Notre Dame Journal of Formal Logic
Volume XIX, Number 4, October 1978
NDJFAM

MODEL THEORETICAL INVESTIGATION OF THEOREM PROVING METHODS

T. GERGELY and K. P. VERSHININ

1 *Introduction* From the literature of the logical deduction theory, several different methods are known for theorem proving in different calculi. Usually, these methods are purely syntactically founded, which, in our opinion, often leads to the mystification of syntax. In the present paper we have tried to discuss these methods from a model-theoretical point of view. The basic facts serving as foundation for our treatment are also discussed.

1.1 *The general concept of language* A language is represented by a triple $\mathcal{L} = \langle \mathsf{F}, \mathbf{M}, \models \rangle$, where F is the syntax of the language, i.e., a certain set of words in a denumerably infinite alphabet and $\langle \mathbf{M}, \models \rangle$ is the semantics with $\mathbf{M}$ being the class of models and $\models$ the validity relation ($\models \subseteq \mathbf{M} \times \mathsf{F}$). Let X be a finite alphabet and X^* the set of all possible words described by the alphabet X. Then the syntax F of the language $\mathcal{L} = \langle \mathsf{F}, \mathbf{M}, \models \rangle$ is usually given as follows:

(1) Some elements of X^* are defined as elements of F;
(2) New elements of F are constructed from the existing ones by using a number of generator rules;
(3) F contains no further element different from those obtained according to (1) and (2).

If $\varphi \in \mathsf{F}$ and $\mathfrak{A} \in \mathbf{M}$, then $\mathfrak{A} \models \varphi$ denotes that φ is valid in $\mathfrak{A}$. In other words, $\mathfrak{A} \models \varphi$ denotes that $\mathfrak{A}$ is a model of φ. If the case $\mathfrak{A} \models \varphi$ is not satisfied, this will symbolically be denoted by $\mathfrak{A} \not\models \varphi$.

1.2 *Basic notations* The notations $\Longrightarrow$ and $\Longleftrightarrow$ stand for "it follows" and "if and only if", respectively. It is to be noted that, in proofs, the notations $\Longrightarrow$ and $\Longleftarrow$ will also be used to indicate in which direction a statement containing $\Longleftrightarrow$ is being proved. The letter "d" above the notations $=$ or $\Longleftrightarrow$ is used to indicate that a new concept is defined. Functions will sometimes be considered as sets of pairs. $\mathrm{Rg}f$ and $\mathrm{Do}f$ denote the range and the domain of a function f, respectively. $\doteq$ denotes the graphical equality, ω denotes the first transfinite ordinal. $|A|$ denotes the cardinal number of the

Received July 16, 1976

set A. Any further necessary notation will be defined in the relevant paragraph.

2 *t-type models* In our approach it will be assumed that every language has a fixed type t. Type t is defined as a pair of functions $\langle t', t''\rangle$ such that

(1) $\mathrm{Rg}\, t' \subseteq \omega - \{0\}$
(2) $\mathrm{Rg}\, t'' \subseteq \omega$
(3) $\mathrm{Do}\, t' \cap \mathrm{Do}\, t'' = \emptyset$, for $|\mathrm{Do}\, t''| \leq \omega$ and $|\mathrm{Do}\, t''| \leq \omega$,
(4) there exists a symbol c_0 for which $\langle c_0, 0\rangle \in t''$.

$\mathrm{Do}\, t'$ is the set of relation symbols and $\mathrm{Do}\, t''$ the set of functional symbols. The function t defines the arity of each symbol in $\mathrm{Do}\, t' \cup \mathrm{Do}\, t''$.

Definition 1 A *t-type model* is such a function $\mathfrak{A}$ for which

(1) $\mathfrak{A}(o) = A$ is a set
(2) for all $\rho \in \mathrm{Do}\, t'$, $\mathfrak{A}(\rho) \subseteq {}^{t'(\rho)} A$
(3) for all $f \in \mathrm{Do}\, t''$, $\mathfrak{A}(f): {}^{t''(f)} A \to A$ and if $t''(f) = 0$, then $\mathfrak{A}(f) \in A$

Remark: 0-ary functional symbols are the constant symbols and the corresponding elements in A are called constants. According to the above definition, A is never empty since one element is provided by $\mathfrak{A}(c_0)$.

The class of t-type models will be denoted by M^t. For convenience we introduce the following notations:

$$\mathfrak{A}(o) \stackrel{d}{=} \mathfrak{A}_0 = A$$
$$\mathfrak{A}(\rho) \stackrel{d}{=} \mathfrak{A}_\rho$$
$$\mathfrak{A}(f) \stackrel{d}{=} \mathfrak{A}_f$$

A t-type model will always be denoted by a German capital and the set $\mathfrak{A}_0$ by the corresponding Roman capital. This set is often called the universe of the model.

Definition 2 Let $\mathfrak{A}, \mathfrak{B} \in \mathsf{M}^t$. $\mathfrak{B}$ is a submodel of the model $\mathfrak{A}$ (symbolically: $\mathfrak{B} \subseteq \mathfrak{A}$) $\stackrel{d}{\Longleftrightarrow}$ for all such ρ that $\langle \rho, n\rangle \in t'$ or $\langle \rho, n - 1\rangle \in t''$ it is true

$$\mathfrak{B}_\rho = \mathfrak{A}_\rho \cap {}^n B \text{ and } \mathfrak{B}_0 \subseteq \mathfrak{A}_0$$

Definition 3 Let $C = \{c \mid \langle c, o\rangle \in t''\}$ be a set of t-type constant symbols. It is said that the model $\mathfrak{B}$ is generated by constants if $\mathfrak{B}_0 = B$ is the smallest set for which

(1) if $c \in C$, then $\mathfrak{B}_c \in B$
(2) if $\langle f, n\rangle \in t''$ and $b_1, \ldots, b_n \in B$, then $\mathfrak{B}_f(b_1, \ldots, b_n) \in B$

In other words the universe B is generated by the constants. In the following such models will be called t-type C models or simply C-models.

Theorem 1 *For an arbitrary t-type model $\mathfrak{A}$, the set of its subsets is partially ordered by the relation $\subseteq$ and has a smallest element.*

Proof: It follows from Definition 2 that the relation $\subseteq$ partially orders the set of all t-type submodels of $\mathfrak{A}$. The intersection of an arbitrary set of submodels of $\mathfrak{A}$ is again a submodel of $\mathfrak{A}$. This submodel is not empty, since all submodels contain constants. Let us now consider the intersection of all submodels of the model $\mathfrak{A}$. This submodel is one of the submodels and it is the smallest one. Q.E.D.

3 *Zero-, first-, and second-order languages* In the following, it will be assumed that the class of models $\mathsf{M} \overset{d}{=} \mathsf{M}^t$, i.e., only t-type or t-languages will be investigated. The zero, first, and second order t-type languages will be denoted by ${}_0\mathcal{L}^t$, ${}_1\mathcal{L}^t$, ${}_2\mathcal{L}^t$, respectively. For each of their M^t is the class of models. ${}_2\mathcal{L}^t$ is introduced as an auxiliary language which will enable some of the results to be expressed in a simpler way. First, the language ${}_2\mathcal{L}^t$ will be defined and the other languages will be obtained using it.

