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AXIOMATIC, SEQUENZEN-KALKUL, AND SUBORDINATE PROOF VERSIONS OF S9

ARNOLD VANDER NAT

1.1 *The System* S9. In [8] the system S9 was presented with primitive connectives $\sim$, &, and $⥽$ as S3 plus the axioms

(*a*) $\sim p \vee ((\sim(p \,\&\, \sim p) ⥽ p) \vee (p ⥽ (p ⥽ p)))$,
(*b*) $\sim p \vee ((p ⥽ p) ⥽ p)$,
(*c*) $(p ⥽ p) ⥽ \sim(\sim(p ⥽ p) ⥽ (p ⥽ p))$,

the rules being Substitution, Strict Detachment, and Adjunction.[1] A simpler formulation of S9 can be given, however, in that (*b*) is redundant and (*a*) and (*c*) can be simplified. If we abbreviate $\sim x ⥽ x$ by $\Box x$ and $\sim(x \,\&\, \sim y)$ by $x \supset y$, then in S3, $x ⥽ y$ is strictly equivalent (s.e.) to $\Box(x \supset y)$ and $x ⥽ x$ is s.e. to $\Box$t, where t is any tautology of classical two-valued logic, PC. Thus, in S3 the axioms (*a*), (*b*), and (*c*) are s.e. to

(*d*) $\sim(p ⥽ \Box \mathrm{t}) \supset (p \supset \Box p)$,
(*e*) $p \supset (\Box \mathrm{t} ⥽ p)$,
(*f*) $\Box \mathrm{t} ⥽ \sim\Box\Box \mathrm{t}$

respectively. Now (*f*) is derivable from $\sim\Box\Box$t and (*e*), and (*e*) is derivable from $\sim\Box\Box$t and (*d*). The latter is shown as follows. The formula $(\sim p ⥽ q) \,\&\, (p ⥽ q) ⥽ \Box q$ is provable in S3, so that $\sim\Box\Box \mathrm{t} ⥽ ((\sim p ⥽ \Box \mathrm{t}) \supset \sim(p ⥽ \Box \mathrm{t}))$ is provable in S3. Hence, by $\sim\Box\Box$t and (*d*) we have $(\sim p ⥽ \Box \mathrm{t}) \supset (p \supset \Box p)$. Substituting $\Box \mathrm{t} \supset p$ for p and detaching $\sim(\Box \mathrm{t} \supset p) ⥽ \Box \mathrm{t}$, we have $(\Box \mathrm{t} \supset p) \supset (\Box \mathrm{t} ⥽ p)$, which yields (*e*) by a two-valued tautology. Hence, in S3 (*a*), (*b*), and (*c*) are derivable from $\sim\Box\Box$t and $\sim(p ⥽ \Box \mathrm{t}) \supset (p \supset \Box p)$, and *vice versa.* A simpler formulation of S9 in $\sim$, &, and $⥽$ is thus S3 plus

(*g*) $\sim(\sim(p ⥽ p) ⥽ (p ⥽ p))$
(*h*) $\sim(p ⥽ (p ⥽ p)) \supset (p \supset (\sim p ⥽ p))$.

1.2 It is desirable to present yet another formulation of S9: a *Lemmon*

1. For a detailed discussion of S9 see [8].

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formulation with primitives $\sim$, $\supset$, and $\Box$. (We shall dispense with the rule of substitution in favor of *axiom-schemes*.)

Axioms: *1.* $\vdash \Box A \supset A$
2. $\vdash \Box(A \supset B) \supset \Box(\Box A \supset \Box B)$
3. $\vdash \sim\Box\Box A$
4. $\vdash \sim\Box(A \supset \Box \mathrm{t}) \supset (A \supset \Box A)$ *where* t *is any tautology of* PC.

Rules: *R1.* *If* A *is a tautology of* PC *then* $\Box A$ *is a theorem*
R2. *From* A *and* $A \supset B$ *infer* B

We note first of all that a formulation of S3 by Lemmon [6] is *R1, R2, 1, 2,* plus the axioms

(1a) $\Box(\Box A \supset A)$
(2a) $\Box(\Box(A \supset B) \supset \Box(\Box A \supset \Box B))$.

We show that in virtue of *3* and *4*, *(1a)* and *(2a)* are derivable, so that S9 is S3 plus *3* and *4*. By *R1* and *R2* and *1* and *2* the formula $\sim\Box\Box \mathrm{t} \supset \sim\Box((\Box A \supset B) \supset \Box \mathrm{t})$ is provable. Hence, by *3* and *R2* we have $\sim\Box((\Box A \supset B) \supset \Box \mathrm{t})$, and by *4* and *R2* we have $(\Box A \supset B) \supset \Box(\Box A \supset B)$, which yields *(1a)* and *(2a)*. We note, thus, that S9 ($\sim, \supset, \Box$) is S7 plus *4*. Moreover, we note that S9 ($\sim, \supset, \Box$) is *deductively equivalent* to S9 ($\sim, \&, \dashv$). If $x \supset y$ and $\Box x$ are defined as before, then S9 ($\sim, \&, \dashv$) *contains* S9 ($\sim, \supset, \Box$). If, on the other hand, $x \& y$ is defined as $\sim(x \supset \sim y)$ and $x \dashv y$ as $\Box(x \supset y)$, then S9 ($\sim, \supset, \Box$) *contains* S9 ($\sim, \&, \dashv$). A more rigorous proof of this is presented in **1.3**, theorem 3.

1.3 In [8] S9 ($\sim, \&, \dashv$) was shown to be *complete* with respect to the matrix N, the values 1 and 2 being designated:[2]

$i \& j$	1	2	3	4	$\sim i$
1	1	2	3	4	4
2	2	2	4	4	3
3	3	4	3	4	2
4	4	4	4	4	1

$i \dashv j$	1	2	3	4
1	2	4	4	4
2	2	2	4	4
3	2	4	2	4
4	2	2	2	2

That is, it was shown that all and only those formulas (wffs) that are satisfied by matrix N are provable in S9 ($\sim, \&, \dashv$). If we define an *N-valuation on A* in the usual way, i.e. a function v from F_A, the set of well-formed parts of A (wfps), into the set $\{1, 2, 3, 4\}$ such that (i) v is defined for V_A, the set of variables of A, and, if $\sim B$, $B \& C$, or $B \dashv C$ are wfps of A then (ii) $v(\sim B) = \sim v(B)$, (iii) $v(B \& C) = v(B) \& v(C)$, and (iv) $v(B \dashv C) = v(B) \dashv v(C)$, where $\sim i$, $i \& j$, and $i \dashv j$ are defined as in matrix N, and say that a wff A is *N-valid* iff for all N-valuations v on A, $v(A) = 1$ or $v(A) = 2$, then we have the following theorem.

