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THE EXISTENCE POSTULATE AND NON-REGULAR SYSTEMS OF MODAL LOGIC

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The reader is advised to read section 6 [8, pp. 178-198] before proceeding with this paper, and have the book in hand. If we refer to a certain page number without indicating its source it always refers to [8]. In this paper we shall use such terms as existence, proposition, etc. rather loosely. This is done on purpose to establish the continuity between Lewis's motivations and these investigations. In future we expect to give precise definitions of these terms and present further results about the systems described here. See also [6, pp. 290-292]. What follows has a close connection with certain remarks made there although those remarks are made in terms of models and "worlds".

1. *Preliminaries.* As we all know the systems of strict implication of Lewis are put forward as rivals to the system of material implication and constructed with the specific purpose of removing the "paradoxes" of the latter system. One such paradox is the following thesis:

P1 $ACpqCpNq$

As an immediate consequence we have:

P2 $NKNCpqNCpNq$

Consequently, "if we take 'p is consistent with q' to mean 'p does not imply the falsity of q' and 'q is independent of p' to mean 'p does not imply q', then in terms of material implication, no two propositions can be at once consistent and independent [p. 122]." This violates our intuitions; in other words, is paradoxical. Lewis thus constructs his systems so that in none of them the strict analogue of *P1*:

P3 $A\mathfrak{C}pq\mathfrak{C}pNq$

is provable. Having constructed his systems Lewis noticed, however, that although *P3* was not a theorem of any of his systems it could be added with impunity to each of them, i.e., no inconsistency results on its addition.

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In fact if *P3* is added even to his weakest system S1 (all systems in this paper are to be thought of as having Lewis's formulations or obtained by the addition of axioms to these formulations), it coincides with the system of material implication:

$Z1$	$ANMKMNpNp\mathfrak{C}MNpNp$	$[P3,\ p/MNp,\ q/p;\ \mathrm{S1}^{\circ}]$
$Z2$	$ANMNp\mathfrak{C}MNpNp$	$[Z1;\ \mathrm{S1}]$
$Z3$	$ANMNpCMNpNp$	$[Z2;\ \mathrm{S1}]$
$Z4$	$CpNMNp$	$[Z3;\ \mathrm{S1}^{\circ}]$
$Z5$	$CCpNMNpNMNCpNMNp$	$[Z4,\ p/CpNMNp]$
$Z6$	$\mathfrak{C}pLp$	$[Z4;\ Z5;\ \mathrm{S1}^{\circ}]$

Hence nothing prevents our viewing his postulates "as an incomplete set for material implication [p. 178]." One can interpret this result by saying that he had not proved the existence of even two consistent and independent propositions in any of his systems; for, had he done so, addition of *P3* to the system would have resulted in an inconsistency. He therefore postulates the existence of two such propositions:

$B9$ $(\exists p,\ q)\ KN\mathfrak{C}pqN\mathfrak{C}pNq$

and shows that *B9* is consistent with his systems (p. 494, p. 498). A reason is given why this existence postulate could not be proved (pp. 189-190). The reader may with justification feel cheated. Bertrand Russell once said that "the method of 'postulating' what we want has many advantages; they are the same as the advantages of theft over honest toil." Note, however, that in subsequent studies of Lewis's systems *B9* has almost always been ignored. One gets the impression that it is an embarrassment.

1.2. When Lewis constructed his systems he considered a postulate:

$C13$ MMp

He did not construct any systems containing it. He does point out, though, that *C13* could be added consistently to S1, S2, or S3 (p. 498). His motivation in considering it was to demonstrate an assumption that would be "contrary [p. 499]" to

$C10$ $\mathfrak{C}MMpMp$

Subsequently the systems that are obtained by adding *C13* to S2 and S3 have been named S6 and S7. Another postulate stronger than *C13* has also been considered:

$C14$ $LMMp$

and the system obtained by its addition to S3 has been called S8. These systems have not found favor with modal logicians and they have not been studied for their own sake until very recently ([3], [4], [6]). *C13* has been described as a "peculiar postulate [10, p. 134]," and the systems containing it have been described as "oddities [7, p. 216]." I do not know why. No reasons are put forward in either of these places.