3.1 *The t-type second order language* (${}_2\mathcal{L}^t$) The language ${}_2\mathcal{L}^t$ is a triple $\langle {}_2\mathsf{F}^t, \mathsf{M}^t, {}_2\models^t\rangle$. Let us define each of the elements in the above triple.

3.1.1 *Definition of the syntax* ${}_2\mathsf{F}^t$ Let us fix the following disjoint sets which are also disjoint from $\mathrm{Do}\,t' \cup \mathrm{Do}\,t'' \cup \{\neg, \wedge, \exists\}$, where symbols $\neg$, $\wedge$, $\exists$ stand for negation, conjunction, and existential quantifier respectively.

V is an infinite set of symbols called individual variables.
V_n^F set of n-ary function variables for each $n < \omega$.
V_n^R set of n-ary relation variables for each $n < \omega$.

Definition 4 (a) ${}_2T^t$ (the set of the terms of the language ${}_2\mathcal{L}^t$) is the smallest set for which

(i) $V \subseteq {}_2T^t$
(ii) if $f \in V_n^F$ or $t''(f) = n$ and $\tau_1, \ldots, \tau_n \in {}_2T^t$, then $f(\tau_1, \ldots, \tau_n) \in {}_2T^t$

(b) Put

$$ {}_2P^t \overset{d}{=} \{\rho(\tau_1, \ldots, \tau_n) \colon \tau_1, \ldots, \tau_n \in {}_2T^t \text{ and } t'(\rho) = n \text{ or } \rho \in V_n^R,\ n < \omega\}. $$

The elements of ${}_2P^t$ are called second order prime formulas;

(c) ${}_2\mathsf{F}^t$ (the set of second order formulas) is a smallest set for which

(i) ${}_2P^t \subseteq {}_2\mathsf{F}^t$
(ii) if $\varphi \in {}_2\mathsf{F}^t$, then $\neg\varphi \in {}_2\mathsf{F}^t$
(iii) if $\varphi, \psi \in {}_2\mathsf{F}^t$, then $\varphi \wedge \psi \in {}_2\mathsf{F}^t$
(iv) if $\varphi \in {}_2\mathsf{F}^t$ and $z \in \bigcup\limits_{n<\omega} (V_n^F \cup V_n^R) \cup V$, then $\exists z\ \varphi \in {}_2\mathsf{F}^t$.

The formula $\psi \in {}_2\mathsf{F}^t$ will be called the *subformula* of the formula $\varphi \in {}_2\mathsf{F}^t$ if the latter is represented in the form $\varphi \equiv \alpha\psi\beta$, where α and β are some sequences of symbols.

We introduce the following notations: $\forall v$ (for all v) will stand for $\neg\exists v\neg$, $\varphi \vee \psi$ for $\neg(\neg\varphi \wedge \neg\psi)$, $\varphi \to \psi$ for $(\neg\varphi \vee \psi)$ and $\varphi \leftrightarrow \psi$ for $(\varphi \to \psi) \wedge (\psi \to \varphi)$ where $\varphi, \psi \in {}_2\mathsf{F}^t$. In the formulas only those brackets will be kept which are necessary for understanding.

3.1.2 *Semantics of the language* ${}_2\mathcal{L}^t$ Let $\mathfrak{A}$ be a given model. We define a function ${}_2k$ which gives a correspondence between variables from

$\bigcup_{n<\omega}(V_n^F \cup V_n^R) \cup V$ and the concrete values from $\mathrm{Rg}\mathfrak{A}$. Such a function is called an assignment function.

Let us introduce now the set ${}_2K$ of second order assignment functions as follows:

$$
{}_2K \overset{\mathrm{d}}{=} \Big\{ {}_2k \mid \mathrm{Do}\,{}_2k = \bigcup_{n<\omega}(V_n^F \cup V_n^R) \cup V \text{ and}
$$

$$
\begin{array}{l}
\text{if } v \in V, \text{ then } {}_2k(v) \in A \\
\text{if } f \in V_n^F, \text{ then } {}_2k(f) \in {}^{({}^nA)}A \\
\text{if } \rho \in V_n^R, \text{ then } {}_2k(\rho) \subseteq {}^nA \Big\}.
\end{array}
$$

Given some assignment function ${}_2k \in {}_2K$ we define its extension $({}_2\bar{k})$ to the set of terms ${}_2T^t$ as follows

$$
{}_2\bar{k}\ (v) \overset{\mathrm{d}}{=} {}_2k\ (v) \text{ for any } v \in V
$$

$$
{}_2\bar{k}\ (f(\tau_1, \ldots, \tau_n)) \overset{\mathrm{d}}{=} \begin{cases} {}_2\bar{k}(f)({}_2\bar{k}(\tau_1), \ldots, {}_2\bar{k}(\tau_n)) \text{ if } f \in V_n^F \\ \mathfrak{A}f({}_2\bar{k}(\tau_1), \ldots, {}_2\bar{k}(\tau_n)) \text{ if } \langle f, n\rangle \in t'' \end{cases}
$$

Now we define the validity relation ${}_2\vDash^t$: $\mathfrak{A}\ {}_2\vDash^t \varphi\,[{}_2k]$ will mean that the formula φ is valid in the model $\mathfrak{A}$ for the assignment function ${}_2k$. This will be defined through induction according to the construction of the formula φ.

Definition 5

(i) $\mathfrak{A}_2 \vDash^t \rho(\tau_1, \ldots, \tau_n)\,[{}_2k] \overset{\mathrm{d}}{\Longleftrightarrow} \begin{cases} \langle {}_2\bar{k}(\tau_1), \ldots, {}_2\bar{k}(\tau_n)\rangle \in {}_2k(\rho) \text{ if } \rho \in V_n^R \\ \langle {}_2\bar{k}(\tau_1), \ldots, {}_2\bar{k}(\tau_n)\rangle \in \mathfrak{A}\rho \text{ if } \langle \rho, n\rangle \in t'' \end{cases}$

(ii) $\mathfrak{A}_2 \vDash^t \neg\varphi\,[{}_2k] \overset{\mathrm{d}}{\Longleftrightarrow} \mathfrak{A}_2 \nvDash^t \varphi\,[{}_2k]$

(iii) $\mathfrak{A}_2 \vDash^t (\varphi \wedge \psi)\,[{}_2k] \overset{\mathrm{d}}{\Longleftrightarrow} \mathfrak{A}_2 \vDash^t \varphi\,[{}_2k]$ and $\mathfrak{A}_2 \vDash^t \psi\,[{}_2k]$