Theorem 1: *A wff A is provable in* S9 ($\sim, \&, \dashv$) *iff A is N-valid.*

2. Matrix N is the Lewis and Langford matrix, Group I, p. 493 of [7].

A similar result is also true for S9 ($\sim, \supset, \Box$). To arrive at this result we shall, using the matrix N, construct a matrix M and show that S9 ($\sim, \supset, \Box$) is complete with respect to M. We construct the matrix such that $\sim i = k$ in M iff $\sim i = k$ in N, $i \supset j = k$ in M iff $\sim(i \,\&\, \sim j) = k$ in N, $\Box i = k$ in M iff $\sim i \mathrel{\mapsto\!\!3} i = k$ in N. Thus we have the matrix M:

$i \supset j$	1	2	3	4	$\sim i$	$\Box i$
1	1	2	3	4	4	2
2	1	1	3	3	3	4
3	1	2	1	2	2	4
4	1	1	1	1	1	4

An *M-valuation on A* is defined for the matrix M just as N-valuations were defined for the matrix N, and we shall say that a wff A is *M-valid* iff for all M-valuations v on A, $v(A) = 1$ or $v(A) = 2$.

Theorem 2: *A wff A is provable in* S9 ($\sim, \supset, \Box$) *iff A is M-valid.*

Proof: We leave it to the reader to verify that the axioms of S9 ($\sim, \supset, \Box$) are M-valid and that the rules preserve M-validity, so that if A is provable in S9 ($\sim, \supset, \Box$) then A is M-valid. To show the converse, we first prove lemmas 1-3 from which the desired result follows immediately.

We introduce a *circle* and a *star* transformation. Given a wff A in $\sim$, $\supset$, and $\Box$, the expression A°, a wff in $\sim$, &, and $\mathrel{\mapsto\!\!3}$, is defined as follows: (i) if B is a variable of A then $B^\circ = B$, and if $\sim B$, $B \supset C$, or $\Box B$ are wfps of A then (ii) $(\sim B)^\circ = \sim B^\circ$, (iii) $(B \supset C)^\circ = \sim(B^\circ \,\&\, \sim C^\circ)$, and (iv) $(\Box B)^\circ = \sim B^\circ \mathrel{\mapsto\!\!3} B^\circ$. Given a wff A in $\sim$, &, and $\mathrel{\mapsto\!\!3}$, we define the expression A^*, a wff in $\sim$, $\supset$, and $\Box$, as follows: (i) if B is a variable of A then $B^* = B$, and if $\sim B$, $B \,\&\, C$, or $B \mathrel{\mapsto\!\!3} C$ are wfps of A then (ii) $(\sim B)^* = \sim B^*$, (iii) $(B \,\&\, C)^* = \sim(B^* \supset \sim C^*)$, and (iv) $(B \mathrel{\mapsto\!\!3} C)^* = \Box(B^* \supset C^*)$.

Lemma 1: *If a wff A is M-valid, then A° is N-valid.*

Proof: The proof is by induction. Let v_M be an M-valuation on A and let v_N be an N-valuation on A° such that if B is a variable of A, $v_M(B) = v_N(B)$. We shall show that if $\Box B$ and $B \supset C$ are wfps of A such that $v_M(B) = v_N(B^\circ)$ and $v_M(C) = v_N(C^\circ)$, then $v_M(\Box B) = v_N((\Box B)^\circ)$ and $v_M(B \supset C) = v_N((B \supset C)^\circ)$, so that we will have shown that $v_M(A) = v_N(A^\circ)$. We show this as follows:

$$v_M(\Box B) = \Box v_M(B) = \Box v_N(B^\circ) = (\sim v_N(B^\circ) \mathrel{\mapsto\!\!3} v_N(B^\circ)) = v_N(\sim B^\circ \mathrel{\mapsto\!\!3} B^\circ) = v_N((\Box B)^\circ)$$
$$v_M(B \supset C) = (v_M(B) \supset v_M(C)) = (v_N(B^\circ) \supset v_N(C^\circ)) = (\sim(v_N(B^\circ) \,\&\, \sim v_N(C^\circ))) =$$
$$v_N(\sim(B^\circ \,\&\, \sim C^\circ)) = v_N((B \supset C)^\circ).$$

Now, let A be M-valid, and let v_N be any N-valuation on A°. Let v_M be the M-valuation on A such that v_M restricted to V_A is identical to v_N restricted to $V_A{}^\circ$. We have just shown that in that case $v_M(A) = v_N(A^\circ)$. But $v_M(A) = 1$ or $v_M(A) = 2$. Hence, $v_N(A^\circ) = 1$ or $v_N(A^\circ) = 2$, and hence, A° is N-valid.

Lemma 2: *If a wff A is provable in* S9 ($\sim$, &, $\mathrel{\mapsto\!\!3}$), *then A^* is provable in* S9 ($\sim, \supset, \Box$).

Proof: From the remarks in **1.1** and **1.2** it follows that if A is provable in S3 $(\sim, \&, \prec\!\!\!\!\!3)$ then A^* is provable in S9 $(\sim, \supset, \Box)$. We leave it to the reader to complete the proof by showing that the starred versions of axioms (g) and (h) are provable in S9 $(\sim, \supset, \Box)$ and that the starred versions of Strict Detachment and Adjunction are derivable in S9 $(\sim, \supset, \Box)$.

Lemma 3: *If a wff* $(A^\circ)^*$ *is provable in* S9 $(\sim, \supset, \Box)$, *then* A *is provable in* S9 $(\sim, \supset, \Box)$.

Proof: We note that by the circle transformation expressions of the form $x \supset y$ and $\Box x$ are replaced by expressions of the form $\sim(x \,\&\, \sim y)$ and $\sim x \prec\!\!\!\!\!3\, x$. The star transformation replaces the latter by $\sim\sim(x \supset \sim\sim y)$ and $\Box(\sim x \supset x)$. Now, since $x \supset y$ is s.e. to $\sim\sim(x \supset \sim\sim y)$, and $\Box x$ is s.e. to $\Box(\sim x \supset x)$, the lemma follows.