1.3. Sobociński [13] noticed that *C13* has a property not shared by the other postulates that Lewis considered, viz., deletion of the modal functors from it gives us a formula which is not a thesis of material implication. Such formulae, he said, are non-regular (Sobociński uses the term irregular—we follow [6] instead and use the term non-regular) modal formulae. And if a system contains a non-regular modal formula as a theorem, it is a non-regular system; otherwise, regular. Consequently S6—S8 are non-regular. He constructed a new non-regular system, T^x, obtained by the addition of

C15 MLp

to T°. He also pointed out that *C15* could be consistently added to S2°, S3°, and S4°. Let us call these systems $S2^x$, $S3^x$, and $S4^x$. Other non-regular systems are S9 of Åqvist [2] (named S7.5 by him but renamed S9 in [6]), S7.5 of Anderson [1], and the systems $S2^n$ of McKinsey [9]. As far as I know these are all the non-regular systems in the literature.

Consider now the systems $S2^y$, $S3^y$, $S4^y$, and T^y obtained by the addition of the formula

Y1 $CpMp$

to the corresponding X-systems. Group IV (p. 494) shows that they are consistent. Next consider the system S7.1 obtained by the addition of $MNMMKpNp$ and $MMKpNp$ to S3. The consistency of S7.1 is established by using the matrix described in [5, p. 233] but taking 1, 2, 3, and 7 as designated values.

2. *Proof of the Existence Postulate*. We shall now show, by honest toil, that the existence postulate can be proved in the systems S6-S9, S7.1, S7.5, $S2^n$, and the Y-systems. We shall prove it in the same way that Lewis proves *20.1-20.4* [pp. 184-186], using "converse substitution [p. 184]." We have been unable to prove it in the X-systems but a weakened form will be established.

2.1. S6-S9, S7.1, S7.5. It is easy to see that each of these systems contain S6. It is then enough to prove it in S6.

Z1	$MMKpNp$	[*C13*, $p/KpNp$]
Z2	$NMKpNp$	[S1°]
Z3	$MKMMKpNpNMKpNp$	[*Z1*; *Z2*; S1]
Z4	$N\mathfrak{C}MMKpNpMKpNp$	[*Z3*; S1°]
Z5	$\mathfrak{C}MKpNpKMMKpNpMKpNp$	[S1]
Z6	$\mathfrak{E}MKpNpKMMKpNpMKpNp$	[*Z5*; S1°]
Z7	$MKMMKpNpMKpNp$	[*Z6*; *Z1*; S1°]
Z8	$N\mathfrak{C}MMKpNpNMKpNp$	[*Z7*; S1°]
Z9	$KN\mathfrak{C}MMKpNpMKpNpN\mathfrak{C}MMKpNpNMKpNp$	[*Z4*; *Z8*; S1°]
B9	$(\exists p, q)\ KN\mathfrak{C}pqN\mathfrak{C}pNq$	[*Z9*, $p \int MMKpNp$, $q \int MKpNp$]

2.2. $S2^n$. McKinsey constructed the systems $S2^n$ to demonstrate that S2 has infinitely many complete extensions. They are defined as follows. Add to S2 the following axioms:

Q^1 $N\mathfrak{C}KpNpMKpNp$
Q^2 $N\mathfrak{C}MKpNpMMKpNp$
. .
. .
. .
Q^n $N\mathfrak{C}M^{n-1}KpNpM^nKpNp$
R^n $\mathfrak{C}M^nKpNpM^{n+1}KpNp$

where "$M^n\alpha$" means the wff formed by putting n "M" symbols in front of α. A curious fact about these systems does not seem to have been noticed. The system $S2^1$, contrary to McKinsey's assertion, is inconsistent.

