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THE HISTORICAL DEVELOPMENT OF GROUP THEORETICAL IDEAS IN CONNECTION WITH EUCLID'S AXIOM OF CONGRUENCE

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*Introduction** Group theoretical ideas have been present in the minds of men since ancient times. What will be discussed in this paper are the thoughts of men concerning group theoretical notions, in connection with Euclid's well known COMMON NOTION 4: *Things which coincide with one another are equal to one another*, as well as two of his propositions. The propositions are: Proposition 4: *If two triangles have the two sides equal to two sides respectively and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will also be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely, those which the equal sides subtend*; and Proposition 5: *In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines will be produced further, then angles under the base will be equal to one another.*

This paper is a summarization of what more than 150 commentaries have to say about these ideas. The commentaries used began with the first printed one and go on past the work of Cayley who gave an explicit definition of the group concept.[1] Few men attempted to come to grips with the problem hidden in the Greek sentence which is translated most frequently: "Things which coincide with one another are equal to one another." It is from hindsight, therefore, that one tries to look at the tangled web formed

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1. See: Article by Arthur Cayley in *Philosophical Magazine*, vol. VII, London (1854) for his definition of a group.

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by the conflicting interpretations of the word: ἐφαρμόζοντα which leaves possible the translation interpretation to be: "superposition vs. agreement" or coincidence. The effort of translating properly the word: ἐφαρμόζοντα proved to be a test of originality and depth of the translator's understanding from the time of the third century, with Pappus, until the time of Legendre in the nineteenth century. The major concern of the paper is to question whether or not it is true that in Common Notion 4 of the text of Euclid there is contained the germ of a new powerful idea which slowly evolved into a plant bearing the likeness of applied group theory in all but the name.

Principal Findings Whereas most of the commentators of Euclid's *Elements* were unwilling to venture past what had been stated by Euclid in the monumental work, the *Elements*, a few like Henry Savile, Jacques Peletier, Isaac Barrow, Edmund Scarburgh, John Playfair, and Adrien Legendre did attempt to distill the idea of motion which is contained in the axiom of congruence so that what remained was the geometrical content. It was, at the beginning of the nineteenth century and later, then, that congruence was studied more intently in connection with the foundations of geometry by mathematicians such as Felix Klein, H. Helmholtz, Bernard Riemann, Sophus Lie, Henri Poincaré and David Hilbert. In 1965 at a meeting in Germany, the Mathematisches Forschungsinstitut, Dr. Hans Wussing stated concern with the development of the emancipation of abstract group theory from the main lines of development, which toward the end of the nineteenth century led to the axiomatically built up abstract group theory. He noted that, whereas it is well known that the concept of the group in the sense of the permutation group evolved from the developing solution of algebraic equations, little is known as to other evolutions. He contended that the path of the mathematical "group" has its historic roots not only in the solution of the equation, but also in number theory and in geometry.[2]

It seems that the notion of the "group" should be considered as a gradually evolving process, a reciprocal activity between historical components, and an event that developed from implicit and explicit group theoretical methods. The formation of the group theory concepts grew and were explicated slowly. Andreas Speiser says that a great part of Euclid's work belongs to the area of higher algebraic number theory and group theory. Although such a theory points to a possible early development, the extant mathematical literature fails to exhibit any explicit group theory developments on the part of the ancients. It has been shown by A. Seidenberg, however, that certain "groups" were used by them. It is well known that the regular figures received much attention from them, both with respect to their ornaments and their philosophy. And, since these cannot be fully comprehended without group theoretic considerations, implicit knowledge of groups must have been perceived.

2. *cf.* [23].

