** and **2.4.2** that if $\alpha \geq 2$, then postulates of $\mathfrak{N}$ together with the cylindric postulates *C2**, *C3*, *C5*, *C6* and *C7* form, *cf.* section **2.2**, an adequate postulate system for Boolean algebra. Therefore, *cf.* [1], p. 179, in the field of system $\mathfrak{N}$ the postulates *C2**, *C3*-*C7* of dimension $\alpha > 1$ yield *C1* for any $\kappa < \alpha$. On the other hand, if $\alpha = 1$, then, *cf.* **2.4.2** and [1], p. 179, we have to add *C1*, as a new axiom, to the list of postulates mentioned above in order to obtain the requested result. Therefore, we can establish the following theorem:

Theorem II. *Let an algebraic structure*

$$\mathfrak{A} = \langle A, +, \times, -, 0, 1, \mathbf{c}_\kappa, \mathbf{d}_{\kappa\lambda}\rangle$$

and the postulates C1, C2, C3-C7 be formulated exactly as they are given in* (**A**) *and* Version B *of the* Problem 1.2. *And, let us modify C0, as follows:*

*C0*** *the structure* $\langle A, +, \times, -, 0, 1\rangle$ *is a dual non-associative Newman algebra* (e.g. defined as in (**F**)).

Then:

(i) *For any ordinal number* $\alpha \geq 1$, *and every* $\kappa, \lambda, \mu < \alpha$, *system* $\mathfrak{A}$ *is a cylindric algebra of dimension* α, *if it satisfies the postulates C0**, C1, C2*, C3-C7 for any* $\kappa, \lambda, \mu < \alpha$.
(ii) *For any ordinal number* $\alpha \geq 1$, *and every* $\kappa < \alpha$, *the substructure*

*$\langle A, +, \times, -, 0, 1, \mathbf{c}_\kappa \rangle$ of $\mathfrak{A}$ is a diagonal-free cylindric algebra of dimension α, if it satisfied the postulates C0**, C1, C2*, C3 and C4 for any $\kappa < \alpha$.*
(iii) *For any ordinal number $\alpha \geq 2$, and every $\kappa, \lambda, \mu < \alpha$, system $\mathfrak{A}$ is a cylindric algebra of dimension α, if it satisfies the postulates C0**, C2*, C3-C7 for any $\kappa, \lambda, \mu < \alpha$.*

The proof of this Theorem is obvious, *cf.* **2.4.1**, **2.4.2** and the proof of the Theorem I in section **1.4** above.

Thus, Version B of the Problem 1.2 is solved affirmatively.

3 The mutual independence of the axioms *M1-M5* of the system $\mathfrak{M}$ is proved in [3], pp. 263-264. In order to establish that the postulates *N1-N5* of $\mathfrak{N}$ are also mutually independent we use the following algebraic tables:

$\mathfrak{M}13$

+	0	α	β	γ	1
0	0	0	0	0	0
α	0	α	0	0	α
β	0	0	β	γ	β
γ	0	0	β	γ	γ
1	0	α	β	γ	1

×	0	α	β	γ	1
0	0	α	β	γ	1
α	α	0	1	1	β
β	β	1	0	0	α
γ	γ	1	0	0	α
1	1	β	α	α	0

x	-x
0	1
α	β
β	α
γ	α
1	0

$\mathfrak{M}14$

+	0	α	β	γ	1
0	0	0	0	0	0
α	0	α	0	0	α
β	0	0	β	β	β
γ	0	0	γ	γ	γ
1	0	α	β	γ	1

×	0	α	β	γ	1
0	0	α	β	γ	1
α	α	0	1	1	β
β	β	1	0	0	α
γ	γ	1	0	0	α
1	1	β	α	α	0

x	-x
0	1
α	β
β	α
γ	α
1	0

$\mathfrak{M}15$

+	α	β	γ
α	β	γ	α
β	γ	α	β
γ	α	β	γ

×	α	β	γ
α	β	α	γ
β	α	γ	β
γ	γ	β	α

x	-x
α	γ
β	β
γ	α

$\mathfrak{M}16$

+	α	1	0
α	α	1	0
1	α	1	0
0	0	0	0

×	α	1	0
α	α	1	α
1	1	1	1
0	α	1	0

x	-x
α	0
1	0
0	1

$\mathfrak{M}17$

+	α	1	0
α	α	α	0
1	1	1	0
0	0	0	0

×	α	1	0
α	α	1	α
1	1	1	1
0	α	1	0

x	-x
α	0
1	0
0	1

Since:

(a) $\mathfrak{M}13$ verifies *N2-N5*, but falsifies *N1* for a/γ, b/α, c/β: (i) $\gamma + (\alpha \times \beta) = \gamma + 1 = \gamma$, (ii) $(\gamma + \alpha) \times (\gamma + \beta) = 0 \times \beta = \beta$;

(b) $\mathfrak{M}14$ verifies *N1*, *N3-N5*, but falsifies *N2* for a/α, b/β, c/γ: (i) $(\alpha \times \beta) + \gamma = 1 + \gamma = \gamma$, (ii) $(\alpha + \gamma) \times (\beta + \gamma) = 0 \times \beta = \beta$;
(c) $\mathfrak{M}15$ verifies *N1*, *N2*, *N4* and *N5*, but falsifies *N3* for a/α and b/α: (i) $\alpha = \alpha$, (ii) $\alpha \times (\alpha + - \alpha) = \alpha \times (\alpha + \gamma) = \alpha \times \alpha = \beta$;
(d) $\mathfrak{M}16$ verifies *N1*, *N2*, *N3* and *N5*, but falsifies *N4* for a/α and $b/1$: (i) $\alpha = \alpha$, (ii) $\alpha + (1 \times - 1) = \alpha + (1 \times 0) = \alpha + 1 = 1$;

and

(e) $\mathfrak{M}17$ verifies *N1-N4*, but falsifies *N5* for a/α and $b/1$: (i) $\alpha = \alpha$, (ii) $(1 \times - 1) + \alpha = (1 \times 0) + \alpha = 1 + \alpha = 1$,

the proof that the axioms *N1-N5* are mutually independent is complete.

Because the sets $\{M1, M2, M14, M7, M8, D1, D2\}$ and $\{N1, N2, N14, N7, N8, Df1, Df2\}$ of the postulates given in (**D**) and (**F**) respectively are rather the artificial axiomatizations of the systems $\mathfrak{M}$ and $\mathfrak{N}$ and they were constructed for a specific purpose connected with the formulation of Problem 1.2, the mutual independence of the formulas belonging to each of these sets will not be discussed in this paper.

3.1 Now, consider the following two algebraic tables:

$\mathfrak{M}18$

+	0	η
0	0	η
η	η	0

×	0	η
0	0	0
η	0	η

x	-x
0	η
η	0

and

$\mathfrak{M}19$

+	0	η
0	0	η
η	η	η

×	0	η
0	η	0
η	0	η

x	-x
0	η
η	0

Concerning $\mathfrak{M}18$, *cf.* [4], p. 266. Since $\mathfrak{M}18$ verifies the postulates *M1-M5* and *P1*, but falsifies *R1* for a/η: (i) $\eta = \eta$, (ii) $\eta + \eta = 0$, it has been proved that the associative Newman algebras are not Boolean algebras, but, obviously, any system of such algebras can be considered as a proper subsystem of a corresponding Boolean algebra. Similarly, $\mathfrak{M}19$ verifies *N1-N5* and *S1*, but falsifies *T1* for $a/0$: (i) $0 = 0$, (ii) $0 \times 0 = \eta$. Therefore, the dual associative Newman algebras are not Boolean algebra, but any system of such algebra can be understood as a proper subsystem of a corresponding Boolean algebra.

3.2 In [2], p. 271, Example 10. Newman has proved that the formulas *P1* and *R1* are not provable in the field of his fully complemented non-associative double algebra. Table $\mathfrak{M}20$ which is given on p. 544 below and which is adjusted to the system $\mathfrak{M}$ is an exact matrix formalization of Example 10 which in [2] is formulated by Newman in a purely mathematical way. The analogous table $\mathfrak{M}21$ for the system $\mathfrak{N}$ is given on p. 545.

Since $\mathfrak{M}20$ verifies *M1-M5*, but falsifies *R1* for a/α: (i) $\alpha = \alpha$, (ii) $\alpha + \alpha = \iota$, and falsifies *P1* for a/δ, b/β and c/γ: (i) $\delta \times (\beta \times \gamma) = \delta \times \delta = \delta$,

(ii) $(\delta \times \beta) \times \gamma = \gamma \times \gamma = \gamma$, it is proved that *P1* and *R1* are not provable in the field of the system $\mathfrak{M}$. On the other hand, $\mathfrak{M}21$ verifies *N1-N5*, but falsifies *T1* for a/β: (i) $\beta = \beta$, (ii) $\beta \times \beta = \alpha$, and falsifies *S1* for a/μ, b/κ, and c/λ: (i) $\mu + (\kappa + \lambda) = \mu + \mu = \mu$, (ii) $(\mu + \kappa) + \lambda = \lambda + \lambda = \lambda$. Hence, the formulas *S1* and *T1* cannot be obtained in the field of the system $\mathfrak{N}$.