$Z1$	$AMKKpNpNMKpNpMKMKpNpNKpNp$	[Q^1; S1°]
$Z2$	$\mathfrak{C}KKpNpNMKpNpKpNp$	[S1°]
$Z3$	$\mathfrak{C}KMKpNpNKpNpMKpNp$	[S1°]
$Z4$	$AMKpNpMMKpNp$	[$Z1$; $Z2$; Z3; S1°]
$Z5$	$CNMKpNpMMKpNp$	[$Z4$; S1°]
$Z6$	$MMKpNp$	[$Z5$; S1°]
$Z7$	$MKpNp$	[$Z6$; R^1; S1°]
$Z8$	$NMKpNp$	[S1°]
$Z9$	p	[$Z7$; $Z8$; S1°]

If we examine the proof which shows the consistency of $S2^n$, the difficulty is easy to find. The matrix which is supposed to verify $S2^n$ is not even a matrix when $n = 1$: It has no designated elements. The proof that S2 has infinitely many complete extensions, of course, remains perfectly good. Next observe that the deduction above shows that $MMKpNp$ follows from $\{S2; Q^1\}$. Also note that Q^1 follows from $\{S2; MMKpNp\}$. This is easily seen by simply reversing the steps from $Z1$ to $Z6$. And it is well-known that $\{S6\} \rightleftarrows \{S2; MMKpNp\}$. Hence $\{S6\} \rightleftarrows \{S2; Q^1\}$. It follows, by **2.1**, that the existence postulate can be proved in $S2^n$ ($n \geq 2$), and also that McKinsey actually has proved a stronger result: S6 has infinitely many complete extensions. Let us then abandon the axioms Q^1 and R^1, view the systems as extensions of S6, and rename them $S6^n$ ($n \geq 1$): $S6^n = S2^{n+1}$. $S6^n$ is, therefore, the system obtained by adding to S6 the following axioms:

B^1 $N\mathfrak{C}MKpNpMMKpNp$
B^2 $N\mathfrak{C}MMKpNpMMMKpNp$
. .
. .
. .
B^n $N\mathfrak{C}M^nKpNpM^{n+1}KpNp$
C^n $\mathfrak{C}M^{n+1}KpNpM^{n+2}KpNp$

We now give an alternative axiomatization of $S6^n$. This will be useful later. We show that $\{S6; B^n\} \rightleftarrows \{S6; MKM^{n+1}KpNpNM^nKpNp\}$. First we show that $\{S6; B^n\} \rightarrow \{S6; MKM^{n+1}KpNpNM^nKpNp\}$.

$Z1$	$AMKM^nKpNpNM^{n+1}KpNpMKM^{n+1}KpNpNM^nKpNp$	[B^n; S1°]
$Z2$	$MAKM^nKpNpNM^{n+1}KpNpKM^{n+1}KpNpNM^nKpNp$	[$Z1$; S2°]

$Z3$ $\mathfrak{C}AKpNqKqNpKApqANpNq$ [S1°]

$Z4$ $\mathfrak{C}AKM^nKpNpNM^{n+1}KpNpKM^{n+1}KpNpNM^nKpNpKAM^nKpNpM^{n+1}KpNpANM^nKpNpNM^{n+1}KpNp$ [$Z3$, p/M^nKpNp, $q/M^{n+1}KpNp$]

$Z5$ $MKAM^nKpNpM^{n+1}KpNpANM^nKpNpNM^{n+1}KpNp$ [$Z2$; $Z4$; S1°]

$Z6$ $\mathfrak{C}AM^nKpNpM^{n+1}KpNpM^{n+1}KpNp$ [S1]

$Z7$ $MKM^{n+1}KpNpANM^nKpNpNM^{n+1}KpNp$ [$Z5$; $Z6$; S1°]

$Z8$ $\mathfrak{C}KM^{n+1}KpNpANM^nKpNpNM^{n+1}KpNpKM^{n+1}KpNpNM^nKpNp$ [S1°]

$Z9$ $MKM^{n+1}KpNpNM^nKpNp$ [$Z7$; $Z8$; S1°]

Now to see {S6; $MKM^{n+1}KpNpNM^nKpNp$} → {S6; B^n}, again, reverse the steps above. Our new axiomatization for S6^n is then obtained by adding to S6 the following axioms:

V^1 $MKMMKpNpNMKpNp$

V^2 $MKMMMKpNpNMMKpNp$

. .