Returning to the proof of theorem 2, if a wff A is M-valid then by lemma 1 A° is N-valid, and by theorem 1 A° is provable in S9 $(\sim, \&, \prec\!\!\!\!\!3)$; by lemma 2 $(A^\circ)^*$ is provable in S9 $(\sim, \supset, \Box)$, and by lemma 3 A is provable in S9 $(\sim, \supset, \Box)$. This completes the proof of theorem 2.

Theorem 3: *A wff* A *is provable in* S9 $(\sim, \supset, \Box)$ *iff* A° *is provable in* S9 $(\sim, \&, \prec\!\!\!\!\!3)$.

Proof: If A is provable in S9 $(\sim, \supset, \Box)$ then by theorem 2 A is M-valid, by lemma 1 A° is N-valid, by theorem 1 A° is provable in S9 $(\sim, \&, \prec\!\!\!\!\!3)$. On the other hand, if A° is provable in S9 $(\sim, \&, \prec\!\!\!\!\!3)$, then by lemma 2 $(A^\circ)^*$ is provable in S9 $(\sim, \supset, \Box)$, and by lemma 3 A is provable in S9 $(\sim, \supset, \Box)$.

1.4 S9 has a very simple *Kripke* semantics. This result is important in that there seems to be no way of understanding some of the theses of S9, such as $(\Diamond A \supset \Box\Diamond A) \,\&\, \sim(\Diamond A \prec\!\!\!\!\!3\, \Box\Diamond A)$, $(A \prec\!\!\!\!\!3\, \Box B) \supset \sim(\sim A \prec\!\!\!\!\!3\, \Box B)$, and the S9 axiom $\sim(A \prec\!\!\!\!\!3\, \Box t) \supset (A \supset \Box A)$, apart from viewing these theses as expressing certain semantic conditions. We give, then, the following *Kripke* semantics for S9 (cf. [5]).

The S9 *model structure* is the set of *possible worlds* $W = \{G, H\}$ where G is a *normal* world and H a *non-normal* world.[3] If an *accessibility* relation R is defined on W, then GRG and GRH, but for no world I, HRI (i.e. G and H are accessible to G, but no world is accessible to H). The assignment of truth values to wffs is as usual, in particular, if IRI, then a wff $\Box A$ is *true* in I iff for all J such that IRJ A is true in J, and, if not IRI, then a wff $\Box A$ is *false* in I. It will be convenient, however, not to introduce an accessibility relation and also to dispense with the usual notion of a *model* (relative to a model structure) since there are only two specific worlds to consider.

3. We follow Kripke here in calling a world *non-normal* if every wff of the form $\Diamond x$, even $\Diamond(A \,\&\, \sim A)$, is true in that world; otherwise, a world is *normal*. Or, alternatively, given an accessibility relation, we call a world *normal* if some world is accessible to it; otherwise a world is *non-normal*. See Hughes and Cresswell [3], especially chapter 15.

Given a wff A, a *K-valuation* on A is a function v from F_A into $\{t, f\} \times \{t, f\}$ such that (i) v is defined for V_A, and, if $\sim B$, $B \supset C$, or $\Box B$ are wfps of A, then, (ii) $v(\sim B) = \sim v(B)$, (iii) $v(B \supset C) = v(B) \supset v(C)$, and (iv) $v(\Box B) = \langle t, f\rangle$ if $v(B) = \langle t, t\rangle$, and $v(\Box B) = \langle f, f\rangle$ if $v(B) \neq \langle t, t\rangle$, where $\sim\langle i, j\rangle = \langle \sim i, \sim j\rangle$ and $\langle i, j\rangle \supset \langle k, l\rangle = \langle i \supset k, j \supset l\rangle$, and where $\sim i$ and $i \supset j$ are defined in the usual two-valued Boolean manner.

The conditions on K-valuations can be expressed in matrix form as follows. Matrix K:

$\langle i, j\rangle \supset \langle k, l\rangle$	$\langle t, t\rangle$	$\langle t, f\rangle$	$\langle f, t\rangle$	$\langle f, f\rangle$	$\sim\langle i, j\rangle$	$\Box\langle i, j\rangle$
$\langle t, t\rangle$	$\langle t, t\rangle$	$\langle t, f\rangle$	$\langle f, t\rangle$	$\langle f, f\rangle$	$\langle f, f\rangle$	$\langle t, f\rangle$
$\langle t, f\rangle$	$\langle t, t\rangle$	$\langle t, t\rangle$	$\langle f, t\rangle$	$\langle f, t\rangle$	$\langle f, t\rangle$	$\langle f, f\rangle$
$\langle f, t\rangle$	$\langle t, t\rangle$	$\langle t, f\rangle$	$\langle t, t\rangle$	$\langle t, f\rangle$	$\langle t, f\rangle$	$\langle f, f\rangle$
$\langle f, f\rangle$	$\langle t, t\rangle$	$\langle t, t\rangle$	$\langle t, t\rangle$	$\langle t, t\rangle$	$\langle t, t\rangle$	$\langle f, f\rangle$

We say that a wff A is *K-valid* iff for all K-valuations v on A, $v(A) = \langle t, t\rangle$ or $v(A) = \langle t, f\rangle$. Inspection of matrices M and K show that M and K are isomorphic under the correspondence of 1 with $\langle t, t\rangle$, 2 with $\langle t, f\rangle$, 3 with $\langle f, t\rangle$ and 4 with $\langle f, f\rangle$. Hence, we have the following theorems.

Theorem 4: *A wff A is M-valid iff A is K-valid.*
Theorem 5: *A wff A is provable in* S9 *iff A is K-valid.*

For the purposes of the next section it will be convenient to be able to separate what a K-valuation does to wffs in the world G from what it does to wffs in the world H. Thus we introduce some additional notions.

f_G and f_H are functions from $\{t, f\} \times \{t, f\}$ into $\{t, f\}$ such that $f_G(\langle i, j\rangle) = i$ and $f_H(\langle i, j\rangle) = j$.