Thus, the non-associative Newman algebras and the dual non-associative Newman algebras are weaker algebraic structures than the associative and dual associative Newman algebras respectively.

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$\mathfrak{M}20$

+	α	β	γ	δ	ε	ξ	η	θ	ι	κ	λ	μ	ν	π	ρ	σ
α	ι	κ	λ	μ	ν	π	ρ	σ	α	β	γ	δ	ε	ξ	η	θ
β	κ	ι	ε	ξ	γ	δ	θ	η	β	α	ν	π	λ	μ	σ	ρ
γ	λ	ε	ι	η	β	θ	δ	ξ	γ	ν	α	ρ	κ	σ	μ	π
δ	μ	ξ	η	ι	θ	β	γ	ε	δ	π	ρ	α	σ	κ	λ	ν
ε	ν	γ	β	θ	ι	η	ξ	δ	ε	λ	κ	σ	α	ρ	π	μ
ξ	π	δ	θ	β	η	ι	ε	γ	ξ	μ	σ	κ	ρ	α	ν	λ
η	ρ	θ	δ	γ	ξ	ε	ι	β	η	σ	μ	λ	π	ν	α	κ
θ	σ	η	ξ	ε	δ	γ	β	ι	θ	ρ	π	ν	μ	λ	κ	α
ι	α	β	γ	δ	ε	ξ	η	θ	ι	κ	λ	μ	ν	π	ρ	δ
κ	β	α	ν	π	λ	μ	σ	ρ	κ	ι	ε	ξ	γ	δ	θ	η
λ	γ	ν	α	ρ	κ	σ	μ	π	λ	ε	ι	η	β	θ	δ	ξ
μ	δ	π	ρ	α	σ	κ	λ	ν	μ	ξ	η	ι	θ	β	γ	ε
ν	ε	λ	κ	σ	α	ρ	π	μ	ν	γ	β	θ	ι	η	ξ	δ
π	ξ	μ	σ	κ	ρ	α	ν	λ	π	δ	θ	β	η	ι	ε	γ
ρ	η	σ	μ	λ	π	ν	α	κ	ρ	θ	δ	γ	ξ	ξ	ι	β
σ	θ	ρ	π	ν	μ	λ	κ	α	σ	η	ξ	ε	δ	γ	β	ι
×	α	β	γ	σ	ε	ξ	η	θ	ι	κ	λ	μ	ν	π	ρ	σ
α	α	β	γ	δ	ε	ξ	η	θ	ι	κ	λ	μ	ν	π	ρ	σ
β	β	β	δ	γ	ξ	ε	η	θ	ι	ι	ξ	ε	δ	γ	θ	η
γ	γ	δ	γ	β	η	ξ	ε	θ	ι	η	ι	ε	δ	θ	β	ξ
δ	δ	γ	β	δ	ε	η	ξ	θ	ι	η	ξ	ι	θ	γ	β	ε
ε	ε	ξ	η	ε	ε	η	ξ	ι	ι	η	ξ	ι	ι	ξ	η	ε
ξ	ξ	ε	ξ	η	η	ξ	ε	ι	ι	η	ι	ε	ε	ι	η	ξ
η	η	η	ε	ξ	ξ	ε	η	ι	ι	ι	ξ	ε	ε	ξ	ι	η
θ	θ	θ	θ	θ	ι	ι	ι	θ	ι	ι	ι	ι	θ	θ	θ	ι
ι	ι	ι	ι	ι	ι	ι	ι	ι	ι	ι	ι	ι	ι	ι	ι	ι
κ	κ	ι	η	η	η	η	ι	ι	ι	κ	σ	σ	σ	σ	κ	κ
λ	λ	ξ	ι	ξ	ξ	ι	ξ	ι	ι	σ	λ	σ	σ	λ	σ	λ
μ	μ	ε	ε	ι	ι	ε	ε	ι	ι	σ	σ	μ	μ	σ	σ	μ
ν	ν	δ	δ	θ	ι	ε	ε	θ	ι	σ	σ	μ	ν	α	α	μ
π	π	γ	θ	γ	ξ	ι	ξ	θ	ι	σ	λ	σ	α	π	α	λ
ρ	ρ	θ	β	β	η	η	ι	θ	ι	κ	σ	σ	α	α	ρ	κ
σ	σ	η	ξ	ε	ε	ξ	η	ι	ι	κ	λ	μ	μ	λ	κ	σ
x	α	β	γ	δ	ε	ξ	η	θ	ι	κ	λ	μ	ν	π	ρ	σ
$-x$	ι	κ	λ	μ	ν	π	ρ	σ	α	β	γ	δ	ε	ξ	η	θ

