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A NOTE ON THE Q-TOPOLOGY

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1 *Preliminaries* In this note we study Abraham Robinson's Q-topology and consider it as a means of constructing counter examples in topology. We shall be interested almost exclusively with separation and disconnectedness conditions in the Q-topology. For instance, we shall show that the Q-topology for a non-discrete completely regular space is a non-discrete zero-dimensional space in every enlargement. The reader is assumed to know what is meant by an enlargement in the sense of Robinson, what is meant by an ultraproduct enlargement, and to be familiar with the rudiments of non-standard analysis. A good short introduction is sections **1-6** of [6]. We generalize the Q-topology somewhat by introducing the notion of a *topological space and the Q-topology for a *topological space. This will help in dealing with subspaces and will give a slightly simpler notation.

Definition 1: A *topological space in a non-standard model $*\mathfrak{M}$ is a pair $(X,\mathfrak{T})$, where X is an internal set in $*\mathfrak{M}$ and $\mathfrak{T} \subset \mathcal{P}(X)$ is an internal family of sets closed under *finite intersections and internal unions, and which contains $\emptyset$ and X.

If $(X,\mathfrak{T})$ is a topological space in a model $\mathfrak{M}$, then $*(X,\mathfrak{T}) = (*X, *\mathfrak{T})$ is a (standard) *topological space in any enlargement $*\mathfrak{M}$ of $\mathfrak{M}$. If $\mathfrak{G}$ is an infinite collection of topological spaces in a model $\mathfrak{M}$, then $*\mathfrak{G}$ contains a non-standard *topological space for any enlargement $*\mathfrak{M}$ of $\mathfrak{M}$.

Definition 2: For any *topological space $(X,\mathfrak{T})$ the topology on X for which $\mathfrak{T}$ is a base is called the Q-topology and is denoted by $\overline{\mathfrak{T}}$.

Let us consider some of the basic relationships between $\mathfrak{T}$ and $\overline{\mathfrak{T}}$ for any *topological space $(X,\mathfrak{T})$. Abraham Robinson showed that for any topological space $(X,\mathfrak{T})$ and arbitrary enlargement $*(X,\mathfrak{T})$, each internal subset of $*X$ is *open iff it is Q-open and *closed iff it is Q-closed, and that for any internal $U \subseteq *X$, the *closure of U coincides with its Q-closure, the *interior of U coincides with its Q-interior, and so on. The proof of these facts relies in no way upon the fact that $*(X,\mathfrak{T})$ is a standard

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*topological space in an enlargement, and these results hold in general. Also, if $(X, \mathfrak{T})$ is a *topological space and $U \subseteq X$ is internal, then the Q-topology on U for the *subspace topology is the subspace topology on U generated by the Q-topology on X. An internal function between *topological spaces is *continuous iff it is Q-continuous, a *homeomorphism iff it is a Q-homeomorphism, *closed iff it is Q-closed, and *open iff it is Q-open when the *topological spaces are given their respective Q-topologies.

2 *Separation axioms* The major relationships between the separation properties for $\mathfrak{T}$ and $\overline{\mathfrak{T}}$ are laid out in this section. There are still interesting unsolved problems in this area and we will indicate some.

Theorem 1 *A *topological space $(X, \mathfrak{T})$ is a $*T_1$-space iff $(X, \overline{\mathfrak{T}})$ is a T_1-space.*

Proof: A *topological space $(X, \mathfrak{T})$ is a $*T_1$-space iff for each $x \in X$, $\{x\}$ is *closed. For each $x \in X$, $\{x\}$ is internal, so $(X, \mathfrak{T})$ is a $*T_1$-space iff for each $x \in X$, $\{x\}$ is Q-closed, or iff $(X, \mathfrak{T})$ is a T_1-space.

Theorem 2 *A *topological space $(X, \mathfrak{T})$ is a $*T_2$-space iff $(X, \overline{\mathfrak{T}})$ is a T_2-space.*

Proof: Suppose that $(X, \mathfrak{T})$ is a $*T_2$-space and let x and y be two points in X. Then there exist disjoint *open (and hence Q-open) *neighborhoods of x and y. Conversely, suppose that $(X, \overline{\mathfrak{T}})$ is a T_2-space. Then for any two points x, $y \in X$ there exist disjoint Q-neighborhoods U and V of x and y respectively containing *open *neighborhoods of x and y respectively.