A *K_G-valuation* is a function $v_G = f_G \circ v$, where v is a K-valuation, and a *K_H-valuation* is a function $v_H = f_H \circ v$. Hence, for a K-valuation v, $v(A) = \langle i, j\rangle$ iff $v_G(A) = i$ and $v_H(A) = j$. We shall say that a wff A is *K_G-valid* iff for all K_G-valuations v_G on A, $v_G(A) = t$.

Theorem 6: *A wff A is K-valid iff A is K_G-valid.*

The conditions on K-valuations given above can just as well be stated in terms of K_G and K_H-valuations.

K0.1. *If* $v_G(A) = f$, *then* $v_G(\sim A) = t$.
K0.2. *If* $v_H(A) = f$, *then* $v_H(\sim A) = t$.
K1.1. *If* $v_G(A) = t$, *then* $v_G(\sim A) = f$.
K1.2. *If* $v_H(A) = t$, *then* $v_H(\sim A) = f$.
K2.1. *If* $v_G(\sim A) = t$ *or* $v_G(B) = t$, *then* $v_G(A \supset B) = t$.
K2.2. *If* $v_H(\sim A) = t$ *or* $v_H(B) = t$, *then* $v_H(A \supset B) = t$.
K3.1. *If* $v_G(A) = t$ *and* $v_G(\sim B) = t$, *then* $v_G(\sim(A \supset B)) = t$.
K3.2. *If* $v_H(A) = t$ *and* $v_H(\sim B) = t$, *then* $v_H(\sim(A \supset B)) = t$.
K4.1. *If* $v_G(A) = t$ *and* $v_H(A) = t$, *then* $v_G(\Box A) = t$.
K5.1. *If* $v_G(\sim A) = t$ *or* $v_H(\sim A) = t$, *then* $v_G(\sim\Box A) = t$.
K5.2. $v_H(\sim\Box A) = t$.

The conditions *K0.1-K5.2* provide us with a *semantic basis* for still another formulation of S9, a *Gentzen* formulation which we shall now introduce.[4]

2.1 *The System* LS9 We shall let α, β, γ, δ range over finite and possibly null sequences of wffs in $\sim$, $\supset$, and $\Box$. If we want to explicitly indicate the null sequence, we shall write Λ. An LS9-*sequent* is a pair of sequences $(\alpha; \beta)$, where at least one of α and β is not null. The sequence α is called the *normal* subsequent and β the *non-normal* subsequent of $(\alpha; \beta)$. (The basic idea of a sequent $A_1, \ldots, A_m; B_1, \ldots, B_n$ is that at least one of the A_i in the normal subsequent is true in the normal world G, *or else*, that at least one of the B_j in the non-normal subsequent is true in the non-normal world H.) If a sequence α contains at least the wffs $A_1, \ldots, A_n$, we shall write $\alpha(A_1, \ldots, A_n)$. The axiom-schemes of LS9 are

PC1. $\quad \alpha(A, \sim A); \beta$
PC2. $\quad \alpha; \beta(A, \sim A)$
$\sim\Box I2.$ $\quad \alpha; \beta(\sim\Box A)$

Examples of LS9-axioms are $(\sim(A \supset B), \Box C, A \supset B; \Box A)$, and $(\sim(A \supset A); B, B, \sim B, B)$, and $(\Lambda; \sim(A \supset A), \sim\Box(B \supset C))$. The rules of LS9 are[5]

$\sim I1.$ $\quad (\alpha, A, \beta; \gamma) \rightarrow (\alpha, \sim\sim A, \beta; \gamma)$
$\sim I2.$ $\quad (\alpha; \beta, A, \gamma) \rightarrow (\alpha; \beta, \sim\sim A, \gamma)$
$\supset I1.$ $\quad (\alpha, \sim A, B, \beta; \gamma) \rightarrow (\alpha, A \supset B, \beta; \gamma)$
$\supset I2.$ $\quad (\alpha; \beta, \sim A, B, \gamma) \rightarrow (\alpha; \beta, A \supset B, \gamma)$
$\sim\supset I1.$ $\quad (\alpha, A, \beta; \gamma)$ *and* $(\alpha, \sim B, \beta; \gamma) \rightarrow (\alpha, \sim(A \supset B), \beta; \gamma)$
$\sim\supset I2.$ $\quad (\alpha; \beta, A, \gamma)$ *and* $(\alpha; \beta, \sim B, \gamma) \rightarrow (\alpha; \beta, \sim(A \supset B), \gamma)$
$\Box I1.$ $\quad (\alpha, A, \beta; \gamma, \delta)$ *and* $(\alpha, \beta; \gamma, A, \delta) \rightarrow (\alpha, \Box A, \beta; \gamma, \delta)$
$\sim\Box I1.$ $\quad (\alpha, \sim A, \beta; \gamma, \sim A, \delta) \rightarrow (\alpha, \sim\Box A, \beta; \gamma, \delta)$

We shall use Γ, Δ, Φ, and Z to range over LS9-sequents. A *derivation* for a sequent Φ in LS9 is a *tree* beginning with the *node* Φ and branching upwards such that if a single node Γ is directly above a node Δ then there is an LS9 rule $\Gamma \rightarrow \Delta$ and if a branch splits at a node Z into nodes Γ and Δ then there is an LS9 rule (Γ *and* Δ) $\rightarrow$ Z; and such that the *terminal* nodes are LS9 axioms.

We shall say that a wff A is provable in LS9 iff there is a derivation for A; Λ in LS9. In the writing of derivations we shall separate the nodes by horizontal lines, to the right of which lines we cite the rule by which the next node comes. We illustrate the proof technique with the following examples.

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4. The author is indebted to Mr. Alasdair Urquhart for his help in formulating the *Kripke* semantics for S9 and the axioms and rules of LS9.

5. The rules (and axioms) are abbreviated mnemonically: "*I*" indicates that the rule is an *introduction* rule; what precedes "*I*" indicates what connective is being introduced; and "*1*" and "*2*" indicate introduction in normal and non-normal subsequents respectively. Later we use "*E*" to indicate *elimination* rules.