(iv) for any $z \in \bigcup_{n<\omega}(V_n^F \cup V_n^R) \cup V$: $\mathfrak{A}_2 \vDash^t \exists z\,\psi\,[{}_2k] \overset{\mathrm{d}}{\Longleftrightarrow}$ there exists such a ${}_2k' \in {}_2K$ that for all $\omega \neq z$ $\left(\omega \in \bigcup(V_n^F \cup V_n^R) \cup V\right) {}_2k'(\omega) = {}_2k(\omega)$ and $\mathfrak{A}_2 \vDash^t \psi\,[{}_2k']$

We shall say that $\varphi \in {}_2\mathsf{F}^t$ is *valid* in the model $\mathfrak{A}$ if for all ${}_2k \in {}_2K$ $\mathfrak{A} \vDash \varphi\,[{}_2k]$, i.e.,

$$
\mathfrak{A}_2 \vDash^t \varphi \overset{\mathrm{d}}{\Longleftrightarrow} (\forall {}_2k \in {}_2K)\ \mathfrak{A}_2 \vDash^t \varphi\,[{}_2k].
$$

The formula $\varphi \in \mathsf{F}^t$ is a *tautology* if it is *valid* in all the models of the class $\mathbf{M}^t$. Symbolically writing,

$$
{}_2\vDash^t \varphi \overset{\mathrm{d}}{\Longleftrightarrow} (\forall \mathfrak{A} \in \mathbf{M}^t)\ \mathfrak{A}_2 \vDash^t \varphi.
$$

The variable v will be called *bound* in a formula φ if it is found in a subformula ψ of form $\exists v\psi$ only. The variable v will be said to be *free* in the formula φ if it is not bound. A formula containing no free variable will be called a *sentence.* We introduce the notation ${}_2\mathsf{S}^t \overset{\mathrm{d}}{=} \{\varphi \in {}_2\mathsf{F}^t \mid \varphi$ contains no free variable$\}$. Note that if $\varphi \in {}_2\mathsf{S}^t$, then $\mathfrak{A}_2 \vDash^t \varphi \overset{\mathrm{d}}{\Longleftrightarrow}$ there exists ${}_2k \in {}_2K$ such that $\mathfrak{A}_2 \vDash^t \varphi\,[{}_2k]$.

Now, the definition of the second order t-type language ${}_2\mathcal{L}^t = \langle {}_2\mathsf{F}^t, \mathbf{M}^t, {}_2\vDash^t\rangle$ is complete.

Let $\varphi, \psi \in {}_2\mathsf{F}^t$. We shall say that ψ is the *logical consequence* of φ (symbolically: $\varphi \vDash \psi$) if each t-model $\mathfrak{A}$, in which φ is valid, is a model of

ψ. In other words:

$$\varphi_2 \models^t \psi \overset{d}{\Longleftrightarrow} (\forall \mathfrak{A} \in \mathbf{M}^t)\ \mathfrak{A}_2 \models^t \varphi \Rightarrow \mathfrak{A}_2 \models^t \psi.$$

Logical consequences are sometimes called semantic consequences. Two formulas φ, $\psi \in {}_2\mathrm{F}^t$ are said to be semantically equivalent (symbolically: $\varphi \equiv \psi$) if $\varphi \models \psi$ and $\psi \models \varphi$. In the following, some new notations will be necessary. Let $\Sigma \subseteq {}_2\mathrm{F}^t$ be a set of formulas $\Sigma \overset{d}{=} \{\sigma_i \mid i < \omega\}$. Conjunction (and disjunction) of the formulas will be denoted by the symbols $\bigwedge\limits_{i<\omega} \sigma_i$ $\left(\bigwedge\limits_{i<\omega} \sigma_i\right)$. Introduce the following notations $\hat{\Sigma} \overset{d}{=} \bigwedge\limits_{i<\omega} \sigma_i$ and $\check{\Sigma} \overset{d}{=} \bigvee\limits_{i<\omega} \sigma_i$.

3.2 *The t-type first order language* $({}_1\mathcal{L}^t)$ The language ${}_1\mathcal{L}^t$ is the triple $\langle {}_1\mathrm{F}^t, \mathbf{M}^t, {}_1\models^t \rangle$. It can be defined with the help of ${}_2\mathcal{L}^t$ as follows. The syntax of ${}_1\mathcal{L}^t$ is obtained from that of ${}_2\mathcal{L}^t$ by supposing in Definition 4 that $V_n^F \cup V_n^R = \emptyset$ $(n < \omega)$. In other words, ${}_1V \overset{d}{=} V$. Then ${}_1T^t$, ${}_1P^t$, and ${}_1\mathrm{F}^t$ are obtained from the corresponding definitions of ${}_2T^t$, ${}_2P^t$, and ${}_2\mathrm{F}^t$ by taking into account the introduced restrictions. The semantics of the language ${}_1\mathcal{L}^t$ is defined in a similar way. For this case

$${}_1k \overset{d}{=} \{{}_1k \mid \mathrm{Do}_1 k = V,\ \mathrm{Rg}_1 k = A\}$$

for a given model $\mathfrak{A} \in \mathbf{M}^t$. The validity relation is defined as

$${}_1\models^t \overset{d}{=} {}_2\models^t \cap (\mathbf{M}^t \times {}_1\mathrm{F}^t)$$

This easily follows from Definition 5. All notions introduced for language ${}_2\mathcal{L}^t$ are valid, with the introduced restrictions, for ${}_1\mathcal{L}^t$ as well.

3.3 *The t-type zero order language* ${}_0\mathcal{L}^t$ ${}_0\mathcal{L}^t$ is a triple $\langle {}_0\mathrm{F}^t, \mathbf{M}^t, {}_0\models^t \rangle$, which is defined with the help of ${}_2\mathcal{L}^t$ as follows. The syntax of ${}_0\mathcal{L}^t$ is obtained from that of ${}_2\mathcal{L}^t$ by assuming that $\bigcup\limits_{n<\omega} (V_n^F \cup V_n^R) \cup V = \emptyset$. ${}_0T^t$, ${}_0P^t$, and ${}_0\mathrm{F}^t$ are obtained from the corresponding definitions of ${}_2T^t$, ${}_2P^t$, and ${}_2\mathrm{F}^t$ by assuming that the language does not contain any variable symbols.

Remark: Remember that ${}_0T^t$ is constructed of constant symbols only, in the following way (according to the Definition 4a): ${}_0T^t$ is a smallest set such that

(i) if $\langle c, 0\rangle \in t''$, then $c \in {}_0T^t$
(ii) if $\langle f, n\rangle \in t''$ and $\tau_1, \ldots, \tau_n \in {}_0T^t$ then $f(\tau_1, \ldots, \tau_n) \in {}_0T^t$.