. .

. .

V^n $MKM^{n+1}KpNpNM^nKpNp$

C^n $\mathfrak{C}M^{n+1}KpNpM^{n+2}KpNp$

Observe further that V^1 is a theorem of S6. But for the sake of symmetry we leave the axioms in the above form.

2.3. S2^y-S4^y, T^y. Evidently each of these systems contain S2^y. We prove the existence postulate in S2^y.

$Z1$ $MLKApNpKpNp$ [$C15$, $p/KApNpKpNp$]

$Z2$ $MKLApNpLKpNp$ [$Z1$; S2°]

$Z3$ $N\mathfrak{C}LApNpNLKpNp$ [$Z2$; S1°]

$Z4$ $CNKpNpMNKpNp$ [$Y1$, $p/NKpNp$]

$Z5$ $MNKpNp$ [$Z4$; S1°]

$Z6$ $LApNp$ [S1°]

$Z7$ $MKLApNpMNKpNp$ [$Y1$, $p/KLApNpMNKpNp$; $Z5$; $Z6$; S1°]

$Z8$ $N\mathfrak{C}LApNpLKpNp$ [$Z7$; S1°]

$Z9$ $KN\mathfrak{C}LApNpLKpNpN\mathfrak{C}LApNpNLKpNp$ [$Z3$; $Z8$; S1°]

$B9$ $(\exists\, p, q)\ KN\mathfrak{C}pqN\mathfrak{C}pNq$ [$Z9$, $p \int LApNp$, $q \int LKpNp$]

2.4. S2^x-S4^x, T^x. Our deductions proceed in S2^x.

$Z1$ $N\mathfrak{C}LApNpNLKpNp$ [As in **2.3.**]

$Z2$ $LApNp$ [S1°]

$Z3$ $\mathfrak{C}NApNpApNp$ [$Z2$; S1°]

$Z4$ $\mathfrak{C}KLApNpNApNpNApNp$ [S1°]

$Z5$ $\mathfrak{C}KLApNpNApNpApNp$ [$Z3$; $Z4$; S1°]

$Z6$ $\mathfrak{C}LApNpApNp$ [$Z5$; S1°]

$Z7$ $\mathfrak{C}MLApNpMApNp$ [$Z6$; S2°]

$Z8$ $MApNp$ [$C15$, $p/ApNp$; $Z7$; S1°]

$Z9$ $KLApNpMApNp$ [$Z2$; $Z8$; S1°]

$Z10$ $NCLApNpNMApNp$ [$Z9$; S1°]

$Z11$	$KNCLApNpLKpNpN\mathfrak{C}LApNpNLKpNp$	$[Z1; Z10; \mathrm{S1}^{\circ}]$
$Z12$	$(\exists p, q)\, KNCpqN\mathfrak{C}pNq$	$[Z11, p \int LApNp, q \int LKpNp]$

This is the promised weakened form.

2.5. Let us say that a *Lewis system* is any system that contains $\mathrm{S1}^{\circ}$. Then, given Lewis's motivations, it is appropriate to define a *Lewis system proper* as any Lewis system in which the existence postulate can be proved. But it is unnecessary to introduce quantifiers into the syntax. Note that the "proof" of the postulate simply consists of proving as a theorem any substitution-instance of $KN\mathfrak{C}pqN\mathfrak{C}pNq$. A Lewis system proper is then a Lewis system in which some substitution-instance of $KN\mathfrak{C}pqN\mathfrak{C}pNq$ can be proved. The perversity of the most popular systems of modal logic: T, S4, and S5, is now vividly brought to us by the results of McKinsey and Sobociński. McKinsey proved that S4 has only one complete extension and Sobociński pointed out that his proof was good for T also. And this extension is material implication. Sobociński also proved that any consistent system which contains T must be regular. From this it follows that not only is T (*a fortiori*, S4 and S5) an incomplete form of material implication but also that it is hopeless to try to extend it to a Lewis system proper. S1, S2, and S3, although incomplete forms of material implication, can at least be extended to Lewis systems proper. Observe that the same is true of T° and $\mathrm{S4}^{\circ}$. We have extended them above to Lewis systems proper: T^{y} and $\mathrm{S4}^{y}$.