Example 1.

$$\dfrac{\dfrac{}{\sim\Box A;\ \sim\Box A}\ {\sim\Box I2}}{\sim\Box\Box A;\ \Lambda}\ {\sim\Box I1}$$

Example 2.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\sim\Box A,\ \Box A;\ \Lambda}\ PC1 \qquad \dfrac{}{\Box A;\ \sim\Box A}\ {\sim\Box I2}}{\Box\sim\Box A,\ \Box A;\ \Lambda}\ \Box I1}{\sim\sim\Box\sim\Box A,\ \Box A;\ \Lambda}\ {\sim I1}}{\sim\Box\sim\Box A \supset \Box A;\ \Lambda}\ {\supset I1}$$

Thus, the wffs $\sim\Box\Box A$ and $\Diamond\Box A \supset \Box A$ are provable in **LS9**. It may be shown that a wff A is provable in **PC** iff there is a derivation for $A;\ \Lambda$ (using only *PC1*, $\sim I1$, $\supset I1$, and $\sim\supset I1$) and there is a derivation for $\Lambda;\ A$ (using only *PC2*, $\sim I2$, $\supset I2$, and $\sim\supset I2$). Hence we have the following *derived* rule:

> *PC3. If* t *is any tautology of* **PC**, *to begin a derivation with* $\alpha(\mathrm{t});\ \beta$ *or with* $\alpha;\ \beta(\mathrm{t})$.

Example 3.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\sim A,\mathrm{t},\sim A,\Box A;\ \Lambda}\ PC3 \quad \dfrac{}{\sim A,\sim A,\Box A;\ \mathrm{t}}\ PC3}{\sim A,\ \Box\mathrm{t},\ \sim A,\ \Box A;\ \Lambda}\ \Box I1}{A\supset\Box\mathrm{t},\ \sim A,\ \Box A;\ \Lambda}\ {\supset I1} \qquad \dfrac{\dfrac{\dfrac{}{\sim A,A;\ \sim A,\Box\mathrm{t}}\ PC1 \quad \dfrac{}{\sim A;\ \sim A,A,\Box\mathrm{t}}\ PC2}{\sim A,\ \Box A;\ \sim A,\ \Box\mathrm{t}}\ \Box I1}{\sim A,\ \Box A;\ A\supset\Box\mathrm{t}}\ {\supset I2}}{\Box(A\supset\Box\mathrm{t}),\ \sim A,\ \Box A;\ \Lambda}\ \Box I1}{\sim\sim\Box(A\supset\Box\mathrm{t}),\ \sim A,\ \Box A;\ \Lambda}\ {\sim I1}}{\sim\sim\Box(A\supset\Box\mathrm{t}),\ A\supset\Box A;\ \Lambda}\ {\supset I1}}{\sim\Box(A\supset\Box\mathrm{t})\supset(A\supset\Box A);\ \Lambda}\ {\supset I1}$$

2.2 As was suggested in **2.1**, **LS9**-sequents are to be thought of in terms of the *Kripke* semantics for **S9**. Given a sequent $(\alpha;\ \beta)$ we are to think of it as the assertion that at least one of the wffs in α is true in G, or else, that at least one of the wffs in β is true in H. With this understanding the axioms and rules of **LS9** correspond exactly with conditions *K0.1-K5.2* given in **1.4.** Thus, it should turn out that the *Kripke* semantics for **S9** is also the semantics for **LS9**, and this being the case, it should also turn out that **LS9** is deductively equivalent to **S9**. The remainder of this section will be devoted to proving these results.

A *full construction on* a sequent Φ in **LS9** is a tree beginning with the node Φ and branching upwards such that if a single node Γ is directly above a node Δ then there is an **LS9** rule $\Gamma \to \Delta$ and if a branch splits at a node Z into nodes Γ and Δ then there is an **LS9** rule (Γ *and* Δ) $\to$ Z; and such that if Z is a terminal node, then Z is an **LS9** axiom or else there is no sequent Γ (or sequents Γ and Δ) such that $\Gamma \to$ Z (or (Γ *and* Δ) $\to$ Z). We note that terminal nodes in full constructions are **LS9** axioms or else non-axiom sequents with only variables or the negations of variables, except that the non-normal subsequents possibly have wffs of the form $\Box x$. Moreover, the following two lemmas are immediate consequences of our definitions.

Lemma 1: *For each* LS9-*sequent* Φ *there is a full construction on* Φ *in* LS9.
Lemma 2: D *is a derivation for* Φ *iff* D *is a full construction on* Φ *all of whose terminal nodes are* LS9 *axioms*.

If $\alpha = A_1, \ldots, A_m$, and v is a K-valuation, we write $v_G(\alpha) = \mathrm{t}$ if for some A_i in α, $v_G(A_i) = \mathrm{t}$; otherwise $v_G(\alpha) = \mathrm{f}$, and similarly for v_H. We also write $v(\alpha; \beta) = \mathrm{t}$ if $\alpha \neq \Lambda$ and $v_G(\alpha) = \mathrm{t}$, or if $\beta \neq \Lambda$ and $v_H(\beta) = \mathrm{t}$; otherwise we write $v(\alpha; \beta) = \mathrm{f}$. If $v(\alpha; \beta) = \mathrm{t}$, we shall say that $(\alpha; \beta)$ is *satisfied by* v, and if $v(\alpha; \beta) = \mathrm{f}$, we shall say that v *falsifies* $(\alpha; \beta)$. Inspection of the axioms and rules will verify the following facts for any K-valuation v:

fact 1. The axioms of LS9 are satisfied by v.
fact 2. If Γ is satisfied by v and $\Gamma \to \Phi$, then Φ is satisfied by v; and, if Γ and Δ are satisfied by v and (Γ *and* Δ) $\to \Phi$, then Φ is satisfied by v.
fact 3. if v falsifies Γ and $\Gamma \to \Phi$, or (Γ *and* Δ) $\to \Phi$, then v falsifies Φ.

These facts give us the following lemmas and theorems.

Lemma 3: *Given a* K-*valuation* v, *if every terminal node in a full construction on* Φ *is satisfied by* v, *then* Φ *is satisfied by* v.
Lemma 4: *Given a* K-*valuation* v, *if* v *falsifies any node in a full construction on* Φ, *then* v *falsifies* Φ.
Theorem 1: *A wff* A *is* K_G-*valid iff there is a derivation for* A; Λ.

