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# MODAL ELABORATIONS OF PROPOSITIONAL LOGICS

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1. *Modally Augmented Systems* One interesting perspective upon modal logic is obtained by beginning with a nonmodal system, and then developing a modal system "around" it, so to speak, by construing necessity in the "surrounding" modal system as provability within the initial system.[1] Modality, so conceived, is obtained in the broader system by bridging rules linking the necessity operator in this system to thesishood at the nonmodal starting point. The aim of the present paper is to trace out one line of thought along which this idea can be implemented.

Let L be an arbitrary system of (nonmodal) propositional logic based upon negation (-), conjunction (&), and implication ($\rightarrow$) as propositional operators. The theses of L are to be derived from certain (at this point unspecified) axioms by the rules of substitution and *modus ponens*. (We shall write $\vdash_{X} A$ to indicate that $A$ is a thesis of the system X.)

To obtain the modal system ML, the modal augmentation of the initial system L, we introduce the modal operator of necessity ($\Box$) subject to the rules and axioms of the following sort:

I. *Modal Axioms Internal to* ML
   (A1) $\vdash_{ML} \Box p \rightarrow p$
   (A2) $\vdash_{ML} \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$

II. ML-*Internal Rules*
   (R1) *Substitution*
   (R2) *Modus Ponens*
   (R3) *Qualified Necessitation*: *If* $\vdash_{ML} A$, *then* $\vdash_{ML} \Box A$, *provided* $A$ *is not modal-free*.

III. L/ML *Bridging Rule*
   (B) *If* $\vdash_{L} A$, *then* $\vdash_{ML} \Box A$

1. The origins of this line of thought may be sought in [1]. Here necessity is identified with provability in a certain system. See also [2] and [3].

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IV. *Metarule of Closure*

(C) $\vdash_{ML} A$ *only if A's being a thesis of* ML *follows from the preceding rules and axioms.*

It is readily shown that:

*If A is a thesis of* L, *then A is a thesis of* ML.

For if $A$ is a thesis of L, then by (B), $\vdash \Box A$ in ML, which by (A1) and (R2) yields $A$. Moreover:

*If A is a thesis of* ML *and A is modal-free, then A is a thesis of* L.

For suppose $A$ is a thesis of ML and $A$ is modal-free. Then by (C) it follows, in view of the nature of the rules and axioms, that $A$ could be the result only of (A1) and (R2). Hence we must have $\Box A$ as a thesis of ML. But this in turn could only be the result of rule (B). Hence $A$ is a thesis of L. It follows from these findings that ML must be a conservative extension of L (whenever ML is consistent[2]).[3]

One obvious consequence of this approach is represented by the theorem:

*If* L = PC, *the classical propositional calculus, then* ML = T, *the well-known system of Feys- von Wright.*

In one well-known axiomatization, T is based on $\Box$ as primitive, subject to the following rules and axioms:

*Rules*

(TI) *If* $\vdash_{PC} A$, *then* $\vdash_T A$

(TII) *If* $\vdash_T A$, *then* $\vdash_T \Box A$

*Axioms*:

(Ti) $\vdash_T \Box p \supset p$

(Tii) $\vdash_T \Box (p \supset q) \supset (\Box p \supset \Box q)$

Throughout, when L = PC, we shall write $\supset$ for $\rightarrow$. Now given (R2), (TI) follows from (A1) and (B). Moreover: (Ti) = (A1), and (Tii) = (A2). Thus ML is at least as strong as T when L = PC. It can also be shown (though we shall not do so here), that it is no stronger, so that ML = T in this case. Finally, (TII) is proved as follows: If $A$ is not modal-free and a thesis of ML, then $\Box A$ follows by (R3). If $A$ is modal-free and a thesis of ML then,

2. The proof that ML is a conservative extension of L depends critically on the use of the rule of closure (C), and such a use is possible only on the assumption that ML is consistent.

3. The system L′ is an extension of L if the vocabulary of L′ contains that of L as a subset and every L-thesis is an L′-thesis. L′ is a conservative extension of L if it is an extension such that all L′ theses formulated in the L vocabulary will also be theses of L′.

using (C), we note that $A$ could come only from (A1) and (R2), and so again we must have $\Box A$ as a thesis of ML.

