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IN SO MANY POSSIBLE WORLDS

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Ordinary modal logic deals with the notion of a proposition being true in at least one possible world. This makes it natural to consider the notion of a proposition being true in k possible worlds for any nonnegative integer k. Such a notion would stand to Tarski's numerical quantifiers as ordinary possibility stands to the existential quantifier.

In this paper[1] I present several logics for numerical possibility. First I give the syntax and semantics for a minimal such logic (sections **1** and **2**); then I prove its completeness (sections **3** and **4**); and finally I show how to extend this result to other logics (section **5**).

1. *The Logic* Kn. The logic Kn is defined as follows.

Formation Rules: Formulas are constructed in the usual way from a set V of propositional variables $p_1, p_2, \ldots$, the binary operator $\vee$ (or), the unary operators - (not), L (necessity) and M_k, $k = 2, 3, \ldots$, and parentheses (and).

Throughout the paper I observe some familiar conventions: A, B, C, D and E, with or without subscripts, range over formulas; $\rightarrow$, $\leftrightarrow$, M (possibility) etc. are given standard definitions; all expressions are used autonomously; and parentheses are omitted from formulas in an obvious way. $M_0 A$ abbreviates $A \rightarrow A$, $M_1 A$ abbreviates MA and $Q_k A$ abbreviates $M_k A$ & - $M_{k+1} A$, $k = 0, 1, \ldots$. $M_k A$ is taken to mean A is true in at least k possible worlds; so $Q_k A$ means A is true in exactly k possible worlds (see section **2**). $\vdash A$ means A is a theorem of Kn.

Transformation Rules:

Axiom-schemes (*where* k, $l = 1, 2, \ldots$)

1. All tautologous formulas

1. The results of this paper are contained in my doctorate thesis, submitted to the University of Warwick in 1969. I am greatly indebted to my supervisor, the late Arthur Prior. Without his help and encouragement this paper would never have been written.

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2. $L(A \to B) \to (LA \to LB)$
3. $M_k A \to M_l A,\ l < k$
4. $M_k A \leftrightarrow \bigvee_{i=0}^{k} M_i (A \ \&\ B) \ \&\ M_{k-i} (A \ \&\ - B)$
5. $L(A \to B) \to (M_k A \to M_k B)$

Rules of Inference.
Modus Ponens. *From* A, $A \to B$ *infer* B
Necessitation. *From* A *infer* $L A$

2. *Semantics.* A *frame* $\mathfrak{F}$ is a pair $\langle W, R\rangle$ where W (worlds) is a non-empty set and R (accessibility) is a binary relation on W. A *structure* $\mathfrak{A}$ is a pair $\langle \mathfrak{F}, \phi\rangle$ where $\mathfrak{F}$ is a frame $\langle W, R\rangle$ and ϕ (valuation) is a map from V (variables) onto $\mathfrak{P}(W)$ (sets of worlds or propositions).

Relative to each structure $\mathfrak{A} = \langle W, R, \phi\rangle$ we define the truth-relation $\vDash$ as follows; for w in W,

(i) $w \vDash P_i$ *iff* $w \in \phi(P_i)$
(ii) $w \vDash -A$ *iff not* $w \vDash A$
(iii) $w \vDash A \vee B$ *iff* $w \vDash A$ *or* $w \vDash B$
(iv) $w \vDash L A$ *iff* $v \vDash A$ *for all* v *such that* wRv
(v) $w \vDash M_k A$ *iff* $\text{card}\{v : w R v \ \&\ v \vDash A\} \geq k$.

A is *valid*, $\vDash A$, if relative to each structure $\mathfrak{A} = \langle W, R, \phi\rangle$ $w \vDash A$ for all w in W. $\mathfrak{A}$ is a model for a set of formulas Δ if for some w in W $w \vDash A$ for each A in Δ.