Proof: Let D be a derivation for A; Λ, and let v be any K-valuation on A. Every terminal node in D is an axiom, and by lemma 2 D is a full construction on A; Λ. By lemma 3 and *fact 1*, A; Λ is satisfied by v so that $v_G(A) = \mathrm{t}$. Hence, if there is a derivation for A; Λ, A is K_G-valid. To show the converse, suppose there is no derivation for A; Λ. By lemma 1 there is a full construction on A; Λ, and by lemma 2 this full construction will have at least one non-axiom terminal node $(\alpha; \beta)$. Since $(\alpha; \beta)$ is terminal and non-axiom, the wffs in α (if there are any) are either variables or negations of variables, and hence are such that each wff can be assigned the value f without violating the conditions on a K_G-valuation. Similarly, since the wffs in β (if there are any; and, there must be some in β if there are none in α) are either variables, the negations of variables, or else wffs of the form $\Box x$, each wff can be assigned the value f without violating the conditions on a K_H-valuation. Let v, then, be a K-valuation such that $v_G(\alpha) = \mathrm{f}$ and $v_H(\beta) = \mathrm{f}$. Then v falsifies $(\alpha; \beta)$, and, by lemma 4, v falsifies A; Λ, so that A is not K_G-valid.

Theorem 2: *A wff* A *is provable in* S9 *iff* A *is provable in* LS9.

3. *The System* S9*. In this section we present S9*, a system of natural deduction.[6] We shall show that S9* is deductively equivalent to S9. The semantical motivation for S9* with regard to the *Kripke* semantics for S9 will be obvious: an NCP is the normal world G, and an NNCP is the non-normal world H.[7]

3.1 We give the notion of *proof* and the *proof technique* of S9*.

A *hypothetical proof* (HP) is a vertical sequence of wffs $A_1, \ldots, A_n$ $(n \geq 1)$ to the right of an *HP line* such that each wff in the sequence comes by an S9* rule.

A *non-normal categorical proof* (NNCP) is a vertical sequence of wffs $A_1, \ldots, A_n$ $(n \geq 0)$ to the right of an *NNCP line* such that each wff in the sequence comes by an S9* rule.

Wffs and proofs will be called *items*. P, Q, and R will range over proofs, and α and β will range over items. We continue to define the notion of *proof*: if P is a *proof* with items $\alpha_1, \ldots, \alpha_n$, and Q is a *proof*, then P' is a *proof* where P' has items $\alpha_1, \ldots, \alpha_n, Q$, *provided that* if P is an NNCP then Q is an HP.

A *normal categorical proof* (NCP) is a sequence of items to the right of an *NCP line* such that each item in the sequence comes by the rules of S9*. Note that a proof *qua item* is said to come by the rules of S9* if each item in the proof comes by the rules of S9*. In diagrams, we have:

$$\begin{array}{|l} A_1 \\ \hline . \\ . \quad \text{HP} \\ . \quad n \geq 1 \\ A_n \end{array} \qquad \begin{array}{||l} A_1 \\ . \\ . \quad \text{NNCP} \\ . \quad n \geq 0 \\ A_n \end{array} \qquad \begin{array}{|l} \alpha_1 \\ . \\ . \quad \text{NCP} \\ . \quad n \geq 0 \\ \alpha_n \end{array}$$

We introduce some more terminology. If P and Q are proofs such that P has items $\alpha_1, \ldots, Q, \ldots, \alpha_n$, then Q is called a (*first-order*) *subproof* of P. If, moreover, R is a subproof of Q, then R is a *second-order subproof* of P. A subproof of R is a *third-order subproof* of P and a second-order subproof of Q, and so on. Finally, if P is an NCP with the wff A as the last item, then we say that P is a *proof for A*.

The *proof technique* of S9* consists of the following sets of rules.

Rules of Auxiliary Transformation

R1. To draw an NCP line.

R2. To draw an HP line at any stage of the proof.

R3. To draw an NNCP line at any stage of the proof provided that we

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6. S9* is styled after the manner of Fitch's *method of subordinate proof*, [2], with which we assume familiarity.

7. For the definitions of NCP and NNCP see section **3.1**. In general, categorical proofs (CP) are possible worlds, an NCP being some normal world and an NNCP being some non-normal world. If Q is a subCP of P, then Q is accessible to P; since non-normal worlds have no worlds accessible to them, no CP is a subproof, of any order, of an NNCP.

do not violate the following restriction on the nesting of NNCPs: no NNCP may be a subproof of any order of another NNCP.

Rules of Inference

HYP. To write A as the first item (the hypothesis) in an HP.

REIT. If Q is a subHP of P, and if A is an item in P, to write A in Q.

$\supset$I. If Q is a subHP of P, where Q has hypothesis A and last item B, to write $A \supset B$ in P.

$\supset$E. If A and $A \supset B$ are items in P, to write B in P.

RAA. If Q is a subHP of P, and Q has hypothesis $\sim A$ and items B and $\sim B$, to write A in P.

AX. To write $\sim\Box A$ in an NNCP.

$\Box$E. If $\Box A$ is an item in P, to write A in P (N$\Box$E), or if, moreover, Q is a subNNCP of P, to write A in Q (NN$\Box$E).

$\Diamond$E. If A and $\sim\Box A$ are items in P, and if Q is a subNNCP of P, to write $\sim A$ in Q.

NN$\sim$I. If P is an HP with hypothesis A and contains a subNNCP with items B and $\sim B$, to write $\sim A$ in P.

We leave it to the reader to verify that the following rules can be derived in S9*.

$\Box$I. If A is an item in P and also in a subNNCP of P, to write $\Box A$ in P.

$\Diamond$I. If $\sim A$ is an item in a subNNCP of P, to write $\sim\Box A$ in P.

PC. If t is a tautology of PC, to write t in any proof and $\Box$t in any HP not nested in an NNCP.

We shall say that a wff A is *provable* in S9* iff there is a proof for A in S9*. In the writing of proofs we shall adopt the convention of consecutively numbering each wff to the left of the NCP line in the order that they are written and of writing a justification for each wff to its right. Also, we shall suppress any use of the rules of auxiliary transformation. We illustrate the proof technique with the following examples.

Example 1.

1.	$\vert\ \Vert \sim\Box A$	AX
2.	$\vert \sim\Box\Box A$	1, $\Diamond$I

Example 2.

1.	$\vert\ \vert\ \underline{\sim\Box\sim A}$	HYP
2.	$\vert\ \vert\ \ \Vert \sim\Box\sim A$	AX
3.	$\vert\ \vert\ \ \Box\sim\Box\sim A$	1, 2, $\Box$I
4.	$\vert\ \Diamond A \supset \Box\Diamond A$	1-3, $\supset$I

Example 3.