The semantics of the language ${}_0\mathcal{L}^t$ is defined from that of the language ${}_2\mathcal{L}^t$ by omitting all the variable symbols. In this case ${}_0K = \emptyset$. At the same time, there exists extended assignment functions applicable for ${}_0T^t$. They can be defined as follows:

Let us suppose that $\langle c, \mathrm{o}\rangle \in t''$, $\langle f, n\rangle \in t''$ and $\tau_1, \ldots, \tau_n \in {}_0T^t$ and consider a model $\mathfrak{A} \in \mathbf{M}^t$. Then:

$${}_0\bar{k}(c) \overset{d}{=} \mathfrak{A}_c$$
$${}_0\bar{k}(f(\tau_1, \ldots, \tau_n)) = \mathfrak{A}_f({}_0\bar{k}(\tau_1), \ldots, {}_0\bar{k}(\tau_n))$$

Since it is easy to see that for any model $\mathfrak{A}$ there exists only one function ${}_0k$ so in the following we shall write $\bar{\mathfrak{A}}$ instead of ${}_0\bar{k}$. This means that the language ${}_0\mathcal{L}^t = \langle {}_0\mathrm{F}^t, \mathbf{M}^t, {}_0\models^t \rangle$ is completely defined.

Remark: It follows from the definition of languages that ${}_0\mathcal{L}^t \subseteq {}_1\mathcal{L}^t \subseteq {}_2\mathcal{L}^t$. While ${}_0\mathsf{F}^t \subseteq {}_1\mathsf{F}^t \subseteq {}_2\mathsf{F}^t$ and ${}_i\models^t = {}_2\models^t \cap (\mathbf{M}^t \times {}_i\mathsf{F}^t)$, for $i = 0, 1$.

Consequently, the validity relation is practically the same for any order language. In the following, therefore, we shall omit indices and simply write $\models$ to denote validity relation.

4 *Some properties of the language* ${}_0\mathcal{L}^t$ In this section, $\models$ always means ${}_0\models^t$.

Theorem 2 *If* $\mathfrak{A} \in \mathbf{M}^t$ *and* $\mathfrak{B} \subseteq \mathfrak{A}$, *then for any formula* $\varphi \in {}_0\mathsf{F}^t$

$$\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{B} \models \varphi.$$

Proof: It goes by induction on the length of the formula φ. For prime formulas the statement follows from the definition of the relations $\subseteq$ and $\models$ and from that of the function $\overline{\mathfrak{A}}$. Suppose that the statement holds for the formulas ψ and χ from ${}_0\mathsf{F}^t$. Then it also holds for the formula $\psi \wedge \chi$ according to the definition of the relation $\models$ (see Definition 5 (iii)). The statement's validity for $\neg\psi$ follows from point (ii) of the same definition.
Q.E.D.

Corollary *Let* $\mathfrak{A} \in \mathbf{M}^t$ *and let* $\mathfrak{B} \subseteq \mathfrak{A}$ *be C-model. Then* $B = \mathrm{Rg}\,\overline{\mathfrak{A}}$.

In other words, the C-submodel of the model $\mathfrak{A}$ consists of exactly those elements which have corresponding terms.

Proof: $\mathrm{Rg}\,\overline{\mathfrak{A}} \subseteq C$ is true for any submodel $\mathfrak{C} \subseteq \mathfrak{A}$. At the same time, $\mathrm{Rg}\,\overline{\mathfrak{A}}$ is closed under the functions of the model $\mathfrak{A}$. This follows from the definition of the set of terms ${}_0T^t$. Consequently $\mathfrak{B}$ is the smallest model and $\mathfrak{B}_0 = B$ contains those elements only, which have corresponding terms. Q.E.D.

Let us now formulate the compactness theorem in dual form. As they are well-known, their proof will not be detailed here.

Theorem 3 (Compactness Theorem) *Let* $\{\varphi_i | i < \omega\} \subseteq {}_0\mathsf{F}^t$. *Then* $\models \bigvee_{i<\omega} \varphi_i$ *if and only if there exists an* $n < \omega$ *for which* $\models \bigvee_{i<n} \varphi_i$.

See proof, e.g., in [1].

Theorem 3′ (Compactness Theorem) *Let* $\{\varphi_i | i < \omega\} \subseteq {}_0\mathsf{F}^t$. $\bigwedge_{i<\omega} \varphi_i$ *is valid if and only if* $\bigwedge_{i<n} \varphi_i$ *is valid for all* $n < \omega$.

See proof, e.g., in [2].

A prime formula or its negation is called *literal.* The complement $(\overline{\pi})$ of the literal π is defined as follows:

$$\overline{\pi} \rightleftharpoons \begin{cases} \neg\pi \text{ if } \pi \text{ is a prime formula} \\ \pi' \text{ if } \pi \rightleftharpoons \neg\pi', \text{ where } \pi' \text{ is prime formula:} \end{cases}$$

The pair of literals $\langle \pi, \pi' \rangle$ is called *contrary.* If Π is a set of literals then $\overline{\pi}$ will denote the set consisting of the complementer elements of Π. A set of literals Π is called consistent if Π contains no contrary pairs. Let $\Pi \subseteq {}_0\mathsf{F}^t$ be a set of literals.

Theorem 4 *If* Π *is consistent, then* $\hat{\Pi}$ *is valid.*

Proof: Let us construct a model $\mathfrak{A} \in \mathbf{M}^t$ such that $\mathfrak{A} \models \hat{\Pi}$. Let $\mathfrak{A}_0 = A \stackrel{d}{=} {}_0T^t$, $\mathfrak{A}_f(\tau_1, \ldots, \tau_n) \stackrel{d}{=} f(\tau_1, \ldots, \tau_n)$ for all $n < \omega$, $\tau_1, \ldots, \tau_n \in {}_0T^t$ and such f, that $t''(f) = n$, $\mathfrak{A}_\rho = \{\langle\tau_1, \ldots, \tau_m\rangle : \rho(\tau_1, \ldots, \tau_m) \in \Pi\}$ for all $m < \omega$ and such ρ that $t'(\rho) = m$. $\mathfrak{A}$ satisfies $\hat{\Pi}$ since, if $\mathfrak{A} \models \pi$ then $\mathfrak{A} \not\models \overline{\pi}$. If $\mathfrak{A} \models \pi'$ and $\mathfrak{A} \models \pi''$ then $\mathfrak{A} \models \pi' \wedge \pi''$ and it follows from $\tau_1 \neq \tau_2$ that $\overline{\mathfrak{A}}_{\tau_1} \neq \overline{\mathfrak{A}}_{\tau_2}$. Q.E.D.

Theorem 5 *If* Π *contains at least one contrary pair then* $\check{\Pi}$ *is tautologic.*

Proof: Let $\pi, \overline{\pi} \in \Pi$ and $\pi \equiv \rho(\tau_1, \ldots, \tau_n)$ where $\tau_1, \ldots, \tau_n \in {}_0T^t$. Then it follows from the definition of the t-type model and of the relation $\models$ that either $\mathfrak{A} \models \pi$ or $\mathfrak{A} \models \overline{\pi}$ is valid in any model $\mathfrak{A} \in \mathbf{M}^t$. Q.E.D.

A set of formulas will often be represented by a sequence of its elements. Then, if φ is a formula then $\Sigma, \varphi \stackrel{d}{=} \Sigma \cup \{\varphi\}$. Let $\varphi_0, \varphi_1, \varphi_2$ be formulas of the language ${}_0\mathcal{L}^t$. Let us formulate the well-known properties of the logical consequence relation $\models$ in the form of a theorem. For convenience, we shall assume that $\Gamma \models \Sigma$ denotes the same as $\hat{\Gamma} \models \check{\Sigma}$ $(\Gamma, \Sigma \subseteq {}_0\mathsf{F}^t)$.