A topological space $(X, \mathfrak{T})$ is said to be completely Hausdorff iff for any two points x, $y \in X$ there exist neighborhoods U of x and V of y such that $\overline{U} \cap \overline{V} = \emptyset$.

We omit the proofs of the following three theorems.

Theorem 3 *A *topological space $(X, \mathfrak{T})$ is a *completely Hausdorff space iff $(X, \overline{\mathfrak{T}})$ is completely Hausdorff.*

Theorem 4 *If $(X, \mathfrak{T})$ is a *semiregular space, then $(X, \overline{\mathfrak{T}})$ is semiregular.*

Theorem 5 *A *topological space $(X, \mathfrak{T})$ is a $*T_3$-space iff $(X, \overline{\mathfrak{T}})$ is a T_3-space.*

There are examples of ultraproduct enlargements of normal spaces for which the Q-topology is not normal, but it is not known whether the normality of $(*X, *\overline{\mathfrak{T}})$ implies the normality of $(X, \mathfrak{T})$. We shall show later that the perfect normality of $(X, \mathfrak{T})$ does not imply that $(*X, *\overline{\mathfrak{T}})$ is perfectly normal. We also remark that a *topological space $(X, \mathfrak{T})$ is *discrete iff $(X, \overline{\mathfrak{T}})$ is discrete. Beyond this point the *separation properties of $(X, \mathfrak{T})$ are interwoven with the disconnectedness properties of $(X, \overline{\mathfrak{T}})$.

3 *Disconnectedness properties in general enlargements* One of the greatest differences between $\mathfrak{T}$ and $\overline{\mathfrak{T}}$ for enlargements lies in the area of disconnectedness properties. In [1] we showed that for any family $\mathfrak{U}$ of open

sets in any topological space $(X,\mathfrak{T})$, $\bigcap\{{}^*U\colon U\in\mathfrak{U}\}$ is Q-open. We also showed that any topological space is regular iff the monad of each point is Q-closed and normal iff the monad of each closed set is Q-closed. While these statements do not apply to non-standard *topological spaces, they show that $({}^*X,{}^*\overline{\mathfrak{T}})$ can be very badly disconnected whether or not $(X,\mathfrak{T})$ is connected. We remind the reader of the following definitions.

A topological space $(X,\mathfrak{T})$ is said to be totally separated iff any two points in X can be separated by a clopen set.

A topological space $(X,\mathfrak{T})$ is said to be zero-dimensional iff each point in X has a neighborhood base of clopen sets.

Theorem 6 *Let $(X,\mathfrak{T})$ be a *Urysohn space in an enlargement ${}^*\mathfrak{M}$. Then $(X,\overline{\mathfrak{T}})$ is totally separated.*

Proof: Let $(X,\mathfrak{T})$ be a *Urysohn space, let x and y be any two points in X, and let $f\colon X\to {}^*\mathbf{R}$ be a *continuous function such that $f(x)=1$ and $f(y)=0$. The regularity of $\mathbf{R}$ implies that $\mu(0)$ is Q-clopen and f is Q-continuous, so $f^{-1}(\mu(0))$ is Q-clopen and separates x and y.

Note that if $(X,\mathfrak{T})$ is a *topological space in an enlargement ${}^*\mathfrak{M}$ and if $(X,\overline{\mathfrak{T}})$ is totally separated, then $(X,\overline{\mathfrak{T}})$ is completely Hausdorff, so $(X,\mathfrak{T})$ is *completely Hausdorff.

Theorem 7 *Let $(X,\mathfrak{T})$ be a *completely regular space in any enlargement ${}^*\mathfrak{M}$. Then $(X,\mathfrak{T})$ is zero-dimensional.*

Proof: Let $(X,\mathfrak{T})$ be *completely regular, let $F\subset X$ be Q-closed and let $x\in X-F$. Then there is a *closed $G\supset F$ such that $x\notin G$ and a *continuous function $f\colon X\to {}^*\mathbf{R}$ such that $f(x)=0$ and $f(y)=1$ for each $y\in G$. As before, $f^{-1}(\mu(0))\subset X-F$ is a Q-clopen neighborhood of x, so there is a neighborhood base at x composed of Q-clopen sets.