Correspondingly, it is also readily seen that if we make a sufficient addition to category I of ML-internal rules (by adding $\vdash_{ML} \Box A_1$, $\vdash_{ML} \Box A_2$, . . . , $\vdash_{ML} \Box A_n$, where $A_1, A_2, \ldots, A_n$ are any syntactically suitable set of PC-axioms) then the modally augmented system ML will have to contain T (no assumptions whatsoever being made about the initial system L).

Suppose now that (R3) were dropped from the construction procedure for ML and that (A2) strengthened to:

(A3) $\vdash_{ML} \Box(p \supset q) \supset \Box(\Box p \supset \Box q)$

we now have the result:[4]

*If* L = PC, *then* ML = S3.

In the face of this finding, it is at once clear from [4] that S4 can also be developed as a modally augmented system by the reinstatement of (R3), and that S5 will then be obtained by further addition of the axiom:

(A4) $\vdash_{ML} \Diamond p \supset \Box\Diamond p$ *where* $\Diamond q = -\Box - q$

These observations indicate how systems of modal logic can be developed via bridging rules from arbitrary systems of nonmodal propositional logic in such a way that in the case of classical propositional calculus (PC) as the starting-system we obtain the spectrum of the most familiar modal systems. This suggests the potentially interesting question, or question family, of the modally augmented systems resulting from initial propositional logics *weaker* than PC such as intuitionistic propositional logic. On the other hand if the initial system is very strong—and specifically is a system of *arithmetic* rather than propositional logic—then a consistent modal augmentation becomes impossible when the modal system is strong enough to contain the thesis: $\Box(\Box p \rightarrow p)$.[5] Since ML has (A1) as one of its axioms, and also the necessitation rule, clearly the system will have $\Box(\Box p \rightarrow p)$; hence modal augmentation along the lines indicated above is impossible when L is so strong a system. But note that the proof that the general necessitation rule is derived in ML depends on (R3). In the following we shall construct a sequence of modally augmented systems which lack (R3), and do not have the necessitation rule, but only some highly restricted form thereof.

2. *Minimality* In considering modal systems as modal augmentations of nonmodal systems we can thus regard the derived modal systems as developed (by a suitable procedure) from the underlying logical system L. This leads to the problem of what modal systems are associated with a

4. This is obvious by inspection of the development of S3 given in [4].

5. See footnote 6 below, as well as its context in the text.

logical system, and to the question of when a modal system of the type of ML, satisfying the bridging-rule (B), could be properly called (in any sense) a modal system *for* L. The following two conditions seem minimally necessary:

(i) ML is consistent if L is so
(ii) ML is a conservative extension of L, so that if $A$ is a thesis of ML and $A$ is modal free (m.f.) then $A$ is a thesis of L.

This is a very weak sort of minimality; thus when L = PC, then all of the Lewis systems will satisfy these conditions. However, for the present, we shall characterize any system satisfying these conditions as a *minimally adequate modal augmentation of L*.

If the initial system L is complete (in the strong sense that for every $A$ in its vocabulary either $A$ or $-A$ is a thesis of L), and ML (constructed by the use of the bridging rule (B)) is consistent, then ML must satisfy the remaining minimality condition if it has (A1) and (R2). For let $A$ be (m.f.) and suppose $A$ is a thesis of ML but not of L. Then, since L is complete $-A$ in L and by (B) $\Box -A$ in ML. This by (A1) and (R2) yields $-A$ in ML, contradicting the consistency of ML.

However, (A1) and (R2) are not sufficient to assure that ML satisfies the minimality property in the more general case when L is incomplete. But in this general case we have the theorem:

*The modally augmented system* ML, *satisfying* (B), (A1), *and* (R2), *is a conservative extension of* L *if and only if it also satisfies the two following rules.*

(R3′) *If* $\vdash_{ML} A$ *then* $\vdash_{ML} \Box A$, *where A is m.f.*
(B′) *If* $\vdash_{ML} \Box A$ *then* $\vdash_{L} A$, *where A is m.f.*

*Proof*: If ML is a conservative extension of L, suppose $A$ is m.f. and a thesis of ML. Then $A$ in L, and hence by (B) $\Box A$ in ML, so that (R3′) obtains. And if $\Box A$ in ML where $A$ is m.f., then by (A1) and (R2) $A$ in ML, which by assumption yields $A$ in L. Hence (B′) also holds. To prove the converse, suppose that ML satisfies both (R3′) and (B′). If $A$ is m.f. and a thesis of ML then by (R3′) $\Box A$ in ML, which by (B′) yields $A$ in L.