Theorem 6

(i) $\Gamma, \varphi_1 \wedge \varphi_2 \models \Sigma \Leftrightarrow \Gamma, \varphi_1, \varphi_2 \models \Sigma$;
(i′) $\Gamma \models \varphi_1 \vee \varphi_2, \Sigma \Leftrightarrow \Gamma \models \varphi_1, \varphi_2, \Sigma$;
(ii) $\Gamma, \varphi_1 \vee \varphi_2 \models \Sigma \Leftrightarrow \Gamma, \varphi_1 \models \Sigma$ *and* $\Gamma, \varphi_2 \models \Sigma$;
(ii′) $\Gamma \models \varphi_1 \wedge \varphi_2, \Sigma \Leftrightarrow \Gamma \models \varphi_1, \Sigma$ *and* $\Gamma \models \varphi_2, \Sigma$;
(iii) $\Gamma, \neg\varphi \models \Sigma \Leftrightarrow \Gamma \models \varphi, \Sigma$;
(iii′) $\Gamma \models \neg\varphi, \Sigma \Leftrightarrow \Gamma, \varphi \models \Sigma$;
(iv) $\Gamma, \varphi_1 \rightarrow \varphi_2 \models \Sigma \Leftrightarrow \Gamma \models \varphi_1, \Sigma$ *and* $\Gamma, \varphi_2 \models \Sigma$;
(v) $\Gamma \models \varphi_1 \rightarrow \varphi_2, \Sigma \Leftrightarrow \Gamma, \varphi_1 \models \varphi_2, \Sigma$.

Proof: We prove one of the above statements, e.g., (ii).

$\Rightarrow$. Let us suppose that, if $\mathfrak{A} \models \hat{\Gamma} \wedge (\varphi_1 \vee \varphi_2)$, then $\mathfrak{A} \models \check{\Sigma}$ for any model $\mathfrak{A}$. Let $\mathfrak{B} \models \hat{\Gamma} \wedge \varphi_1$. Then, according to the definition of the relation $\models$, $\mathfrak{B} \models \varphi_1$. Consequently $\mathfrak{B} \models \varphi_1 \vee \varphi_2$, and hence $\mathfrak{B} \models \hat{\Gamma} \wedge (\varphi_1 \vee \varphi_2)$ which means that $\mathfrak{B} \models \check{\Sigma}$. It can be proved in a similar way that, if $\mathfrak{B} \models \hat{\Gamma} \wedge \varphi_2$, then $\mathfrak{B} \models \check{\Sigma}$.

$\Leftarrow$. Let us suppose that

(1) if $\mathfrak{A} \models \hat{\Gamma} \wedge \varphi_1$, then $\mathfrak{A} \models \check{\Sigma}$
(2) if $\mathfrak{A} \models \hat{\Gamma} \wedge \varphi_2$, then $\mathfrak{A} \models \check{\Sigma}$

for any model $\mathfrak{A}$. Let $\mathfrak{B} \models \hat{\Gamma} \wedge (\varphi_1 \vee \varphi_2)$. Then $\mathfrak{B} \models \hat{\Gamma}$ and $\mathfrak{B} \models \varphi_1 \vee \varphi_2$. The latter statement is equivalent with $\mathfrak{B} \models \varphi_1$ or $\mathfrak{B} \models \varphi_2$.

Case 1: Let $\mathfrak{B} \models \varphi_1$. Then, according to (1), $\mathfrak{B} \models \check{\Sigma}$
Case 2: Let $\mathfrak{B} \models \varphi_2$. Then, according to (2), $\mathfrak{B} \models \check{\Sigma}$. Q.E.D.

Theorem 7

(i) *if* $\hat{\Gamma}_1 \models \check{\Sigma}$ *and* $\Gamma_1 \subseteq \Gamma_2$, *then* $\hat{\Gamma}_2 \models \check{\Sigma}$;
(ii) *if* $\hat{\Gamma} \models \check{\Sigma}_1$ *and* $\Sigma_1 \subseteq \Sigma_2$, *then* $\hat{\Gamma} \models \check{\Sigma}_2$.

Proof is evident, therefore it is omitted.

The following properties of the relation $\models$ follow also simply from the definition:

Theorem 8 *Let* Γ, Σ *be consistent sets of literals. Then:*

(i) $\hat{\Gamma} \models \hat{\Sigma} \Leftrightarrow \Sigma \subseteq \Gamma$;
(ii) $\hat{\Gamma} \models \check{\Sigma} \Leftrightarrow \Gamma \cap \Sigma \neq \emptyset$;
(iii) $\check{\Gamma} \models \check{\Sigma} \Leftrightarrow \Gamma \subseteq \Sigma$;
(iv) $\hat{\Gamma} \not\models \hat{\Sigma} \Leftrightarrow \Gamma \cap \bar{\Sigma} \neq \emptyset$;
(v) $\hat{\Gamma} \not\models \check{\Sigma} \Leftrightarrow \bar{\Sigma} \subseteq \Gamma$.

(N.B.: $\varphi_1 \not\models \varphi_2$ means that no model of φ_1 is a model of φ_2).

We also refer to a theorem which, although trivial in its proof, plays an important role in some later considerations.

Theorem 9 *Let* $\Gamma, \Sigma \subseteq {}_0\mathsf{F}^t$, $\varphi \in {}_0\mathsf{F}^t$. *Then:*

(i) $\Gamma, \varphi \models \Sigma$ *and* $\Gamma, \neg\varphi \models \Sigma \Leftrightarrow \Gamma \models \Sigma$
(ii) $\Gamma \models \Sigma, \varphi$ *and* $\Gamma \models \Sigma, \neg\varphi \Leftrightarrow \Gamma \models \Sigma$.

Proof: (i) $\Leftarrow$. It follows immediately from Theorem 7. $\Rightarrow$. Suppose that, if $\mathfrak{A} \models \hat{\Gamma} \wedge \varphi$, then $\mathfrak{A} \models \check{\Sigma}$, and, if $\mathfrak{B} \models \hat{\Gamma} \wedge \neg\varphi$, then $\mathfrak{B} \models \check{\Sigma}$. Let $\mathfrak{L} \models \hat{\Gamma}$. It is obvious that always either $\mathfrak{L} \models \varphi$ or $\mathfrak{L} \models \neg\varphi$. In both cases, $\mathfrak{L} \models \check{\Sigma}$.

(ii) $\Leftarrow$. It is trivial from Theorem 7. $\Rightarrow$. Suppose the contrary. Let $\mathfrak{A} \models \hat{\Gamma}$ and $\mathfrak{A} \not\models \check{\Sigma}$. Since $\varphi \in {}_0\mathsf{F}^t$, then either (*) $\mathfrak{A} \models \varphi$ or (**) $\mathfrak{A} \models \neg\varphi$. In the (*) case $\mathfrak{A} \not\models \check{\Sigma} \vee \neg\varphi$, and in the (**) case $\mathfrak{A} \not\models \check{\Sigma} \vee \varphi$ what contradicts our initial assumption. Q.E.D.