Similarly, we have the following:

Theorem 8 *Let $(X,\mathfrak{T})$ be a ${}^*\mathsf{T}_4$-space in an enlargement ${}^*\mathfrak{M}$. Then any two disjoint *closed subsets of X can be separated by a Q-clopen set.*

If $(X,\mathfrak{T})$ is a *topological space such that $(X,\overline{\mathfrak{T}})$ is zero-dimensional, then $(X,\overline{\mathfrak{T}})$ *is a T_3-space, so $(X,\mathfrak{T})$ is a ${}^*\mathsf{T}_3$-space.* We shall show in the next section that every T_3-space $(X,\mathfrak{T})$ has an enlargement ${}^*\mathfrak{M}$ for which $({}^*X,{}^*\overline{\mathfrak{T}})$ is zero-dimensional.

Note that if $(X,\mathfrak{T})$ is a *totally separated space, then $(X,\overline{\mathfrak{T}})$ is totally separated, and if $(X,\mathfrak{T})$ is *totally disconnected, then $(X,\overline{\mathfrak{T}})$ is totally disconnected. Also, if $(X,\overline{\mathfrak{T}})$ is extremally disconnected, then $(X,\mathfrak{T})$ is *extremally disconnected, and if $(X,\overline{\mathfrak{T}})$ is scattered, then $(X,\mathfrak{T})$ is *scattered. The converses of these four statements can fail. We shall see later how the first three converses can fail and the failure of the latter is shown by the following example. Unfortunately we do not know of an example of a scattered Hausdorff space whose Q-topology in some enlargement is not scattered, but regard this as an interesting question.

Example 1: Let $X = \{x \in \mathbf{Z}^+ \mid x \geq 2\}$ together with the topology $\mathfrak{T}$ generated by sets of the form $U_n = \{x \in X \mid x \text{ divides } n\}$. This topology, called the divisor topology, is scattered because each non-empty $U \subset X$ has a least element x, which is an isolated point in U [7]. Note that for each point p in X, U_p is the smallest neighborhood of p in X. Let $(*X, *\mathfrak{T})$ be any enlargement of $(X, \mathfrak{T})$, let $U = \{2^i \mid i \in \mathbf{Z}^+ \text{ is infinite}\}$, and let $k = 2^j$ be an arbitrary element of U. Then $2^{j-1} \in U_k$, and $(U_k - \{k\}) \cap U \neq \emptyset$, so $(*X, *\overline{\mathfrak{T}})$ is not scattered.

We can now give some counterexamples in topology.

Counterexample 1: If $(X, \mathfrak{T})$ is a Urysohn space which is not regular, then $(*X, *\overline{\mathfrak{T}})$ in any enlargement is an example of a totally separated space which is not zero-dimensional. For instance, let X be the set of real numbers, let $A = \left\{\frac{1}{n} \mid n = 1, 2, 3, \ldots\right\}$ and let $\mathfrak{T} = \{G \mid G = U - B$, where $B \subseteq A$ and U is an open set in the Euclidean topology on $X\}$. $\mathfrak{T}$ is called the Smirnov topology on X, and $(X, \mathfrak{T})$ is a connected Urysohn space which is not regular [7]. Hence, $(*X, *\mathfrak{T})$ is *connected and $(*X, *\overline{\mathfrak{T}})$ cannot be scattered or extremally disconnected, so $(*X, *\overline{\mathfrak{T}})$ is an example of a topological space which is totally separated but not zero-dimensional, extremally disconnected or scattered.

Counterexample 2: Any enlargement of the real numbers with the Euclidean topology is an example of a totally separated zero-dimensional space which is not extremally disconnected or scattered when given the Q-topology. We shall see in the next section that this counterexample can be improved by considering enlargements with special properties.