Euclid's Elements But who among the commentators grappled with group theoretic ideas when they examined and analyzed the axiom of congruence as it is found in Euclid's *Elements*: καὶ τά' ἐφαρμόζοντα ἐπ' ἀλλήλα ἴσα ἀλλήλοισ; and how did they go about answering their own queries? Euclid began his *Elements* by enunciating a certain number of axioms, but it must not be imagined that the axioms which he enunciated explicitly are the only ones to which he appealed. If one analyzes his demonstrations there will be found in them, in a more or less masked form, a certain number of hypotheses which are in reality disguised axioms. Euclid's geometry began with declaring that two figures are equal if they are superposable. This assumes that they can be displaced and also that among all the changes which they may undergo, one can distinguish those which may be regarded as displacements without deformation. Again, this definition implies that two figures which are equal to a third are equal to each other. That is tantamount to saying that if there be a displacement which puts the figure A upon the figure B, and a second displacement which superposes the figure B upon the figure C, there will also be a third, the resultant of the first two, which will superpose the figure A upon the figure C. In other words, it is presupposed that the displacements form a "group." From this it could be implied that the notion of a "group" was introduced from the outset in Euclid's *Elements*.

Length, Angle, Triangle When one pronounces the word "length," a word which is frequently not necessary to define, one implicitly assumes that the figure formed by two points is not always superposable upon that which is formed by two other points, for otherwise any two lengths whatever would be equal to each other. This important property of a group was recognized by the earliest commentators, as well as by Euclid himself.

One also implicitly enunciates a similar hypothesis when one pronounces the word "angle." This, too, was realized by the commentators. Peletarius recognized and delineated this point in his 1557 commentary. By displacing figures and causing them to execute certain movements, one wishes to show that at a given point in a straight line a perpendicular can always be erected. To accomplish this one conceives a movable straight line turning about the point in question. But one presupposes here that the movement of this straight line is possible, that it is continuous and that in so turning it can pass from the position in which it is lying on its prolongation. Here again is a hypothesis touching the properties of the group, one of which Peletarius was aware.

To demonstrate the cases of the equality of triangles, the figures are displaced so as to be superposed one upon the other. The meaning of this took centuries to define. The method employed in demonstrating that from a given point one and only one perpendicular can always be drawn to a straight line was not well understood until Playfair and Legendre endeavored to explain it in their commentaries.

Modern Explanation It was demonstrated in the nineteenth century that what happens is that the figure is turned 180 degrees around the given straight line and in this manner the point symmetrical to the given point with respect to the straight line is obtained.[3] One has here a feature, most characteristic, and here appears the part which the straight line most frequently plays in geometrical demonstrations, namely, that of an axis of rotation.

There is implied in all of this the existence of a sub-group, that which is a subset of a given group when the elements of the subset form a group with respect to a binary operation defined for the group. When, and this frequently happens, a straight line is made to slide along itself (continuing to suppose that it can serve as an axis of rotation), one implicitly takes the existence of the helicoidal sub-group for granted. The principal foundation of Euclid's demonstrations, then, is really the existence of the group and its properties.

What the Ancients Thought The properties of group theoretic notions of congruence that have been mentioned, although implicitly known by the ancients, took a great amount of time to be communicated. The earliest commentators of Euclid used the idea that a figure was congruent to itself, but it was only Pappus who described this phenomenon in his demonstration of the isosceles triangle theorem (I.5) of Euclid. It seems, from the study that has been made, that the Greeks and their successors were reluctant to bring into geometry an idea of change that would belong to mechanics or kinematics rather than to geometry. Whatever the opinion of the ancients really was, the idea slowly evolved that since displacements and motions seemed to be at the basis of every notion of geometry, these movements must be analyzed.

Distillation of Motion It was not an easy task for geometers to free the axiom of congruence from all non-geometrical aspects. Most were unwilling to venture past the literal translation that Euclid had made or that his translators produced. Before the time of the analytical development of geometry of René Descartes, the concepts of Euclidean geometry afforded no way of clearly expressing the mathematics inherent in the axiom of congruence. Because of this, perhaps, attempts to solve the problem of congruence failed. However, persons like Isaac Barrow in his "Lectures on Mathematics" gave explanations of the properties of group theory for the rigid motions even though he was unable to translate these ideas into the symbolical language of mathematics. What Barrow did was to ask himself: What are those properties of motion on which congruence depends? His conclusions were: *First*, rest is a special kind of motion. If we assign to every point of the plane that same point, obviously distances are preserved. Thus, the definition of a rigid motion is satisfied. Today one calls such an assignment or mapping an identity transformation. He described it although he did not define it. *Secondly*, he said, if one assigns

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3: *cf.* [20], p. 34.

a point P of the plane to a point P', one may also assign a point P' to a point P and in this way there exists an inverse mapping. It may also be considered thus: If one assigns figure A to a figure B, one may assign figure B to figure A. This explanation satisfied the notion of inverse. *Thirdly*, if there exists a point or figure A which is equal or corresponds to point or figure C, that is, if there is a mapping which takes A into C and if there exists point or figure B which also may be mapped into C in the same manner, then one says that there exists the possibility of the combination of two mappings.[4]

It may be concluded from Barrow's *Lectures* that the properties of congruence that depend on rigid motion, viz., the existence of the identity mapping, the existence of the inverse mapping and the existence of the combinations of two mappings, were recognized by him.