5 *Set of literals as a model-theoretical tool* Let Γ be a finite consistent set of literals. The formula $\hat{\Gamma}$ ($\check{\Gamma}$) is called a *conjunct* (*disjunct*). A conjunct (or disjunct) containing no literal will be said to be *empty* and denoted by the symbol Δ (∇).

If no ambiguity may occur, a conjunct (disjunct) will often be identified with the corresponding set of literals and the notations and terminology of set theory will be treated freely. We shall say that $\varphi \in {}_0\mathsf{F}^t$ is represented in conjunctive (disjunctive) normal form if φ has the form $\bigwedge_{i=1}^{n} \varphi_i$ where each φ_i is disjunct $\left(\text{and } \bigvee_{i=1}^{n} \varphi_i \text{ where each } \varphi_i \text{ is conjunct, respectively}\right)$.

Theorem 10 *For any formula* $\varphi \in {}_0\mathsf{F}^t$, *there exist formulas* φ' *and* φ'' *in* ${}_0\mathsf{F}^t$ *such that* $\varphi' \equiv \varphi$ *and* $\varphi'' \equiv \varphi$ *and* φ' *is represented in conjunctive normal form and* φ'' *in disjunctive normal form.*

Proof: It will be presented for the first statement of the theorem only. The proof goes by induction on the length of φ. For φ we can construct a formula φ' being semantically equivalent to φ and whose subformulas beginning with the symbol $\neg$ do not contain the symbols $\wedge$, $\vee$, and $\neg$. This formula can be obtained by repeated application of the following equalities:

$$\neg\neg\theta \equiv \theta$$
$$\neg(\theta \wedge \psi) \equiv \neg\theta \vee \neg\psi$$
$$\neg(\theta \vee \psi) \equiv \neg\theta \wedge \neg\psi$$

where $\theta, \psi \in {}_0\mathrm{F}^t$. These equalities follows easily from Theorem 6. Now, the wanted representation of φ can be obtained by using the equality $\varphi \vee (\psi \wedge \theta) \equiv (\varphi \vee \psi) \wedge (\varphi \vee \theta)$ (where $\varphi, \psi, \theta \in {}_0\mathrm{F}^t$), which is readily seen from Theorem 6, and by taking into consideration that the symbols $\vee$ and $\wedge$ are commutative. Since the proof was carried out by using semantic equalities only, then $\varphi \equiv \varphi'$. The second statement of the theorem can be proved in a similar way. Q.E.D.

The formula φ' will be called conjunctive normal form (CNF) of the formula φ and φ'' the disjunctive normal form (DNF) of the formula φ. Let $\Sigma \subseteq {}_0\mathrm{F}^t$. P_Σ denotes the set of prime formulas occurring in Σ.

Theorem 11 *Let* $\mathrm{P} \subseteq {}_0\mathrm{P}^t$. *Then, for any model* $\mathfrak{A} \in \mathbf{M}^t$, *there exists a set of literals* Π *such that* $\mathrm{P}_\Pi \subseteq \mathrm{P}$ *and for any* $\varphi \in {}_0\mathrm{F}^t$, *for which* $\mathrm{P}_{\{\varphi\}} \subseteq \mathrm{P}$ *it holds that*

$$\mathfrak{A} \models \varphi \Leftrightarrow \hat{\Pi} \models \varphi$$

Proof: Given a model $\mathfrak{A}$, the set P can linearly be ordered. Let $\mathrm{P} \stackrel{d}{=} \langle p_1, \ldots, p_n, \ldots\rangle$. Each element $p_i (i \geqslant 1)$ has the form of $\rho_i(t_1, \ldots, t_{n_i})$. Let us add the literal p_i to Π if $\langle \overline{\mathfrak{A}}_{t_1^i}, \ldots, \overline{\mathfrak{A}}_{t_{n_i}^i}\rangle \in \mathfrak{A}_{\rho_i}$ and the literal $\neg p_i$ in the opposite case. The obtained set Π will obviously be consistent.

$\Rightarrow$. Let $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \models \hat{\Pi}$. Then it is obvious that $\mathfrak{B}$ satisfies each element of Π and thus the value of the assignment function $\overline{\mathfrak{B}}$ for the prime subformulas φ agrees with that of $\overline{\mathfrak{A}}$. Consequently, $\mathfrak{B} \models \varphi$. $\Leftarrow$. It is evident, since $\mathfrak{A} \models \hat{\Pi}$ according to the construction. Q.E.D.

Thus, a "restriction" of any model to a fixed set of prime formulas P can be given by a conjunct. The validity of any formula, the prime subformulas of which are contained in P, is equivalent to the logical consequences of this conjunct. On the other hand, any conjunct defines a class of models coinciding (as functions, see section **2.3**) in the prime formulas contained in this conjunct. Evidently the fewer elements a conjunct contains, the more "extended" the corresponding class of models is. The empty conjunct Δ "defines" the class $\mathbf{M}^t$ of all models.

Theorem 12 *Let* $\mathrm{P} \subseteq {}_0\mathrm{P}^t$. *Then for any model* $\mathfrak{A} \in \mathbf{M}^t$ *there exists a set of literals* Π *such that* $\mathrm{P}_\Pi \subseteq \mathrm{P}$ *and for any formula* $\varphi \in {}_0\mathrm{F}^t$, $\mathrm{P}_{\{\varphi\}} \subseteq \mathrm{P}$, *it holds that*

$$\mathfrak{A} \not\models \varphi \Leftrightarrow \varphi \models \check{\Pi}.$$

Proof: Let $\mathfrak{A}$ be a given model. Let us construct a set of literals Π' which satisfies the preceding theorem and suppose that $\Pi = \overline{\Pi}'$. $\Rightarrow$. Let $\mathfrak{A} \not\models \varphi$ and $\mathfrak{B} \models \varphi$. Then, obviously, there exists $\rho(\tau_1, \ldots, \tau_n) \in \mathsf{P}_{\{\varphi\}}$ such that

$$\langle \overline{\mathfrak{A}}_{\tau_1}, \ldots, \overline{\mathfrak{A}}_{\tau_n} \rangle \in \mathfrak{A}_\rho \Leftrightarrow \langle \overline{\mathfrak{B}}_{\tau_1}, \ldots, \overline{\mathfrak{B}}_{\tau_n} \rangle \notin \mathfrak{B}_\rho.$$

Then the corresponding literal from Π is valid in $\mathfrak{B}$ and $\mathfrak{B} \models \check{\Pi}$. $\Leftarrow$. It is evident, since $\mathfrak{A} \models \check{\Pi}$. Q.E.D.