4 *Disconnectedness properties in special enlargements* We shall now consider the disconnectedness properties of the Q-topology for comprehensive and $\aleph_1$-saturated enlargements. Briefly, an enlargement $*\mathfrak{M}$ of $\mathfrak{M}$ is comprehensive iff for each set $C \in \mathfrak{M}$ and internal set $D \in *\mathfrak{M}$, each function $f: C \to D$ has an internal extension $g: *C \to D$. An enlargement $*\mathfrak{M}$ of $\mathfrak{M}$ is α-saturated, where α is an infinite cardinal, if whenever b is an internal binary relation in $*\mathfrak{M}$ concurrent on a subset A of its domain and $\text{card}(A) < \alpha$, then there exists an entity y in the range of b such that $b(x, y)$ holds for all $x \in A$. Note that A need not be internal and that b is only required to be concurrent on A and need not be concurrent on its entire domain. Note also that by the definition of a concurrent relation, every model is $\aleph_0$-saturated. It has been shown that an α-saturated model $*\mathfrak{M}$ exists for every model $\mathfrak{M}$ and infinite cardinal α. We will use Luxemburg's notation and write $\mathfrak{M} = \mathfrak{M}(Z, M)$, where Z is the set of individuals in $\mathfrak{M}$ and M is the set of entities in $\mathfrak{M}$. If $*\mathfrak{M}$ is an α-saturated model of $\mathfrak{M} = \mathfrak{M}(Z, M)$, where α is a cardinal greater than $\text{card}(M)$, then $*\mathfrak{M}$ is an enlargement of $\mathfrak{M}$ in the sense of Robinson.

A completely regular space $(X, \mathfrak{T})$ is said to be strongly zero-dimensional if any two disjoint zero-sets can be separated by a clopen set. A completely regular space is said to be a P-space if every prime ideal in

the ring of continuous real-valued functions is maximal. A convenient characterization for our purposes is that a completely regular space is a P-space iff every G_δ-set is open [3]. For this reason we shall assume that the models which we deal with in this section contain a countably infinite well-ordered set which we identify with the natural numbers.

Theorem 9 *If $(X,\mathfrak{T})$ is a $*T_3$-space in a comprehensive enlargement $*\mathfrak{M}$, then $(X,\overline{\mathfrak{T}})$ is a P-space.*

Proof: We shall show first that each G_δ-set in $(X,\overline{\mathfrak{T}})$ is open. Let $F \subset X$ be a G_δ-set, say $F = \bigcap_{i\in\mathbf{N}} G_i$, where each G_i is Q-open, and let p be any point in F. Also, let $\mathfrak{F} = \{U \in \mathfrak{T} \mid p \in U\}$ so that $\mathfrak{F}$, the *neighborhood filter at p, is an internal collection of sets in $*\mathfrak{M}$. Then we can find a sequence $s\colon \mathbf{N} \to \mathfrak{F}$ of *open *neighborhoods of p such that $G_i \supset s(i) \supset s(i+1)$ for each $i \in \mathbf{N}$. Let $s'\colon *\mathbf{N} \to \mathfrak{F}$ be any internal extension of s in $*\mathfrak{M}$.

If for some $i \in \mathbf{N}$, $s'(i) \not\supset s'(i+1)$, then let n be the least *natural number j such that $s'(j) \not\supset s'(j+1)$. If $s'(i) \supset s'(i+1)$ for each $i \in *\mathbf{N}$, then let n be an arbitrary infinite *natural number. Now, for each $m \in *\mathbf{N}$ with $m < n$, $s'(m) \supset s'(n)$, and for each finite $i \in *\mathbf{N}$, $G_i \supset s(i) = s'(i) \supset s'(n)$, or $\bigcap_{i\in\mathbf{N}} G_i \supset s'(n)$, so $s'(n)$ is a Q-open neighborhood of p contained in F.

Finally, we show that $(X,\overline{\mathfrak{T}})$ is completely regular. If $F \subset X$ is Q-closed and $x \in X - F$, then there is a sequence $\{U_i \mid i \in \mathbf{N}\}$ of Q-open neighborhoods of x such that $x \in \overline{U_{i+1}} \subset U_i \subset X - F$, and $\bigcap_{i\in\mathbf{N}} U_i = \bigcap_{i\in\mathbf{N}} \overline{U_{i+1}}$ is Q-clopen, so is a zero-set separating x and F.

Theorem 10 *If $(X,\mathfrak{T})$ is a $*T_3$-space in an $\aleph_1$-saturated enlargement $*\mathfrak{M}$, then $(X,\overline{\mathfrak{T}})$ is a P-space.*

Proof: We need only show that the intersection of a countably infinite collection of *open sets is Q-open. Again let $F = \bigcap \mathfrak{U}$, where $\mathfrak{U}$ is a countably infinite collection of *open sets, and let $p \in F$. Then for each finite subset $\mathfrak{V}$ of $\mathfrak{U}$ there is an *open *neighborhood V of p contained in $\bigcap \mathfrak{V}$. We have in internal relation concurrent on $\mathfrak{U}$, so there is an *open *neighborhood V of p contained in $\bigcap \mathfrak{U}$.