John Playfair and Adrien Legendre, more than a hundred years later, were also able to mathematize some of the properties of group theory mentioned, but until Evariste Galois (1830) devised his method of expressing group theoretic ideas in algebraic notation, it was not possible for the commentators to free themselves from the mechanical notion of superposition. On the other hand, it was probably due to the fact that more than a few commentators expressed concern and dissatisfaction with the ideas set forth about "congruence" that the question remained an open one and continued to give rise to theorizing, as well as disputation. This theorizing led, in time, to developing one of the most powerful tools of modern mathematics.

The Elements of Thomas L. Heath Thomas L. Heath has an historical survey of *Common Notion 4* that is extensive even though it may not all be taken as factual. Sir Thomas makes clear that the axiom of congruence as stated by Euclid is not incontestably geometrical in character. He supposes that superposition is a legitimate way of proving the equality of two figures which have the necessary parts respectively equal. Arguing in this way, however, he overlooks the testimony of the commentators mentioned before, viz., P. Ramus, Henry Savile, Edmund Scarburgh and others, all of whom went to great lengths to show that the Greek sentence had been misinterpreted and mistranslated by many of the commentators.[5] It is Heath's contention that: "The phraseology of the propositions, e.g., I.4 and I.8, in which Euclid employs the method indicated, leaves no room for doubt that he regarded one figure as actually moved and placed upon the other."[6]

Nineteenth Century Thought on Congruence When Bertrand Russell wrote on the foundations of geometry in 1897 he asked: What geometrical knowledge must be the logical starting place for a science of space and must also be logically necessary to the experience of any form of

4. *cf.* [1], pp. 163-245 and 188-197.

5. See: Heath, pp. 249-250.

6. Ibid., pp. 225-228.

externality?[7] Russell's own answer was that this was homogeneity of space or free mobility, an axiom about distance and the basic realization that two points determine a distance that is unaltered in any motion of the two points as a single figure. This was asserted to be a congruence, a motion or rigid transformation.

There were and always had been many philosophers of science who had debated the meaning of congruence, of superposition, of rigid bodies, rigid motions, free mobility and empty space since the problem had first been posed by the ancients. These included Aristotle, Plato, Proclus, Augustine, Aquinas, Nicholas of Cusa, Nicole Oresme, Descartes, Leibniz, Barrow, Newton, Kepler, Kant, Helmholtz, Russell, and later Einstein. The one notion that most of them agreed upon is that the axiom of congruence is fundamental to the explanation of motion and of space when it is associated with the foundations of geometry. Immanuel Kant had stated this when he said that space is represented as an infinite given quantity, an essential of which is congruence.[8]

Foundations of Geometry With respect to the question: What is geometry and what are the foundations of geometry, Arthur Schopenhauer said:

> I am surprised that, instead of the eleventh axiom (The Parallel Postulate) that the eighth is not attacked: "Figures which coincide (sich deckon) are equal to one another." For coincidence (das Sichdecken) is either mere tautology, or something entirely empirical, which belongs, not to pure intuition (Anschauung) but to external sensuous experience. It presupposes in fact the mobility of figures; but that which is movable in space is matter and nothing else. Thus this appeal to coincidence means leaving pure space, the sole element of geometry, in order to pass over to the material and the empirical.[9]

With these words, Schopenhauer expressed the viewpoint of those who were claiming that geometry was an empirical subject rather than a theoretical one.