The fundamental intuition behind modal logic—whether we accept it or not—is the distinction between actuality and necessity, the distinction between propositions that are as a matter of fact true and those that are necessarily so. Hence any decent modal logic ought to have propositions that are true but not necessary, and its counterpart, false but not impossible. Formally stated, there must be theorems which are substitution-instances of $KpNLp$. And this is a non-regular modal formula. So we have theorems of the above form only in non-regular modal systems. In S6, for example, we have $KNMKpNpNLNMKpNp$.

Finally observe that in those systems in which the existence postulate can be proved the remaining theorems of section 6 (pp. 178-198), in particular, **20.6** (p. 188), go through. Consider S6. Then the four distinct propositions of **20.6** turn out to be $MMKpNp$, $NMMKpNp$, $MKpNp$, and $NMKpNp$.

3. *Finite model property*. We shall now show that each of the systems discussed above, with the exception of S9, has the finite model property (f.m.p.). The question as to whether S9 has the property remains open. For this section we presuppose acquaintance with [11] and [12], and we shall use the terminology and notation of these papers. The f.m.p. for S6 has been established in [5], for S7 and S8 in [12]. That S7.1 has the f.m.p. is implicit in [12]. The property for S7.5 can be established by a simple modification of Theorem 6 [12]. We leave this to the reader. We treat then the X-systems, the Y-systems, and $\mathrm{S6}^{n}$. It is easy to see that instead of

C15, *MLp*, we can use the postulate *MNMApNp*. This is more convenient for our purpose here. We shall, of course, use matrices $\mathfrak{M} = \langle M, D, \cap, -, \mathsf{P}\rangle$, and as our stock of conditions on these matrices we list the following:

(A) $\langle M, \cap, -, \mathsf{P}\rangle$ is a weak modal algebra;
(B) D is an additive ideal of M;
(C) $x = 0$ if and only if $-\mathsf{P}(x) \in D$;
(D) $\mathsf{P}0 \leq \mathsf{P}x$;
(E) $\mathsf{P}(\mathsf{P}x \cap -\mathsf{P}0) \leq \mathsf{P}x$;
(F) $\mathsf{P}\mathsf{P}x \leq \mathsf{P}x$;
(G) $-\mathsf{P}\mathsf{P}0 \in D$;
(H) $x \rightarrow \mathsf{P}x \in D$;
(I) $x \leq \mathsf{P}x$;
(J) $\mathsf{P} - \mathsf{P}1 \in D$;
(K) $\mathsf{P}\mathsf{P}0 \in D$;
(L) $\mathsf{P}(\mathsf{P}^{n+1}0 \cap -\mathsf{P}^n 0) \in D$ $(n \geq 1)$;
(M) $\mathsf{P}^{n+1}0 = \mathsf{P}^{n+2}0$ $(n \geq 1)$.

We omit the proofs of the two theorems that follow.

Theorem 1. $\mathfrak{M} = \langle M, D, \cap, -, \mathsf{P}\rangle$ *is a* σ*-regular* $S2^x(S3^x, S4^x, T^x, S2^y, S3^y, S4^y, T^y, S6^n)$*-matrix if and only if the following conditions hold for the respective systems.*

(1) (A) -- (D), (J); [$S2^x$]
(2) (A) -- (E), (J); [$S3^x$]
(3) (A) -- (F), (J); [$S4^x$]
(4) (A) -- (D), (G), (J); [T^x]
(5) (A) -- (D), (H), (J); [$S2^y$]
(6) (A) -- (E), (H), (J); [$S3^y$]
(7) (A) -- (F), (H), (J); [$S4^y$]
(8) (A) -- (D), (G), (H), (J); [T^y]
(9) (A) -- (D), (I), (K) -- (M). [$S6^n$]

