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THE ELIMINABILITY OF THE ACTUALITY OPERATOR IN PROPOSITIONAL MODAL LOGIC

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The modal logic S5 is determined by the class of Kripke models in which the accessibility relation is the universal relation over the set of possible worlds or is an equivalence, or by the class of "pure" Kripke models, with no epicycles or accessibility relation, *Lp*(*Mp*) being counted true at a world in a model structure just in case *p* is true at every (at least one) world of that structure. It has been argued (*cf.* [4], in which ideas related to the actuality operator are also introduced, especially page 475) that it is the correct modal logic for the notions of metaphysical or logical necessity and possibility.

S5 may be given a natural deduction formulation in the manner of Fitch (*cf.* [1]) by adding to a natural deduction formulation of the classical propositional calculus (e.g., that of [1] or of [5]) a special kind of subordinate proof, called a *strict* subproof, into which formulas whose initial symbols are modal operators (we assume a notation like that of Łukasiewicz, in which the initial symbol of a complex formula is always its main connective) may be reiterated without change and other formulas not at all, and the following rules:

(*L*-introduction): *Lp* is a direct consequence (**d.c.**) of a strict subordinate proof with no hypothesis whose last item is *p*.
(*L*-elimination): *p* is a **d.c.** of *Lp*.
(*M*-introduction): *Mp* is a **d.c.** of *p*.
(*M*-elimination): *Mq* is a **d.c.** of *Mp* and a strict subordinate proof whose only hypothesis is *p* and whose last item is *q*.
(Negative modality rules): *NLp*, *MNp*, *NMp*, and *LNp* are **d.c.**'s respectively of *Mnp*, *NLp*, *LNp*, and *NMp*.

These rules are readily seen to be equivalent to a set presented in the first appendix to [3], where it is shown that they are at least as strong as an axiomatic formulation proved elsewhere in [3] to be complete relative to the Kripke semantics for S5. That the natural deduction system is sound

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relative to the intended semantics or, equivalently, that it is not stronger than the axiomatic formulation, may be proven by a straightforward but tedious argument involving a double induction on the depth of nesting of subordinate proofs in and the length of proofs (The proof, in sections **5.21** to **5.24** of [1], that any proof in the natural deduction system of implicational logic given there can be reproduced in the Hilbert-style system of the first chapter is of this sort, though obviously much simpler than the corresponding proof for a system with half a dozen connectives, like S5).

The logic S5A—S5 plus an actuality operator—is obtained by adding to the language of S5 the singulary operator R (for Really—I would prefer A, for actually, but in the notation of Łukasiewicz A has been preempted for disjunction) with the semantic rule that Rp is true at any world of a model structure just in case p is true at the actual world of that model structure. A natural deduction formulation of S5A may be obtained from that of S5 as follows: R is counted as a modal operator for the purposes of licensing reiteration, and a new kind of strict subproof, the actuality subproof, is introduced with the same reiteration rules as the ordinary strict subproofs. We have six new rules (or four, if you count rule-and-converse pairs as single rules):

(R-introduction): Rp is a **d.c.** of an actuality subproof with no hypothesis whose last item is p.
(R-elimination): Rq is a **d.c.** of Rp and an actuality subproof whose only hypothesis is p and whose last item is q.
(Negative actuality rules): NRp and RNp are **d.c.**'s of each other.

Before stating the last two rules we introduce a classification of the various subproofs of a proof into two kinds, **a** (for actuality) and **f** (for foreign world). This is done by simultaneous induction:

The main proof (i.e., a proof not subordinate to another) is an **a**-proof,
any actuality subproof is an **a**-proof,
any strict subproof other than an actuality subproof is an **f**-proof, and
any non-strict proof immediately subordinate to an **a**-proof (**f**-proof) is an **a**-proof (**f**-proof).

The last two rules may be used *only* in **a**-proofs:

(Actual actuality rules): *In* **a**-*proofs only*, Rp and p are **d.c.**'s of one another.

We now prove a series of metatheorems. The first is a semantic completeness theorem for the natural deduction system, and the rest, including the eliminability of the actuality operator, will be proven as theorems about the natural deduction system by purely proof theoretic and constructive means. As theorems about the logic, semantically construed, they will follow from the completeness theorem.