Theorem 12 is apparently the dual pair of the preceding one and it states that the non-validity of a formula in a given model is equivalent to the existence of a disjunct semantic consequence of it. Consequently, we can say that each disjunct $\check{\Pi}$ also defines a class of models, namely, the complementer of the class determined by the conjunct $\hat{\Pi}$. That is, the fewer elements the disjunct contains, the "poorer" the corresponding class is. The empty disjunct ∇ "defines" an empty class of models.

6 *The semantical basis of resolution-like proof proceedures* Now we have shown that a consideration on the validity (or non-validity) of a formula $\varphi \in {}_0\mathsf{F}^t$ may be substituted by a consideration on the logical consequence of φ from a conjunct (or on the logical consequence of a disjunct from φ) while the range of such a conjunct (disjunct) enables us to conclude on the range of the class of models in which φ is valid (non-valid). It also has to be noted that any conjunct (disjunct) CNF (DNF) of the formula φ logically follows from (is the logical consequence of) φ and, in addition Theorems 8 and 9 permit the construction of new conjuncts (disjuncts) having such property. Later we shall show that they exhaust all the conjuncts (disjuncts) possessing this property.

Two sets of literals Γ_1 and Γ_2 will be said to be *compatible* if $\Gamma_1 \cup \Gamma_2$ is consistent. We also shall speak of compatible pairs of conjuncts (disjuncts) by regarding them as a set of literals.

Definition 6 Two conjuncts (disjuncts) Γ_1 and Γ_2 form a *contrary pair* if there exists a literal ρ such that $\Gamma_1 = \Gamma_1' \cup \{\rho\}$, $\Gamma_2 = \Gamma_2' \cup \{\bar{\rho}\}$ and Γ_1' is compatible with Γ_2'. If these conjuncts (disjuncts) form a contrary pair then conjunct (disjunct) $\Gamma_1' \vee \Gamma_2'$ is called their *resolvent* and denoted by $\mathbf{R}(\Gamma_1, \Gamma_2)$.

Theorem 13 *Let* $\varphi \in {}_0\mathsf{F}^t$ *and let* Γ_1, Γ_2 *form a contrary pair. Then:*

(i) $\hat{\Gamma}_1 \models \varphi$ *and* $\hat{\Gamma}_2 \models \varphi \Rightarrow \hat{\mathbf{R}}(\Gamma_1, \Gamma_2) \models \varphi$;
(ii) $\varphi \models \check{\Gamma}$ *and* $\varphi \models \check{\Gamma}_2 \Rightarrow \varphi \models \check{\mathbf{R}}(\Gamma_1, \Gamma_2)$.

Proof: It follows immediately from Theorems 8 and 9. Q.E.D.

Let S be a set of conjuncts (disjuncts). Assume that $\mathbf{R}^0(S) \stackrel{d}{=} S$, and that $\mathbf{R}^{n+1}(S) \stackrel{d}{=} \mathbf{R}^n(S) \cup \{\Gamma \mid$ there exist $\Gamma_1, \Gamma_2 \in \mathbf{R}^n(S)$, which form a contrary pair, and $\Gamma = \mathbf{R}(\Gamma_1, \Gamma_2)\}$. Let $\mathbf{R}(S) \stackrel{d}{=} \bigcup_{n<\omega} \mathbf{R}^n(S)$.

If S is finite, then $\mathbf{R}(S)$ is finite too, since P_S is finite. More exactly:

$$|\mathbf{R}(S)| \leq \sum_{A \subseteq P_S} 2^{|A|}$$

Let $\varphi \in {}_0F^t$ be valid, and $\pi_1 \vee \ldots \vee \pi_r$ a DNF formula of φ. The symbol Π^φ stands for a set of conjuncts $\{\pi_1, \ldots, \pi_r\}$.

Theorem 14 *Let* Γ *be a consistent set of literals. Then:* $\Gamma \models \varphi \Longleftrightarrow$ *there exists* $\Pi \in \mathbf{R}(\Pi^\varphi)$ *such that* $\Pi \subseteq \Gamma$.

Proof: $\Longleftarrow$. Let $\Pi \in \mathbf{R}(\Pi^\varphi)$ and $\Pi \subseteq \Gamma$. Let $\mathfrak{A} \models \Gamma$. Then, according to Theorem 8, $\mathfrak{A} \models \Pi$. Obviously, if $\mathfrak{A} \models \hat{\mathbf{R}}(\Gamma_1, \Gamma_2)$, then either $\mathfrak{A} \models \hat{\Gamma}_1$ or $\mathfrak{A} \models \hat{\Gamma}_2$. Thus, one of the "initial" conjuncts $\pi_i (i = 1, \ldots, r)$ is valid in $\mathfrak{A}$. Consequently, $\mathfrak{A} \models \check{\Pi}^\varphi$ and $\mathfrak{A} \models \varphi$.

$\Longrightarrow$. Let $\Gamma \models \varphi$. Assume that $P = P_{\{\varphi\}} - P_\Gamma$. If $P = \emptyset$ then the statement of the theorem is obvious. Let $P = \{\rho_1, \ldots, \rho_n\}$: Put that $\mathbf{R}_0 = \{\Pi (\Pi \in \mathbf{R}(\Pi^\varphi)\}$ and Π is compatible with $\Gamma\}$; $\mathbf{R}_{i+1} = \{\Pi \mid \Pi \in \mathbf{R}_i$ and $\rho_{i+1} \notin \Pi$ and $\neg\rho_{i+1} \notin \Pi\}$; where $i = 0, \ldots, n-1$. We shall show by induction with respect to i

$$\text{that for any } i (i = 0, \ldots, n) \Gamma \models \check{\mathbf{R}}_i \text{ and } \mathbf{R}_i \neq \emptyset$$

which means that the statement of the theorem is true.

Basis of the induction: Let $i = 0$. $\mathbf{R}_0 \neq \emptyset$ since $\Gamma \models \check{\Pi}\varphi$, which means that $\Gamma \models \mathbf{R}(\Pi^\varphi)$, but it $\Pi \notin \mathbf{R}_0$ then $\Gamma \not\models \Pi$. Consequently, there exist $\Pi \in \mathbf{R}_0$. It is also obvious that if $\mathfrak{A} \models \hat{\Gamma}$ then $\mathfrak{A} \models \check{\mathbf{R}}_0$.

Induction step: Let the statement be valid for $\mathbf{R}_j (j = 0, \ldots, i)$. Let us consider $\mathbf{R}_{i+1}$. Let $\mathfrak{A} \models \hat{\Gamma}^t$. The symbol $\mathfrak{A}^{i+1}$ stands for a model which differs from $\mathfrak{A}$ only in that $\mathfrak{A}^{i+1} \models \rho_{i+1} \Longleftrightarrow \mathfrak{A} \not\models \rho_{i+1}$. Obviously, if $\psi \in {}_0F^t$ and $\rho_{i+1} \notin P_{\{\psi\}}$, then $\mathfrak{A} \models \psi \Longleftrightarrow \mathfrak{A}^{i+1} \models \psi$. In particular, $\mathfrak{A}^{i+1} \models \hat{\Gamma}$, therefore $\mathfrak{A}^{i+1} \models \varphi$. Let:

$$\mathbf{R}^+_{i+1} = \{\Pi \mid \Pi \in \mathbf{R}_i \text{ and } \rho_{i+1} \in \Pi\}$$
$$\mathbf{R}^-_{i+1} = \{\Pi \mid \Pi \in \mathbf{R}_i \text{ and } \neg\rho_{i+1} \in \Pi\}$$

Then: $\mathbf{R}_i = \mathbf{R}^+_{i+1} \cup \mathbf{R}^-_{i+1} \cup \mathbf{R}_{i+1}$. Let us consider the following cases:

Case 1. $\mathbf{R}^+_{i+1} = \mathbf{R}^-_{i+1} - \emptyset$. Then $\mathbf{R}_{i+1} = \mathbf{R}_i$ and the theorem is proved.