Theorem 2. *A is provable in each of the systems mentioned in Theorem 1 if and only if A is verified by all matricies* $\mathfrak{M} = \langle M, D, \cap, -, \mathsf{P}\rangle$ *fulfilling the respective conditions for each of the systems mentioned in Theorem 1.*

Theorem 3. *Let* $\mathfrak{M} = \langle M, D, \cap, -, \mathsf{P}\rangle$ *be a* σ*-regular* $S2^x(S3^x$, *etc.)-matrix, and let* $a_1, \ldots, a_r$ *be a finite sequence of elements of M. Then there is a finite* σ*-regular* $S2^x(S3^x$, *etc.)-matrix* $\mathfrak{M}_1 = \langle M_1, D_1, \cap_1, -_1, \mathsf{P}_1\rangle$ *with at most* $2^{2^{r+n+3}}$ *elements such that*

(i) *for* $1 \leq i \leq r$, $a_i \in M_1$;
(ii) *for* $x, y \in M_1$, $x \cap_1 y = x \cap y$;
(iii) *for* $x \in M_1$, $-_1 x = -x$;
(iv) *for* $x \in M_1$ *such that* $\mathsf{P}x \in M_1$, $\mathsf{P}_1 x = \mathsf{P}x$;
(v) *for* $x \in M_1$, *if* $x \in D_1$, *then* $x \in D$.

Proof. See Theorem IV.1 [11], Theorem IV.4 [11], and Theorem 6 [12]. For our theorem here M_1 is the set of elements of M obtained from

$P0, PP0, \ldots, P^{n+1}0$ $(n \geq 1)$, $P1$, $P - P1, a_1, a_2, \ldots, a_r$ by any finite number of applications of the operations - and $\cap$. Evidently the only thing which remains to be shown is that $\mathfrak{M}_1$ satisfies conditions (D) -- (M) given that $\mathfrak{M}$ satisfies the corresponding conditions. It should be noted that we do not need all that many elements in M_1 for each individual system. But we treat them all at once. The refinements in the statement of the theorem for the various systems is left to the reader.

D: See [12].
E: See [12].
F: See [11, pp. 173-175].
λ: We pause now and prove by induction that $P^m0 = P_1{}^m0$ $(1 \leq m \leq n + 1)$. Observe first that $P0 = P_10$ (see [11, p. 170]). Next let $P^t0 = P_1{}^t0$ $(1 \leq t \leq n)$. Then $PP^t0 = PP_1{}^t0$, i.e., $P^{t+1}0 = PP_1{}^t0$. Evidently $P_1{}^t0 \in M_1$, and $PP_1{}^t0 = P^{t+1}0 \in M_1$ by construction. Hence by condition (iv) of the theorem, $P_1P_1{}^t0 = PP_1{}^t0$, i.e., $P_1{}^{t+1}0 = PP_1{}^t0 = P^{t+1}0$.
G: Let $-PP0 \in D$. Then by (λ) $-P_1P_10 = -PP0 \in D$. Also $-P_1P_10 \in M_1$. Hence $-P_1P_10 \in D_1$.
H: See [12].
I: See [12].
J: Let $P - P1 \in D$. Since $1 \in M_1$ and $P1 \in M_1$, we get $P_11 = P1$. So $-P1 = -P_11$. Hence $P - P1 = P - P_11$. Next note that $-P_11 \in M_1$ and $P - P_11 = P - P1 \in M_1$. Then $P_1 - P_11 = P - P_11 = P - P1 \in D$. Also $P_1 - P_11 \in M_1$. Hence $P_1 - P_11 \in D_1$.
K: Let $PP0 \in D$. Then arguing as in (G), $P_1P_10 \in D_1$.
L: Let $P(P^{n+1}0 \cap -P^n0) \in D$. Then by ($\lambda$), $P(P_1{}^{n+1}0 \cap -P_1{}^n0) \in D$. But $P(P_1{}^{n+1}0 \cap -P_1{}^n0) \leq P_1(P_1^{n+1}0 \cap -P_1{}^n0)$ (see Theorem IV.1(2) [11] and the remark that follows). By Theorem III.8(c) [11], $P_1(P_1{}^{n+1}0 \cap -P_1{}^n0) \in D$. Also $P_1(P_1{}^{n+1}0 \cap -P_1{}^n0) \in M_1$. Hence $P_1(P_1{}^{n+1}0 \cap -P_1{}^n0) \in D_1$.
M: Let $P^{n+1}0 = P^{n+2}0$. Now $P^{n+2}0 = PP^{n+1}0 = PP_1{}^{n+1}0$ (by (λ)). Also $P_1{}^{n+1}0 \in M_1$ and $PP_1{}^{n+1}0 = P^{n+2}0 = P^{n+1}0 \in M_1$. Hence $P_1P_1{}^{n+1}0 = PP_1{}^{n+1}0 = P^{n+2}0 = P^{n+1}0 = P_1{}^{n+1}0$ (by (λ)). So $P_1{}^{n+1}0 = P_1{}^{n+2}0$.