3. *A Preliminary Result.* A set of formulas Δ is a *theory* if each theorem of Kn is in Δ and Δ is closed under modus ponens. Δ is *consistent* if $-A$ is in Δ only when A is not in Δ, and Δ is *complete* if $-A$ is in Δ whenever A is not in Δ.

Let W be the set of consistent and complete theories. For $k = 1, 2, \ldots$, we define the relations R_k on W. For w, v in W:

$w R_k v$ *iff whenever* $A \in v$ *then* $M_k A \in w$.

First we note three straightforward lemmas:

Lemma 1. *If* $\vdash A \leftrightarrow B$ *then* $\vdash C \leftrightarrow C(A/B)$.
Lemma 2. $w R_k v$ *iff* $\{A: - M_k - A \in w\} \subseteq v$.
Lemma 3. *If* $w R_k v$ *then* $w R_l v$, $k > l$.

Lemma 1 is proved with the help of axiom-scheme 5; lemma 2 follows from lemma 1; and lemma 3 is proved by axiom-scheme 3. Use of lemmas 1 to 3 will often be tacit.

The next result states the crucial property of the relations R_k. Let $T_w(A)$ (the *truth-set* of A) be

$$\{(v, l) : l > 0 \ \&\ w R_l v \ \&\ A \in v\}.$$

Theorem 1. *For* $k = 1, 2, \ldots,$ *and* w *in* W, $M_k A \in w$ *iff* $\text{card}\ T_w(A) \geq k$.

Proof. $\Longrightarrow$ By induction on k.

$k = 1$. Assume $M_1 A \in w$. Clearly it suffices to show that for some v, $w R_1 v$ and $A \in v$.

Let $\mathfrak{L} = \{B : L B \in w\} \cup \{A\}$. Suppose $\mathfrak{L}$ is not consistent. Then by axiom-scheme 1 and the Deduction Theorem, there are formulas B_1, $B_2, \ldots, B_n$ such that $L B_1, L B_2, \ldots, L B_n \in w$ and $\vdash B_1 \rightarrow (B_2 \rightarrow \ldots (B_n \rightarrow -A) \ldots)$. So by the logic Kn, $\vdash L B_1 \rightarrow (L B_2 \rightarrow \ldots \rightarrow (L B_n \rightarrow L - A) \ldots)$. Hence $L - A \in w$, i.e. $- M_1 A \in w$, contrary to the consistency of w.

So $\mathfrak{L}$ is consistent. By Lindenbaum's Lemma $\mathfrak{L}$ is contained in a consistent and complete theory v. But $A \in v$ and, by lemma 2, $w R_1 v$.

$k > 1$. Assume that the theorem holds for all $i < k$. Now assume $M_k A \in w$. By scheme 4, for each B there is an $i \leq k$ such that $M_i (A \& B)$, $M_{k-i} (A \& -B) \in w$. We distinguish two cases:

(a) For some B, $0 < i < k$. By the induction hypothesis, card $T_w(A \& B) \geq i$ and card $T_w (A \& -B) \geq k - 1$. But $T_w(A) = T_w(A \& B) \cup T_w(A \& -B)$. So card $T_w (A) \geq i + (k - i) \geq k$.

(b) For each B, $i = 0$ or $i = k$. Suppose $i = 0$. Then $M_k (A \& -B) \in w$. But then $L (A \rightarrow -B) \in w$. For otherwise, by scheme 6, $M_1 (A \& -B) \in w$; and so by scheme 3 we can put $i = 1$. Similarly, if $i = k$, $L(A \rightarrow B) \in w$. So either $L (A \rightarrow -B) \in w$ or $L(A \rightarrow B) \in w$.