1.	$\vert\ \vert\ \underline{\sim\Box(A \supset \Box\text{t})}$	HYP
2.	$\vert\ \vert\ \ \vert\ \underline{A}$	HYP
3.	$\vert\ \vert\ \ \vert\ \sim\Box(A \supset \Box\text{t})$	1, REIT
4.	$\vert\ \vert\ \ \vert\ \ \vert\ \underline{A}$	HYP
5.	$\vert\ \vert\ \ \vert\ \ \vert\ \Box\text{t}$	PC

6.	$\vert\ \vert\ \ \vert A \supset \Box\mathrm{t}$	4-5, $\supset$I
7.	$\vert\ \vert\ \ \vert\ \ \Vert \sim(A \supset \Box\mathrm{t})$	3, 6, $\Diamond$E
8.	$\vert\ \vert\ \ \vert\ \ \Vert \sim(A \supset \Box\mathrm{t}) \supset A$	PC
9.	$\vert\ \vert\ \ \vert\ \ \Vert A$	7, 8, $\supset$E
10.	$\vert\ \vert\ \ \vert \Box A$	2, 9, $\Box$I
11.	$\vert\ \vert\ A \supset \Box A$	2-10, $\supset$I
12.	$\vert\ \sim\Box(A \supset \Box\mathrm{t}) \supset (A \supset \Box A)$	1-11, $\supset$I

3.2 We shall show that S9* is deductively equivalent to S9. We leave it to reader to show that the axioms of S9 are derivable in S9*. Moreover, the rules of S9 are rules of S9*. Hence, if a wff is provable in S9, it is provable in S9*. To show the converse, we introduce the notion of *quasi-proof* and prove some lemmas.[8]

Lemma 1: *The following theorems of* S9 *are derivable in any* HP *and any* NNCP:

1. $A \supset A$
2. $A \supset (B \supset A)$
3. $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$
4. $(A \supset \Box B) \supset (A \supset B)$
5. $(\sim A \supset B) \supset ((\sim A \supset \sim B) \supset A)$
6. $A_1 \supset (A_2 \supset \ldots (A_{n-1} \supset (A_n \supset T)) \ldots)$, where $n \geq 1$ and T is one of the theorems 1-5.

The following theorems of S9 are derivable in any HP not nested within an NNCP.

7. $\Box(\sim\Box\mathrm{t} \supset \sim\Box A)$
8. $\Box A \supset \Box(\sim\Box\mathrm{t} \supset A)$
9. $A \supset (\sim\Box A \supset \Box(\sim\Box\mathrm{t} \supset \sim A))$
10. $\Box(\sim\Box\mathrm{t} \supset A) \supset (\Box(\sim\Box\mathrm{t} \supset (A \supset B)) \supset \Box(\sim\Box\mathrm{t} \supset B))$
11. $A \supset (\Box(\sim\Box\mathrm{t} \supset B) \supset (\Box(\sim\Box\mathrm{t} \supset \sim B) \supset \sim A))$
12. $\Box(\sim\Box\mathrm{t} \supset T)$, where T is one of the theorems 1-6
13. $A_1 \supset (A_2 \supset \ldots (A_{n-1} \supset (A_n \supset T)) \ldots)$, where $n \geq 1$ and T is one of the theorems 7-12.

Proof: Using the methods of **1** or **2** we show that the formulas are S9 theorems. Then using the method of **3.1** we establish the lemma. By lemma 1 come two *derived* rules which will be helpful in showing the equivalence of S9* to S9.

THM1. To insert any of the theorems 1-6 in an HP or an NNCP.

THM2. To insert any of the theorems 7-13 in a HP not nested within an NNCP.

Now, a *quasi-proof* is a proof such that each item either comes by the rules

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8. The method of proof used to prove lemmas 2-4 and theorem 1 of this section is an adaption of Anderson and Belnap [1], which in turn stems from Fitch [2].

of S9* or else comes by the rules THM1 or THM2. Lemma 1 guarantees that whenever we use THM1 or THM2 we could have used the rule REIT instead.

Lemma 2: *If* Q, *with items* $A_1, \ldots, A_n$, *is a* subHP *of* P, *where* Q *itself contains no proofs, then* P *can be converted into a quasi-proof in which* Q *is replaced by the items* $A_1 \supset A_1, \ldots, A_1 \supset A_n$ *and some (appropriate) theorems of* S9.

Proof: Let P be a proof containing the proof Q, an HP with items $A_1, \ldots,$ A_n. Q contains no subproofs. Let Q' be the sequence of wffs $A_1 \supset A_1, \ldots,$ $A_1 \supset A_n$, and let P' be the result of replacing Q by Q' in P. We shall show that P' can be converted into a quasi-proof P'' by showing how to insert theorems of S9 among the items of P'. Assuming that the latter holds for items $A_1, \ldots, A_{i-1}$ in $Q\,(i \geq 1)$, we show that it holds for the item A_i in Q, thereby proving the lemma.

If A_i is an item in Q, then A_i cannot have come by $\supset$I, RAA, AX, $\Diamond$E, nor NN$\sim$I, but only by HYP, REIT, $\supset$E, or N$\Box$E.

Case 1. A_i comes by HYP. Then $A_i = A_1$. Insert $A_1 \supset A_1$ in P' (THM1).

Case 2. A_i comes by REIT. Then A_i is already in P. Insert $A_i \supset (A_1 \supset A_i)$ in P' (THM1), and use $\supset$E to get $A_1 \supset A_i$.

Case 3. A_i comes by $\supset$E. Then A_i comes from B and $B \supset A_i$ in Q. By hypothesis $A_1 \supset B$ and $A_1 \supset (B \supset A_i)$ are already in P'. Insert $(A_1 \supset B) \supset ((A_1 \supset (B \supset A_i)) \supset (A_1 \supset A_i))$ in P' (THM1) to get $A_1 \supset A_i$ by $\supset$E.

Case 4. A_i comes by N$\Box$E. Then A_i comes from $\Box A_i$ in Q. By hypothesis $A_1 \supset \Box A_i$ is already in P'. Insert $(A_1 \supset \Box A_i) \supset (A_1 \supset A_i)$ in P' (THM1) to get $A_1 \supset A_i$ by $\supset$E.