Note that in the previous section we took ML as satisfying (A1), (A2), (R1), (R2), (B) and the metarule (C). We now realize that all that was needed in order to show that ML is a conservative extension of L apart from (B), (A1), and (R2) was the metarule (C). Hence (C) entails (R3′) and (B′) under these conditions.

3. *A Modal Hierarchy* As a result of the foregoing, in order to satisfy the minimality requirement, a system ML has to have a restricted form of the necessitation rule (R3′), but need not have the general unrestricted rule, (R3). For if ML is minimal, and has $\Box A$ as a thesis then it does *not* follow that $\Box\Box A$ is a thesis in ML. This suggests that we can construct yet another modal system "around" the minimal modal system ML, by construing necessity in it as provability in ML. As we shall see, this

yields a hierarchy of modal systems each a conservative extension of the previous one. The system which is the "least upper bound" of the hierarchy has the rule of necessitation in it, and hence if we add to it (A2) and (R1), it becomes T-like in the sense that when L = PC the system is T.

In order to show all this, we shall prove it for systems assumed to have (A2) and (R2), but since the proof does not make use of these, the corresponding general assertion about minimal systems follows. Moreover, the consistency (supposing that L is a consistent system of propositional logic) of each of the modal systems of the hierarchy follows immediately from the consistency of T, since they must all be fragments of T.

To facilitate the discussion, we shall introduce the notion of the *degree of a formula*.

1. A formula $A$ (of ML) is of degree 0, if it is modal free. We shall write this as $\deg(A) = 0$.
2. If $A$ has the form $\Box B$, and $\deg(B) = n$ then $\deg(A) = n + 1$.
3. $\deg(-A) = \deg(A)$.
4. $\deg(A \,\&\, B) = \mathrm{Max}(\deg(A), \deg(B))$.
5. $\deg(A \rightarrow B) = \mathrm{Max}(\deg(A), \deg(B))$.

The modal system ML is said to be of degree $n$, (in this case we shall say ML = $M_nL$) if it has theses of degree $n$ but none of greater degree, so that $\vdash_{M_nL} A$ implies that $\deg(A) \leq n$. We allow $n$ to be any non-negative integer, understanding $M_0L$ to be the nonmodal system L.

Given the nonmodal system L = $M_0L$, we can construct simultaneously by induction on $n (n = 1, 2, \ldots)$ a hierarchy of systems where each is the modal augmentation of the preceding one; thus:

I. *Modal Axioms Internal to $M_nL$*

($A_n1$) $\vdash_{M_nL} \Box p \rightarrow p$
($A_n2$) $\vdash_{M_nL} \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$

II. *$M_nL$ Internal Rules*
($R_n1$) *Substitution*
($R_n2$) *Modus Ponens*

III. *Bridging Rules*
($B_n$) *If* $\vdash_{M_{n-1}L} A$ *then* $\vdash_{M_nL} \Box A$.

A system $M_nL$ satisfying these rules and axioms is not in general a conservative extension of $M_{n-1}L$. However, by a similar proof to that given for the basic case, one can prove (by induction on $n$) that $M_nL$ is a conservative extension of $M_{n-1}L$ if and only if it satisfies the following:

($R_n3$) *If* $\vdash_{M_nL} A$ *then* $\vdash_{M_nL} \Box A$, *where* $\deg(A) \leq n - 1$
($B_n'$) *If* $\vdash_{M_nL} \Box A$ *then* $\vdash_{M_{n-1}L} A$, *where* $\deg(A) \leq n - 1$

Still, there is nothing to assure us that $M_nL$ is a system of degree $n$, i.e., all its theses have degree $\leq n$. This effect may be introduced by a restriction on the language of $M_nL$, allowing it to have as well-formed formulas (wffs) only those whose degree is not greater than $n$.

A possible motivation for such an approach is the following. Just as in the basic case, we consider a formula with modalities as foreign to the non-modal system L, because the language of L does not have the degree one necessity operator, so in the general case: if $A$ is a formula itself of degree $n$, then $\Box A$ is not a meaningful formula of $M_nL$ because (although we here use the same necessity symbol) this additional modality represents a *new*, deeper modal operator foreign to the language of $M_nL$. For example, one might want to claim that if $A$ is a thesis of a logical system L then it is logically necessary, so that $\Box A$ is a thesis of a modal system, but that $\Box\Box A$ is meaningless (in the sense of undefined).[6] Alternatively, one might consider this iteratedly modal formula as meaningful, but hold that the first occurrence of the necessity operator has a meaning different from that of the second one, so that one should properly write it as $\Box_2\Box_1 A$ rather than simply $\Box\Box A$.