This completes the proof.

4. *Halldén-style results.* We now presuppose acquaintance with [5] and section 1 of [12]. Consider the systems S3, S3.1, S4, S7, S8, and S7.1. The relation between these systems is visualised by the following diagram:

$$\begin{tikzcd}
\mathrm{S8} \arrow[dr] & & \mathrm{S7.1} \arrow[dl] \arrow[dr] & & \mathrm{S4} \arrow[dl] \\
& \mathrm{S7} \arrow[dr] & & \mathrm{S3.1} \arrow[dl] & \\
& & \mathrm{S3} & &
\end{tikzcd}$$

Recall [12] that S3.1 = {S3; *MNMMKpNp*}, S4 = {S3; *NMMKpNp*}, S7 = {S3; *MMKpNp*}, S8 = {S3; *NMNMMKpNp*}, S7.1 = {S3; *MMKpNp*; *MNMMKpNp*}.

Hence the containments are obvious. The following matrices show that the containments are proper: (1) Group I (p. 493) verifies S3 and S7 but falsifies S3.1 and S7.1; (2) Group II (p. 493) verifies S3 but falsifies S7; (3) Halldén's matrix [5, p. 232] with 1 and 2 as designated values verifies S3.1 but falsifies S7.1 and S4; (4) The same matrix with 1, 2, 3, and 7 as designated values verifies S7 but falsifies S8.

Let us now denote the class of theorems of a system by writing the name of the system in bold type. And let us say that a trio of distinct, consistent systems S_1, S_2, S_3 are in *Halldén-relation* if and only if (1) $\mathbf{S}_1 = \mathbf{S}_2 \cap \mathbf{S}_3$; (2) the number of complete extensions of S_1 is equal to the sum of the complete extensions of S_2 and S_3. Halldén proved that S3, S7, S4 are in Halldén-relation. Now note that we have *ApNp*, $\mathfrak{E}KpNpKqNq$, and the substitutability of strict equivalents in each of the systems of the previous paragraph. And see [6, p. 244, n. 231]. Hence as a consequence of the above axiomatizations and Theorems 2, 7, and 8 of [5] we have that the following trios are in Halldén-relation: (1) S3, S8, S3.1; (2) S7, S8, S7.1; (3) S3.1, S7.1, S4. It follows that the number of complete extensions of S3 is equal to the number of complete extensions of S8 plus the number of complete extensions of S7.1 plus the number of complete extensions of S4. This further refines Halldén's partial solution to McKinsey's open question about the number of complete extensions of S3 [9, p. 42].

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(*Added in proof.*) I saw [15] after completing this paper, and the reader is urged to read it in conjunction with this. Note the similarity of theme of both although none of the results of either is duplicated in the other. See also the remaining items of the

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