Study of the Axioms Until David Hilbert published his *Grundlagen der Geometrie* in 1899 geometers had turned to the study of a kind of research instigated in 1868 by Helmholtz.[10] But Bernhard Riemann, in 1854, had first initiated such a study.[11] In this paper, Riemann introduced a space as a topological manifold of an arbitrary number of dimensions; a metric was distinguished in such a manifold by means of a quadratic differential form.

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7. *cf.* [6], p. 6.

8. Kant made this statement in his *Critique of Pure Reason.* For a full discussion of "Kant and Modern Mathematics," see Prof. Fang's article in *Philosophia Mathematica,* Vol. II, No. 2 (1965), pp. 47-67.

9. See: Heath, p. 227.

10. *cf.* [13].

11. *cf.* [21].

He defined the character of space by its local behavior. In this way he gave a unifying principle for space that had enabled him to classify all existing forms of geometry. The classification included non-Euclidean geometry and allowed him to open the way for the creation of any number of new types of space, hence, of new types of geometries. Two revolutionary remarks made in his paper were that for a discrete manifold the principle of measurement is already contained in the concept of the manifold, but that for a continuous one it must come from elsewhere, and the empirical concepts upon which the spatial metric is based, the concepts of rigid body and the light ray, cease to be valid in the domain of the infinitely small. Riemann's paper changed the mode of questioning for the geometer. In the opening sentences of his essay he summed up the problem of the foundations of geometry and also disclosed the fundamental principles for its solution.

> It is known that geometry assumes, as things given, both the notions of space and the first principles of construction in space. She gives definitions of them which are merely nominal, while the true determinations appear in the form of axioms. The relation of these assumptions remains consequently in darkness; we neither perceive whether and how far their connection is necessary, nor, *a priori* whether it is possible.[12]

He went on:

> From Euclid to Legendre (to name the most famous of geometers) this darkness was cleared up neither by mathematicians nor by philosophers who concerned themselves with it. The reason for this is doubtless that the general notion of multiple extended magnitudes (in which space magnitudes are included) remained entirely unworked.[13]

Riemann had set out to discover the simplest matters of fact from which measure-relations of space may be determined where measure consists in the superposition of the magnitudes to be compared by means of the notion of congruence. The congruence that Riemann would determine was accomplished by means of the total curvature of space. He distinguished relations of extension or partition from relations of measure. Doing this he found that, with the same extensive properties, different measure-relations were conceivable. This new interpretation of congruence and of metric relations paved the way for inquiry which, in a relatively short time, then would result in the group theoretic notion of space, itself, as well as of congruence. If Riemann's plan is followed, one finds a metric geometry of congruence as the invariant theory for elementary geometry and arrives at a six parameter subgroup. This enables one to take as the definition of a congruence relation that which defines the same things as a rigid motion.

Helmholtz, in 1868, had analyzed Riemann's conception of space, partly by looking for a geometrical image of his own famous theory of colors and

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12. *cf.* [21], p. 14.

13. Ibidem.

partly by inquiring into the origin of ocular measure, all of which led him to investigate the nature of the geometrical axioms as well as Riemann's quadratic measure.[14] Camille Jordan, in 1867, played his part in the development of group theory, publishing an article in which the groups of infinite order play an important role. The main problem that Jordan set forth was the study of all the sub-groups of the groups of movements. This initiated research concerning the groups of movements.[15] Felix Klein took up this research. In 1872, Felix Klein initiated his famous *Erlanger Program*. For Klein, the criterion that distinguishes one geometry from another is the group of transformations under which the propositions remained true. He regarded all of geometry as a problem of group theory, and he connected the idea of superposition to the essential idea of a group of space transformations. He said:

> An example of a group of transformations is afforded by the totality of motions, every motion being regarded as an operation performed on the whole of space. A group contained in this group is formed, say, by the rotations about one point. On the other hand, a group containing the group of motions is presented by the totality of the collineations. But the totality of the dualistic transformations does not form a group; for the combination of two dualistic transformations is equivalent to a collineation. A group is, however, formed by adding the totality of the dualistic to that of the collinear transformations.[16]