Thus if the initial system L were of a very strong sort (specifically, a system of arithmetic), then the augmented modal systems (if they are to be plausible as modal systems) should be cut off at some finite level, see [5]. Alternatively this result could be evaded by construing iterated modalities as equivocal, along the lines of the present proposal. This suggests also that we can consider modal augmentation of modal systems, i.e., where L is itself a modal system. Consider for example the case where L = PC. From it we construct $M_1L$ as before, only instead of using the symbol '$\Box$' we write '$\Box_1$'. Thus $\Box_1 A$ in $M_1L$ if and only if $A$ in PC. Now we add to $M_1L$ the symbols and axioms of arithmetic and considering this system as our base we construct its modal augmentation, using this time the symbol '$\Box_2$'. Now in this system, call it $M_2L$, $\Box_1 A$ means $A$ is a PC-thesis, $\Box_2 A$ means $A$ is either a thesis of PC or a thesis of $M_1L$ or a thesis of $M_1L$ + arithmetic. In short, since the latter is an extension of the first two systems, it is a thesis of $M_1L$ + arithmetic. Clearly in $M_2L$ no formula of the form $\Box_1\Box_2 A$ is a thesis, since no rule allows construction of it. Moreover, for the same reason, $M_2L$ does not have $\Box_2\Box_2 A$, which yields the inconsistency indicated by Montague [5].

Yet another way of assuring ourselves that the system $M_nL$ will be of degree $n$ is by introducing the following:

*Metarule of Closure*

($C_n$) $\vdash_{M_nL} A$ *only if* $A$*'s being a thesis of* $M_nL$ *follows from the preceding rules and axioms* (*of groups* (I), (II), *and* (III))

6. On this approach, modalities are to be viewed as classifiers of sentences (or as predicates thereof) rather than as sentential operators like negation. Correspondingly, modal iteration makes no sense. Thus if necessity is construed as provability "is provably provable" could—from a suitable perspective—be viewed as a nonsense.

It can be easily shown that by adding this rule to the system, $M_nL$ is a conservative extension of $M_{n-1}L$. (The proof is similar to the one given in the first section.) Thus in order that our system will satisfy the minimality condition it suffices that we introduce ($C_n$), and then ($R_n3$) and ($B'_n$) will be derived rules of the system.

By the construction of the hierarchy, each system is an extension of the preceding one. Since each system allows as theses formulas of degree $n$ or less, an "upper bound" of this set will have the same rules and axioms as $M_nL$, except for the bridging-rule and the metarule of closure. If we call this system M*L, it transpires that this system may be defined by the stipulation that

$\vdash_{M^*L} A$ *is to obtain just in case there exists an n such that* $\vdash_{M_nL} A$.

Now clearly, M*L will have the rule of necessitation. For if $A$ is a thesis of M*L, then there will be an $n$ such that $A$ is a thesis in $M_nL$, and hence $\Box A$ in $M_{n+1}L$, so that $\Box A$ in M*L. Moreover, M*L will clearly have (A1), (A2), (R1), (R2), and hence M*L, thus defined, will be an extension of ML, the modal augmentation of L introduced in the previous section. But ML is an extension of each of the $M_nL$, and therefore M*L = ML. From this it follows immediately that when L = PC, then M*L = T.[7]

## REFERENCES

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7. This incidentally exhibits (and the fact may be obvious enough from other points of view) how there is an infinite sequence of distinct modal systems intermediate between the straightforward modalization of PC (viz. the system $M_1L$ with L = PC), and the system T. This suggests a further question: It has been shown that if we successively add to the system T axioms of the form $\Box^n p \supset \Box^{n+1} p$ ($n$ = 1, 2,...) then the result is a descending sequence of systems $T_1, \ldots T_n, \ldots$ between S4 and T with $T_1$ = T+($\Box p \supset \Box\Box p$) = S4, and each $T_{n+1}$ a proper subsystem of $T_n$. (This is an unpublished result of B. Sobociński, see [6].) This suggests the following question: Starting with any formulation of S4, could we construct along lines similar to those in this paper, a kind of *descending* modal hierarchy of weaker and weaker systems of which the lower bound would yield T?

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