Similarly, if $*\mathfrak{M}$ is an α-saturated model and $(X,\mathfrak{T})$ is a *topological space in $\mathfrak{M}$, then for each family $\mathfrak{U}$ of Q-open subsets of X with cardinal less than α, $\bigcap \mathfrak{U}$ is Q-open.

A completely regular space X is said to be basically disconnected iff the closure of every cozero-set is open. Every P-space is basically disconnected and every basically disconnected space is zero-dimensional.

Corollary 1 *Let $*\mathfrak{M}$ be a comprehensive or $\aleph_1$-saturated model of the model $\mathfrak{M}$, and let $(X,\mathfrak{T})$ be a *topological space in $*\mathfrak{M}$. Then the following conditions are equivalent:*

(i) *$(X,\mathfrak{T})$ is a $*T_3$-space.*
(ii) *$(X,\overline{\mathfrak{T}})$ is completely regular.*
(iii) *$(X,\overline{\mathfrak{T}})$ is zero-dimensional.*

(iv) $(X, \overline{\mathfrak{T}})$ *is strongly zero-dimensional.*
(v) $(X, \overline{\mathfrak{T}})$ *is basically disconnected.*
(vi) $(X, \overline{\mathfrak{T}})$ *is a* P*-space.*

Returning to our second counterexample, we now see that any comprehensive or $\aleph_1$-saturated enlargement of the real numbers with the Euclidean topology is an example of a totally separated P-space which is not extremally disconnected or scattered when given the Q-topology.

Corollary 2 *Let* $*\mathfrak{M}$ *be a comprehensive or* $\aleph_1$*-saturated enlargement of the model* $\mathfrak{M}$*, and let* $(X, \mathfrak{T})$ *be a* $*$*topological space in* $*\mathfrak{M}$*. Then* $(X, \overline{\mathfrak{T}})$ *is totally separated iff it is Urysohn.*

Example 2: We now give an example of an extremally disconnected space $(X, \mathfrak{T})$ such that $(*X, *\overline{\mathfrak{T}})$ is not extremally disconnected.

Let $X = \mathbf{Z}^+ \cup \{\mathfrak{F}\}$, where $\mathfrak{F}$ is a free ultrafilter on $\mathbf{Z}^+$ and let $\mathfrak{T}$ be the topology on X generated by all the subsets of $\mathbf{Z}^+$ together with sets of the form $A \cup \{\mathfrak{F}\}$, where $A \in \mathfrak{F}$. This is called the single ultrafilter topology, and $(X, \mathfrak{T})$ is an extremally disconnected space, for the only possible limit point of any set A is $\mathfrak{F}$, but $\mathfrak{F}$ is a limit point of A iff $A \in \mathfrak{F}$, and then $A \cup \{\mathfrak{F}\}$ is open. The single ultrafilter topology is also perfectly normal [7], so $(*X, *\overline{\mathfrak{T}})$ is a P-space in each comprehensive enlargement. $(X, \mathfrak{T})$ is not discrete, so $(*X, *\overline{\mathfrak{T}})$ is not discrete in any enlargement.

Let $\mathfrak{M} = \mathfrak{M}(X, M)$ be the standard model for X, where M denotes the set of all entities of $\mathfrak{M}$, so $\operatorname{card}(M) = \aleph_\omega$. If $\mathbf{I} = \mathcal{P}_f(\mathcal{P}(\aleph_\omega^+))$ is the family of finite subsets of $\mathcal{P}(\aleph_\omega^+)$, then there is an ultrafilter $\mathfrak{U}$ on $\mathbf{I}$ such that the ultraproduct $*\mathfrak{M} = \mathfrak{U}\text{-prod } \mathfrak{M}$ is an enlargement of $\mathfrak{M}$, and $*\mathfrak{M}$ is comprehensive because it is an ultraproduct [4]. For $*X$ in $*\mathfrak{M}$, $\operatorname{card}(*X) \leqslant \operatorname{card}\left(\prod_{\mathbf{I}} X\right) = \operatorname{card}(\mathbf{I})$, which is non-measurable, so $\operatorname{card}(*X)$ is non-measurable. Every extremally disconnected P-space of non-measurable cardinal is discrete, $(*X, *\overline{\mathfrak{T}})$ is not extremally disconnected [3].