Now let $\mathfrak{L} = \{B : -M_k -A \in w\}$ and suppose $\mathfrak{L}$ is inconsistent. Then there are formulas $B_1, \ldots, B_n$ such that $-M_k -B_1, \ldots, -M_k -B_n \in w$ and $\vdash (B_1 \& \ldots \& B_n) \rightarrow -A$. Either $L (A \rightarrow B_i) \in w$ for $i = 1, 2, \ldots, n$ or for some $i = 1, 2, \ldots, n$, $\vdash L (A \rightarrow -B_i)$. In the first case, $L (A \rightarrow B_1 \& \ldots \& B_n) \in w$; but $L (B_1 \& \ldots \& B_n \rightarrow -A) \in w$; and so $-M_1 A \in w$—a contradiction. In the second case, since $-M_k -B_i \in w$, $-M_k A \in w$ by scheme 3—again a contradiction.

So $\mathfrak{L}$ is consistent. By Lindenbaum's Lemma, $\mathfrak{L}$ is contained in a $v \in W$. But then by lemmas 2 and 3, $\langle v, i \rangle \in T_w(A)$ for $i = 1, 2, \ldots, k$. So card $T_w(A) \geq k$.

By induction on k.

$k = 1$. Assume card $T_w(A) \geq 1$. Suppose $\langle v, l \rangle \in T_w(A)$. Now $A \in v$ and $w R_1 v$. So $M_1 A \in w$. Hence $M_1 A \in w$ by scheme 3.

$k > 1$. Assume card $T_w(A) \geq k$. We distinguish two cases:

(a) For some $\langle v_1, l_1 \rangle$, $\langle v_2, l_2 \rangle$ in $T_w(A)$, $v_1 \neq v_2$. So for some B, $B \in v_1$ and $-B \in v_2$. But then for some i, $0 < i < k$, $T_w(A \& B) \geq i$ and $T_w(A \& -B) \geq k - i$. So by the induction hypothesis, $M_i (A \& B)$, $M_{k-i} (A \& -B) \in w$. Hence by scheme 4, $M_k A \in w$.

(b) For each $\langle v_1, l_1 \rangle$, $\langle v_2, l_2 \rangle$ in $T_w(A)$, $v_1 = v_2$. But then clearly, $\langle v_1, l \rangle \in T_w(A)$ for some $l \geq k$. So $M_l A \in w$. Hence $M_k A \in w$ by scheme 3.

4. *Canonical Models*. The intuitive interpretation of $w R_k v$ is that there are at least k v-type worlds accessible from w, i.e. k worlds which are accessible from w and which are copies of, have the same truth-value assignments as, v. So let us say that f is a *canonical mapping* for a structure $\mathfrak{B} = \langle X, R, \phi \rangle$ if f maps X onto W and

(i) *if* $f(x) = w$ *and* $v \in W$, *then* card $\{y : f(y) = v \& x R y\} \geq k$ *iff* $w R_k v$, *and*

(ii) $\phi(p_i) = \{x : f(x) = w \& p_i \in w\}$.

We now have:

Theorem 2. *If f is a canonical mapping from a structure* $\mathfrak{B} = \langle X, R, \phi\rangle$, *then for any x in X and any formula:* $x \models A$ *(relative to* $\mathfrak{B}$*) if and only if* $A \epsilon f(x)$.

Proof. By induction on the length of A. The main case is $A = M_k B$. Now $M_k B \epsilon w = f(x)$ iff card $\{\langle v, l\rangle : l > 0 \ \& \ w R_l v \ \& \ B \epsilon v\} \geq k$ (by theorem 1) iff for some $\langle v_i, l_i\rangle$, $v_i \neq v_j (i < j)$, $\sum_{i=1}^{m} l_i \geq k$, $w R_{l_i} v_i$ and $B \epsilon v_i$, $i, j = 1, 2, \ldots, m$ iff for some $\langle v_i, l_i\rangle$, $v_i \neq v_j$ $(i < j)$, $\sum_{i=1}^{m} l_i \geq k$, card $\{y : f(v) = v_i \ \& \ x Ry\} \geq l_i$ and $y \models B$ for $f(y) = v_i$ $i, j = 1, 2, \ldots, m$ (by f canonical and induction hypothesis) iff card $\{y : y \models B \ \& \ x Ry\} \geq k$ iff $x \models M_k B$.