We note, finally, that if an item B in P follows from items in Q, then it does so by either $\supset$I or by RAA. If the former, then $B = A_1 \supset A_n$. But $A_1 \supset A_n$ is in P' by cases 1-4. So use REIT to get B. If the latter, then Q has hypothesis $A_1 = \sim B$ and some items A_j and $\sim A_j$. By cases 1-4 $\sim B \supset A_j$ and $\sim B \supset \sim A_j$ are in P'. Insert $(\sim B \supset A_j) \supset ((\sim B \supset \sim A_j) \supset B)$ in P' (THM1) and use $\supset$E to get B.

Lemma 3: *If an* NNCP Q, *with items* $A_1, \ldots, A_n$, *is an item in a proof* P, *where* Q *itself contains no proofs, then* P *can be converted into a quasi-proof in which* Q *is replaced by the items* $\Box(\sim\Box\mathrm{t} \supset A_1), \ldots, \Box(\sim\Box\mathrm{t} \supset A_n)$ *and some (appropriate) theorems of* S9.

Proof: As in lemma 2 we shall show how to insert theorems of S9 among the wffs $\Box(\sim\Box\mathrm{t} \supset A_1), \ldots, \Box(\sim\Box\mathrm{t} \supset A_n)$ in P' such that all the items in P'' are justified by the rules of S9* or by the rules THM1 or THM2. Again, the proof is by induction on A_i. A_i cannot have come by HYP, REIT, $\supset$I, RAA, nor NN$\sim$I, but only by AX, NN$\Box$E, $\Diamond$E or $\supset$E.

Case 1. A_i comes by AX. Then $A_i = \sim\Box B$. Insert $\Box(\sim\Box\mathrm{t} \supset \sim\Box B)$ in P' (THM2).

Case 2. A_i comes by NN$\Box$E. Then A_i in Q comes from $\Box A_i$ in P. Insert $\Box A_i \supset \Box(\sim\Box\mathrm{t} \supset A_i)$ in P' (THM2) and use $\supset$E to get $\Box(\sim\Box\mathrm{t} \supset A_i)$.

Case 3. A_i comes by $\Diamond$E. Then $A_i = \sim B$ and $\sim B$ in Q comes from B and

$\sim\Box B$ in P. Insert $B \supset (\sim\Box B \supset \Box(\sim\Box \mathrm{t} \supset \sim B))$ in P' (THM2) and use $\supset$E to get $\Box(\sim\Box \mathrm{t} \supset A_i)$.

Case 4. A_i comes by $\supset$E. Then A_i comes from B and $B \supset A_i$ in Q. By hypothesis $\Box(\sim\Box \mathrm{t} \supset B)$ and $\Box(\sim\Box \mathrm{t} \supset (B \supset A_i))$ are already in P'. Insert $\Box(\sim\Box \mathrm{t} \supset B) \supset (\Box(\sim\Box \mathrm{t} \supset (B \supset A_i)) \supset \Box(\sim\Box \mathrm{t} \supset A_i))$ in P' (THM2) and use $\supset$E to get $\Box(\sim\Box \mathrm{t} \supset A_i)$.

Finally, if an item B in P follows from items in the NNCP Q, then it does so only by NN~I. In that case P has hypothesis A and Q has items C, $\sim C$, where $B = \sim A$. By cases 1-4 $\Box(\sim\Box \mathrm{t} \supset C)$ and $\Box(\sim\Box \mathrm{t} \supset \sim C)$ are in P'. Insert $A \supset (\Box(\sim\Box \mathrm{t} \supset C) \supset (\Box(\sim\Box \mathrm{t} \supset \sim C) \supset \sim A))$ in P' and use $\supset$E to get B.

Lemma 4: *If P is a proof for the wff A, then P can be converted into a quasi-proof P'' such that every item in P'' is either a theorem of* S9 *or else comes by the rules* REIT, $\supset$E, *or* N$\Box$E.

Proof: Let P be a proof for A. We shall suppose for convenience that P contains only the proofs $Q_1, \ldots, Q_n$, where Q_i is an i-order subproof of P.

Step 1. By lemmas 2 and 3 Q_{n-1} can be converted into a quasiproof Q''_{n-1} containing no subproof and containing some S9 theorems.

Step 2. Let us assume that Q_i was converted into a quasi-proof Q''_i containing no subproof and containing some S9 theorems. We show that Q_{i-1} can be converted into such a quasiproof also.

Case 1. Q''_i is a quasi-HP with hypothesis A. Every item in Q''_i came by the rules of S9* or by THM1, or else, every item in Q''_i came by the rules of S9* or by THM2.

Case 1.1. If an item T in Q''_i came by THM1, then we insert $A \supset T$ in Q'_{i-1} by THM1.

Case 1.2. If an item T in Q''_i came by THM2, then we insert $A \supset T$ in Q'_{i-1} by THM2.

Case 1.3. If an item B in Q''_i came by an S9* rule, then we insert theorems of S9 as in lemma 2 by THM1, replacing B in Q''_i by $A \supset B$ in Q'_{i-1}.

Case 2. Q''_i is a quasi-NNCP. Every item in Q''_i came by the rules of S9* or by THM1.

Case 2.1. If an item T in Q''_i came by THM1, then we insert $\Box(\sim\Box \mathrm{t} \supset T)$ in Q'_{i-1} by THM2.

Case 2.2. If an item A in Q''_i came by an S9* rule, then we insert theorems of S9 as in lemma 3 by THM2, replacing A in Q''_i by $\Box(\sim\Box \mathrm{t} \supset A)$ in Q'_{i-1}.

Hence, if Q''_i is a quasi-proof containing no subproof, then cases 1 and 2 show how to convert Q_{i-1} into a quasi-proof containing no subproof, such that every item is either a theorem of S9 or else comes by the rules of S9*. By *Step 1* and *Step 2*, it follows that if P is a proof for A, then P can be converted into a quasi-proof P'' such that every item in P'' is either a theorem of S9 or else comes by the rules REIT, $\supset$E, or N$\Box$E.

Theorem 1: *A wff A is provable in* S9 *iff A is provable in* S9*.

Proof: From lemma 4 it follows that if P is a proof for A, then P can be converted into a quasi-proof P'' such that *every* item in P'' is a theorem of S9 (since $\Box A \supset A$ is a theorem of S9), in which case A is also a theorem of

S9. Thus, we have shown that if a wff A is provable in S9*, then it is provable in S9 so that we have the theorem.

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University of Pittsburgh
Pittsburgh, Pennsylvania