This is essentially a cardinality argument, and can be used for extremally disconnected spaces other than the single ultrafilter topology, which was picked for its simplicity and small cardinality. If we assume that there are no measurable cardinals, then in every comprehensive enlargement the Q-topology for a non-discrete regular space is not extremally disconnected.

Counterexample 3: We begin with the rational extension in the plane, which is the plane given the topology $\mathfrak{T}$ gotten by adding to the Euclidean topology each subset of $\mathbf{Q} \times \mathbf{Q}$ and each set of the form $\{x\} \cup ((\mathbf{Q} \times \mathbf{Q}) \cap U)$, where $x \in U$ and U is open in the Euclidean topology on $\mathbf{R}^2$. This topology is Urysohn and completely Hausdorff but is not regular [7]. Since $*(\mathbf{Q} \times \mathbf{Q})$ is $*$discrete, it is a discrete subset of $(*\mathbf{R}^2, *\overline{\mathfrak{T}})$, and any dense-in-itself subset must be contained in $*\mathbf{R}^2 - *(\mathbf{Q} \times \mathbf{Q})$; but each point $p \in *\mathbf{R}^2 - (\mathbf{Q} \times \mathbf{Q})$ has a neighborhood contained in $\{p\} \cup *(\mathbf{Q} \times \mathbf{Q})$, so no nonempty subset can be dense-in-itself. Correspondingly, $(*\mathbf{R}^2, *\overline{\mathfrak{T}})$ is an example of a scattered, basically disconnected and totally separated space which is not extremally

disconnected or zero-dimensional in comprehensive or $\aleph_1$-saturated enlargements.

We note that if $(X,\mathfrak{T})$ is a *topological space in a comprehensive or $\aleph_1$-saturated enlargement and $(X,\overline{\mathfrak{T}})$ is perfectly normal, then every *closed subset of X is a G_δ-set and is *open, so $(X,\mathfrak{T})$ is *perfectly normal. Obviously the converse fails unless every *closed set is also *open.

5 *Net convergence properties* In view of the disconnectedness properties of Q-topologies for Urysohn and T_3-spaces it is not surprising that these spaces have weak net convergence properties. It is more surprising to find that the nature of the enlargement involved is at least as important as the separation properties. This is illustrated by the following:

Theorem 11 *Let $\mathfrak{M}$ be a model, D be a directed set in $\mathfrak{M}$, and let $^*\mathfrak{M}$ be an α-saturated model of $\mathfrak{M}$, where α is an infinite cardinal greater than* $\mathrm{card}(D)$. *Then for each $^*T_1$-space $(X,\mathfrak{T})$ in $^*\mathfrak{M}$ every net $(x_d \mid d \in D)$ in X is convergent in $(X,\overline{\mathfrak{T}})$ iff it is constant on a tail.*

Proof: If $(x_d \mid d \in D)$ is constant on a tail then it is convergent. Suppose that $(x_d \mid d \in D)$ is convergent to $z \in X$ but is not constant on a tail. Set $D' = \{d \in D \mid x_d \neq z\}$ and consider the internal binary relation $b \subseteq X \times \mathfrak{T}$ defined by: $b(y,U)$ iff U is a *neighborhood of z and $x \notin U$. Clearly b is concurrent on D' and $\mathrm{card}(D') < \alpha$, so there is a *neighborhood U of z such that if $x_d \neq z$, then $x_d \notin U$ for each $d \in D$, a contradiction.

Similarly, we have:

Theorem 12 *Let $\mathfrak{M}$ be a model, let D be a directed set in $\mathfrak{M}$ and let $^*\mathfrak{M}$ be an α-saturated model of $\mathfrak{M}$, where α is an infinite cardinal greater than* $\mathrm{card}(D)$. *Then for each $^*T_1$-space in $^*\mathfrak{M}$ every net $(x_d \mid d \in D)$ clusters to $z \in X$ in the Q-topology iff $x_d = z$ frequently.*

We believe that the Q-topology can be useful not only as a stenographic space as in [1], but as a tool for constructing counterexamples in topology. We also believe that its value can be enhanced by further work in model theory. For instance, we must look to an enlargement which is neither comprehensive nor $\aleph_1$-saturated to find an example of a Q-topology which is zero-dimensional but not strongly zero-dimensional. We do not know of an example of such an enlargement and consider the question of its existence to be interesting.

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