Now define $\mathfrak{B} = \langle X, R, \phi\rangle$ by: $X = W \times N$ where $N = \{1, 2, \ldots\}$, $\langle w, l\rangle R \langle v, k\rangle$ iff $w R_k v$ and $\phi(p) = \{\langle w, l\rangle : p \epsilon w\}$.
Let $f(\langle w, l\rangle) = w$. Then clearly f is a canonical mapping for $\mathfrak{B}$. So we have:

Theorem 3. (*Completeness*) *A set of formulas* Δ *is consistent if and only if* Δ *has a model.*

Proof. $\Longrightarrow$ Assume Δ consistent. By Lindenbaum's Lemma, for some $w \epsilon w$, $\Delta \subseteq w$. So by theorem 2, $\langle w, 1\rangle \models A$ for all A in Δ, and Δ has a model. $\Longleftarrow$ Straightforward.

Note that there are alternative ways of defining R above. For example, we could let $\langle w, l\rangle \ R \ \langle v, k\rangle$ iff $k > l$ and $w R_{k-l} v$. In this case the canonical structure $\mathfrak{B}$ would be asymmetric.

5. *Other Logics.* The above method can be applied to other logics L besides Kn. First we relativise to L all of the constructions and results up to theorem 2. Then we prove the analogue of theorem 3. This requires that R have certain properties, which will follow from the definition of $\mathfrak{B}$ and the fact that each theory in W contains L. I shall outline this procedure for some logics below.

(I) Tn, given by Kn plus the axiom-scheme

7. $L A \rightarrow A$,

and complete for all reflexive structures. The definition of the canonical mapping f for $\mathfrak{B}$ is as before, but with

$\langle w, l\rangle \ R \ \langle v, k\rangle$ iff $w = v, k \geq l$ and $w R_{k+1-l} v$ or $w \neq v$, $k > l$ and $w R_{k-l} v$.

Notice that R, so defined, is antisymmetric.

(II) K Bn, given by Kn plus the axiom-scheme

8. $A \rightarrow L M A$,

and complete for all symmetric structures. We now let X be the set of all sequences $w_1 k_1 w_2 k_2 \ldots k_{n-1} w_n$, $n \geq 1$, such that $w_i, w_n \epsilon W$, $k_i \epsilon N$ and $w_i R_{k_i+1} w_{i+1}$ if w_{i-1} exists and $w_{i-1} = w_i$, and $w_i R_{k_i} w_{i+1}$ otherwise.

xRy iff $y = xkw$ or $x = ykw$, and $f(x)$ is the last term of x.

The above construction may be modified to show that KTBn is complete for all reflexive and symmetric structures.

(III) We may also determine the logics which are complete for R being reflexive and transitive, reflexive and transitive and antisymmetric, linear etc. However, in all of these cases the completeness proofs are very much more difficult. It is worth noting that imposing antisymmetry on a reflexive and transitive relation makes a difference to one's logic. For example, $A \mathbin{\&} M\,(\text{-}A \mathbin{\&} M_k A) \rightarrow M_{k+1} A$ becomes valid.

(IV) S5n, given by Tn plus the axiom-scheme

9. $M_k A \rightarrow L\, M_k A$,

complete for all reflexive, symmetric and transitive structures. Completeness for S5n can be proved by the above method and also by normal forms.[2]

Let S5π+ be the logic obtained from S5 by adding propositional quantifiers which range over all sets of possible worlds. Then S5n has the interesting property that any formula of S5π+ is equivalent to a formula of S5n (see [2]).

Finally, it should be noted that standard techniques, or modifications of them, may be used to prove the decidability of most of the logics mentioned above.

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2. The second method was carried out independently by Kaplan [3] and myself. The case in which one adds only M_2 (or Q_1) to S5 was axiomatized by Prior [4] and proved complete, independently, by Bull [1], Kaplan and myself.