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+	α	β	γ	δ	ε	ξ	η	θ	ι	κ	λ	μ	ν	π	ρ	σ
α	α	α	α	α	α	α	α	α	α	α	α	α	α	α	α	α
β	α	β	θ	θ	θ	θ	β	β	β	α	ρ	ρ	ρ	ρ	α	α
γ	α	θ	γ	θ	θ	γ	θ	γ	γ	π	α	π	π	α	π	α
δ	α	θ	θ	δ	δ	θ	θ	δ	δ	ν	ν	α	α	ν	ν	α
ε	α	θ	θ	δ	ε	ι	ι	δ	ε	μ	μ	σ	α	ν	ν	σ
ξ	α	θ	γ	θ	ι	ξ	ι	γ	ξ	λ	σ	λ	π	α	π	σ
η	α	β	θ	θ	ι	ι	η	β	η	σ	κ	κ	ρ	ρ	α	σ
θ	α	β	γ	δ	δ	γ	β	θ	θ	ρ	π	ν	ν	π	ρ	α
ι	α	β	γ	δ	ε	ξ	η	θ	ι	κ	λ	μ	ν	π	ρ	σ
κ	α	α	π	ν	μ	λ	σ	ρ	κ	κ	μ	λ	π	ν	ρ	σ
λ	α	ρ	α	ν	μ	σ	κ	π	λ	μ	λ	κ	ρ	π	ν	σ
μ	α	ρ	π	α	σ	λ	κ	ν	μ	λ	κ	μ	ν	ρ	π	σ
ν	α	ρ	π	α	α	π	ρ	ν	ν	π	ρ	ν	ν	ρ	π	α
π	α	ρ	α	ν	ν	α	ρ	π	π	ν	π	ρ	ρ	π	ν	α
ρ	α	α	π	ν	ν	π	α	ρ	ρ	ρ	ν	π	π	ν	ρ	α
σ	α	α	α	α	σ	σ	σ	α	σ	σ	σ	σ	α	α	α	σ
×	α	β	γ	δ	ε	ξ	η	θ	ι	κ	λ	μ	ν	π	ρ	σ
α	α	β	γ	δ	ε	ξ	η	θ	ι	κ	λ	μ	ν	π	ρ	σ
β	β	α	ν	π	λ	μ	σ	ρ	κ	ι	ε	ξ	γ	δ	θ	η
γ	γ	ν	α	ρ	κ	σ	μ	π	λ	ε	ι	η	β	θ	γ	ξ
δ	δ	π	ρ	α	σ	κ	λ	ν	μ	ξ	η	ι	θ	β	γ	ε
ε	ε	λ	κ	σ	α	ρ	π	μ	ν	γ	β	θ	ι	η	ξ	δ
ξ	ξ	μ	σ	κ	ρ	α	ν	λ	π	δ	θ	β	η	ι	ε	γ
η	η	σ	μ	λ	π	ν	α	κ	ρ	θ	δ	γ	ξ	ε	ι	β
θ	θ	ρ	π	ν	μ	λ	κ	α	σ	η	ξ	ε	δ	γ	β	ι
ι	ι	κ	λ	μ	ν	π	ρ	σ	α	β	γ	δ	ε	ξ	η	θ
κ	κ	ι	ε	ξ	γ	σ	θ	η	β	α	ν	π	λ	μ	σ	ρ
λ	λ	ε	ι	η	β	θ	δ	ξ	γ	ν	α	ρ	κ	σ	μ	π
μ	μ	ξ	η	ι	θ	β	γ	ε	δ	π	ρ	α	σ	κ	λ	ν
ν	ν	γ	β	θ	ι	η	ξ	δ	ε	λ	κ	σ	α	ρ	π	μ
π	π	δ	θ	β	η	ι	ε	γ	ξ	μ	σ	κ	ρ	α	ν	λ
ρ	ρ	θ	δ	γ	ξ	ε	ι	β	η	σ	μ	λ	π	ν	α	κ
σ	σ	η	ξ	ε	δ	γ	β	ι	θ	ρ	π	ν	μ	λ	κ	α
x	α	β	γ	δ	ε	ξ	η	θ	ι	κ	λ	μ	ν	π	ρ	δ
$-x$	i	κ	λ	μ	ν	π	ρ	σ	α	β	γ	δ	ε	ξ	η	θ