M1: *Semantic completeness*. The soundness of the system with respect to the intended interpretation should be obvious, and will be left as an exercise for the reader (it may help to note that a formula with an initial modal operator, including *R*, will have the same truth value at all the possible worlds of a model). For completeness we prove that, for any set of formulas in the language of S5A, if no contradiction is derivable from them, then there is a Kripke-model in which every member of the set is true at the actual world. This is proven by the method of Henkin (there is an excellent elementary exposition of this method on pp. 149-159 of [3]). We will simply sketch the proof. In particular, appeal will be made to the maximal consistency of various sets at a number of points, and the discovery of the relevant proof-schemata in the natural deduction system will be left as an exercise (none of them are difficult). First expand the given set to maximal consistency; call this maximal consistent set a. For each member of a of the form *Mp*, start a new set containing *p* and all those members of a of the form *Lq*. Expand these new sets (their consistency follows from that of *a* and the presence of the rule *M*-elimination) to maximal consistency *with respect to the natural deduction system S5A***f**, which is like the natural deduction formulation of S5A given above except for counting the main proof as an **f**-proof (i.e., the auxiliary deductive system S5A**f** differs from the system whose completeness is being proven only in that in it the actual actuality rules are licensed only in actuality subproofs and non-strict subproofs immediately subordinate to proofs in which the actual actuality rules are licensed). Let a and the other maximal consistent sets be the possible worlds of a Kripke model, with a the actual world, and let an atomic formula be true at a world just in case it is a member of the world. We complete the proof by showing that every formula which is a member of a world is true at it, which we show by induction on the complexity of formulas. The base case—the atomic formulas—is given us by the definition of the model; the cases of truth-functional compounds by the fact that all of the sets are maximal complete relative to systems which include the classical propositional calculus. Suppose a world, w, contains the formula *Lp* as a member. Then the actual world, a, must also contain it. For suppose *Lp* were not a member of a. Then, by maximal completeness, *NLp* and hence *LNLp* would be members of a, and so, by the construction of the model, *LNLp*, and, by maximal consistency, *NLp* would be members of w, contradicting the consistency of w. But since *Lp* is a member of a, it is a member of every world, and so is *p*. By hypothesis of induction, then, *p* is true at every world, and so *Lp* is true at w. Suppose *Mp* is a member of w. Then *Mp* is a member of a, for otherwise *NMp* and *LNMp* would be members of a, *LNMp* and *NMp* members of w, and w would not be consistent. But since *Mp* is a member of a, there is, by construction of the model, some world having *p* as a member, at which world, by hypothesis of induction, *p* is true. Therefore, *Mp* is true at w. Suppose *Rp* is a member of w. Then *Rp* is a member of a, for if it were not, *NRp* and *LNRp* would be, making *LNRp* and *NRp* members of w, making w inconsistent. But since *Rp* is a member of a, so is *p*, by maximal consistency with respect to S5A. By hypothesis of induction, then, *p* is true at a, and *Rp* is true at w.

M2: *Any substitution instance in the language of* S5A *of a theorem of* S5 *is a theorem of* S5A.

Proof: Every rule of the natural deduction system for S5 is also a rule of S5A, so the proof in the natural deduction formulation of S5 can be reproduced.

Let us define *p* and *q* to be provably strictly equivalent just in case *LCpq* and *LCqp* are theorems of S5A, and to be provably equivalent just in case *Cpq* and *Cqp* are theorems. Then:

M3: *If p and q are provably strictly equivalent and r results from s by substitution of p for q in zero or more places, r and s are provably strictly equivalent,*

and

M4: *If p and q are provably equivalent, and r results from s by substitution of p for q in zero or more places, none within the scope of a modal operator, then r and s are provably equivalent,*

may be proven by inductive arguments similar to those whereby analogous metatheorems are proven for other systems.

Comment: S5A differs from such logics as S5 without the actuality operator in having theorems whose necessitations are not theorems—*CRpp* and *CpRp* being short examples. Since most modern axiomatic formulations of logics like S5 use a rule of necessitation, this fact will make the adaptation of such an axiomatic formulation to S5A more difficult.

M5: *The actuality operator commutes with negation, in the sense that NRp and RNp are provably strictly equivalent.*

Proof: Immediate from negative actuality rules.

M6: *The actuality operator distributes over binary truth-functional connectives.*

Proof: We display proof schemata for the case of conjunction. The other cases follow from this case, **M5**, and the definability of the other truth functions in terms of negation and conjunction (substitution of definientia for definienda being guaranteed by **M2** and **M3**).