In his lectures, Klein defined congruence as a rigid motion or an isometry. These he considered as special cases of a more general kind of transformation of space than is usually considered. From this idea of congruence he was able to determine the geometry of a certain structure that differed from that of another structure because of the arbitrary definition given to congruence. Klein continued and showed that geometry could be developed according to two different plans, depending on the approach given to congruence. One can begin to study geometry through the analysis of a group of motions, in particular a group of translations and use the pure group theoretical approach, or one can begin with the axioms of congruence and push parallelism to a later place. Sophus Lie and Felix Klein worked together at first, but then Lie independently began to publish his ideas concerning the problem in 1896. Lie showed that the geometry of the Euclidean and the non-Euclidean space can be represented by three transformation equations.[17] He was convinced that for the examinations on the foundations of geometry it was of great value to specify the simple qualities of the groups of movements. He also showed that these groups could be characterized in a simple way if one introduced the concept of free mobility (congruence).

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14. *cf.* [14], Mind I, p. 309.

15. *cf.* [18], p. 446.

16. *cf.* [15].

17. *cf.* [16].

Contributions of Poincaré and Hilbert After Lie's paper of 1896 it seemed possible that the "last word" had been spoken, but such was not the case. Henri Poincaré, in 1898, revealed his own insights.[18] Herein he presented geometry as the study of a certain continuous group and space itself is a group, which serves us not to represent things to ourselves, but to reason upon things. He also pointed out that the distinctive feature of the Euclidean motion group was that it had a large abelian normal subgroup formed by the translations. More precisely than anyone before him, Poincaré explained *why* he reasoned that Euclid had implied in the *Elements* the idea of group theory.

When David Hilbert published his *Grundlagen der Geometrie*, he revolutionized mathematics by dispensing with the idea that the axioms were self-evident truths having relation to experience or intuition. It was with this seemingly simple statement that he severed geometry from the experience of physical reality. What he did was to create a new form of axiomatics which was the starting point for the extension of the axiomatic method to many branches of the exact sciences. With particular respect to the questions put forth in this paper his *Appendix* is of primary interest. Herein, he discussed geometry as a continuation of the arguments set up by Riemann, Helmholtz, and Lie. He said:

> The investigations by Riemann and Helmholtz of the foundations of geometry led Lie to take up the problem of the axiomatic treatment of geometry as introductory to the study of groups . . . . As the basis of his transformation groups Lie made the assumption that the functions defining the group can be differentiated. Hence in Lie's development the question remains uninvestigated as to whether this assumption as to the differentiability of the functions in question is really unavoidable in developing the subject according to the axioms of geometry, or whether, on the other hand, it is not a consequence of the group conception and of the remaining axioms of geometry.[19]

Hilbert's axioms, deducible from or a division of Lie's were simple and easily seen, geometrically. He went on from there to analyze the groups of motions in order to retrieve the axioms of Euclid. Hilbert's *Grundlagen* may, then, be considered the culmination of an investigation of the axioms of congruence and space that evolved into research that studies the group theoretic concept of space in order to realize the axiomatic basis for it. After him, a group of mathematicians interested in the foundations of geometry worked out mathematical systems showing that beginning with the groups of motions, the axioms of Euclid and, in particular, the axioms of congruence can be retrieved.[20] However, it remains an open question whether or not the converse of this is true.

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18. *cf.* [20].

19. *cf.* [5], p. 133.

20. See: [7], [10], [19] and the works on group theory by such mathematicians as G. Bachmann, J. Hjelmsev, G. Thomson. Sr. Mary Justin Markham, B.S.M. in

One may glean from this essay more about the fundamental concept of congruence, how it developed and why. Although there may not be available to them the entire historical and philosophical studies that accompany the evolvement of the subject, access to further study may be given by the appended bibliographic sources. The paper has meant to suggest that it can be both interesting and enlightening to make further inquiries and historical investigations of the fundamental principles of mathematics so that we may understand more about how the experiences of the past evolved into the modern mathematical ideas of today and of the future.[21]

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her doctoral dissertation, published in this journal, vol. VII (1966), pp. 209-238, "A Group Theoretic Characterization of the Ordinary and Isotropic Euclidean Planes" worked out such a system as did Hans Zassenhaus in an unpublished manuscript, "The Euclidean Plane", worked out for the Academic Year Institute of the University of Notre Dame, 1963.

21. Work to complete this paper was carried on while the author was a postdoctoral Research Associate at the Office of Educational Research, University of Notre Dame.

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