1	□	*KRpRq*		hyp
2		*Rp*		*K* elim, 1
3		*Rq*		*K* elim, 1
4		*A*	*Rp*	reit, 2
5			*Rq*	reit, 3
6			*p*	act *R* elim, 4
7			*q*	act *R* elim, 5
8			*Kpq*	*K* int, 6, 7
9		*RKpq*		*R* int, 4-8
10		*CKRpRqRKpq*		*C* int, 1-9
11	*LCKRpRqRKpq*			*L* int, 1-10

1	□ *RKpq*	hyp
2	*A* *Kpq*	hyp
3	*p*	*K* elim, 2
4	*Rp*	*R* elim, 1, 2-3
5	*A* *Kpq*	hyp
6	*q*	*K* elim, 5
7	*Rq*	*R* elim, 1, 5-6
8	*KRpRq*	*K* int, 4, 7
9	*CRKpqKRpRq*	*C* int, 1-8
10	*LCRKpqKRpRq*	*L* int, 1-9

M7: *LCMRpRp, LCRpLRp, LCMpRMp, LCLpRLp, LCRpRRp, LCRMpMp, LCRLpLp, and LCRRpRp are all theorems of* S5A.

Proof: We display a proof of *LCRMpMp*. The others are left as an exercise for the reader.

1	□ *RMp*	hyp
2	*NMp*	hyp
3	*LNp*	neg *M* elim, 2
4	*A* *RMp*	reit, 1
5	*Mp*	act *R* elim, 4
6	*LNp*	reit, 3
7	*NMp*	neg *M* int, 6
8	*KMpNMp*	*K* int, 5, 7
9	*RKMpNMp*	*R* int, 4-8
	A ⋮	some series of propositional
10+*n*	*NKMpNMp*	steps within the subproof.
11+*n*	*RNKMpNMp*	*R* int, 10-10+*n*
12+*n*	*NRKMpNMp*	neg *R* int, 11+*n*
13+*n*	*Mp*	indirect proof, 2-12+*n*
14+*n*	*CRMpMp*	*C* int, 1-13+*n*
15+*n*	*LCRMpMp*	*L* int, 1-14+*n*

(The rule of indirect proof is readily derivable in the systems of [1] and [5].)

Corollary: *Without strengthening the system, the rule of actuality elimination could be strengthened to: If the initial symbol of q is a modal operator, then q is a* d.c. *of Rp and an actuality subproof whose sole hypothesis is p and whose last item is q.*

(In virtue of the theoremhood of *LCMMpMp*, *LCMLpLp*, and *LCMRpRp*, a similarly strengthened version of the rule of possibility elimination is also derivable.)

M8: S5A *shares with* S5 *the following property ("reduction of iterated modalities"): if p results from q by the erasure of all but the rightmost of a string of modal operators, then p and q are provably strictly equivalent.*

Proof: By **M2**, **M7**, and **M3**.

M9: *LCMKpRqKMpRq, LCKMpRqMKpRq, LCLApRqALpRq, and LCALpRqLApRq are all theorems of* S5A.

Proof left to the reader (difficulty comparable to that of proof of **M7**).

M10: *Every formula of the language of* S5A *is provably strictly equivalent to one in which no modal operator occurs within the scope of another (to a first-degree formula).*

Proof: On pp. 51-54 of [3], it is proven that there is an effective procedure for reducing an arbitrary formula of S5 to an equivalent first degree formula. Metatheorems 2 through 9 suffice for the proof that the procedure described there can be extended to S5A.

M11: *Every formula of the language of* S5A *is provably equivalent to a formula containing no occurrences of the actuality operator.*

Proof: Since p is provably equivalent to Rp (though they are not provably strictly equivalent), this follows from **M10** and **M4**.

Comment: This result does not carry over to the quantificational extensions of S5A. Indeed, it is shown in [2] that the formula $M\Sigma xNR\Sigma y(x = y)$ is not equivalent to any R-free formula in any of a range of natural quantificational extensions of S5A. The ineliminability of the actuality operator in quantified modal logic is among the most interesting facts about it, and a prime motivation for its study. The eliminability of the actuality operator in propositional modal logic, therefore, seems to me to be little more than a curiosity, though it may help to explain the neglect of the actuality operator by early twentieth century students of modal logic.

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