Case 2. $\mathbf{R}^+_{i+1} = \emptyset$, $\mathbf{R}^-_{i+1} \neq \emptyset$. Let $\mathfrak{A} \models \hat{\Gamma}$. If $\mathfrak{A} \models \rho_{i+1}$, then $\mathfrak{A} \not\models \mathbf{R}^-_{i+1}$. But, $\mathfrak{A} \models \mathbf{R}_i$ (according to the assumption of the induction), hence $\mathfrak{A} \models \mathbf{R}_{i+1}$ and $\mathbf{R}_{i+1} \neq \emptyset$. If $\mathfrak{A} \models \neg\rho_{i+1}$, then let us take $\mathfrak{A}^{i+1}$ and, by analogous considerations, we can see that $\mathfrak{A}^{i+1} \models \mathbf{R}_{i+1}$ but, because of $\rho_{i+1} \notin P_{\mathbf{R}i+1}$, $\mathfrak{A} \models \mathbf{R}_{i+1}$. The theorem has been proved for Case 2.

Case 3. $\mathbf{R}^+_{i+1} \neq \emptyset$, $\mathbf{R}^-_{i+1} = \emptyset$. This case is the symmetric equivalent of the former one.

Case 4. $\mathbf{R}^+_{i+1} \neq \emptyset$, $\mathbf{R}^-_{i+1} \neq \emptyset$. Suppose the contrary. Let $\mathfrak{A} \models \Gamma$ and $\mathfrak{A} \not\models \mathbf{R}_{i+1}$.

As from the assumption of the induction it follows that $\mathfrak{A} \models \mathbf{R}_i$, then $\mathfrak{A} \models \check{\mathbf{R}}^+_{i+1} \vee \check{\mathbf{R}}^-_{i+1}$. Consequently, there exists a conjunct $\Pi \in \mathbf{R}^+_{i+1} \vee \mathbf{R}^-_{i+1}$ such that $\mathfrak{A} \models \Pi$. Let $\Pi^+ \in \mathbf{R}^+_{i+1}$ be such a conjunct (the considerations are symmetrical for $\Pi \in \mathbf{R}^-_{i+1}$). It is seen that $\Pi^+ \rightleftharpoons \Pi^+_1 \vee \{\rho_{i+1}\}$ and $\mathfrak{A} \models \Pi^+_1$ and $\mathfrak{A} \models \rho_{i+1}$.

Let us investigate $\mathfrak{A}^{i+1} \cdot \mathfrak{A}^{i+1} \models \Gamma$ and $\mathfrak{A}^{i+1} \not\models \mathbf{R}_{i+1}$, since $\rho_{i+1} \in \mathrm{P}_{\mathbf{R}_{i+1}}$. Consequently, $\mathfrak{A}^{i+1} \models \check{\mathbf{R}}^+_{i+1} \vee \check{\mathbf{R}}^-_{i+1}$. But $\mathfrak{A}^{i+1} \models \neg\rho_{i+1}$ hence $\mathfrak{A}^{i+1} \models \check{\mathbf{R}}^-_{i+1}$. Let $\Pi^- \in \mathbf{R}^-_{i+1}$ be *such a conjunct that* $\mathfrak{A}^{i+1} \models \Pi^-$, $\Pi^- = \Pi^-_1 \cup \{\neg\rho_{i+1}\}$ *and* $\mathfrak{A}^{i+1} \models \Pi^-_1$. Since $\rho_{i+1} \in \mathrm{P}_{\Pi^-_1}$, then $\mathfrak{A} \models \Pi^-_1$. Consequently, Π^+_1 and Π^-_1 are compatible and there exists $\mathbf{R}(\Pi^+, \Pi^-)$ such that $\mathbf{R}(\Pi^+, \Pi^-) \in \mathbf{R}_{i+1}$ and, according to Theorem 13, $\mathfrak{A} \models \mathbf{R}(\Pi^+, \Pi^-)$. That is, $\mathfrak{A} \models \mathbf{R}_{i+1}$. The investigation of Case 4 leads to a contradiction, which means that the proof of the theorem has been completed. Q.E.D.

Corollary *Let* $\varphi \in {}_0\mathrm{F}^t$. *Then:*

$$\models \varphi \Leftrightarrow \Delta \in \mathbf{R}(\Pi^\varphi)$$

Now let $\varphi \in {}_0\mathrm{F}^t$ be non-tautologic. Let $\sigma_1 \wedge \ldots \wedge \sigma_s$ be a CNF of the formula φ. Let us suppose that $\Sigma^\varphi \stackrel{d}{=} \{\sigma_1, \ldots, \sigma_s\}$. It will be shown that the *dual equivalent of the statement of Theorem 14 is also true.*

Theorem 15 *Let* Γ *be a finite consistent set of literals. Then:*

$$\varphi \models \hat{\Gamma} \Leftrightarrow \text{there exists } \Sigma \in \mathbf{R}(\Sigma^\varphi) \text{ such that } \Sigma \subseteq \Gamma.$$

Proof: Since φ is not-tautologic, so $\neg\varphi$ is valid. Let us consider the disjunct $\hat{\overline{\Gamma}}$. It is obvious that $\overline{\Gamma}$ is consistent and $\varphi \models \hat{\Gamma} \Leftrightarrow \hat{\overline{\Gamma}} \models \neg\varphi$. It is also evident that $\Pi^{\neg\varphi} = \overline{\Sigma}^\varphi$. According to Theorem 14, there exists $\Pi \in \mathbf{R}(\Pi^\varphi)$ such that $\Pi \subseteq \overline{\Gamma}$. Then it is evident that $\Sigma = \overline{\Pi} \in \mathbf{R}(\Sigma^\varphi)$ and $\Sigma \subseteq \Gamma$. Q.E.D.

Corollary *Let* $\varphi \in {}_0\mathrm{F}^t$. *Then:*

$$\not\models \varphi \Leftrightarrow \nabla \in \mathbf{R}(\Sigma^\varphi)$$

Let $\mathfrak{A} \in \mathbf{M}^t$. The pair of conjuncts (disjuncts) Π_1, Π_2 is called $\mathfrak{A}$-*resolvable*, if

(i) Π_1, Π_2 are resolvable in the usual sense;
(ii) $\mathfrak{A} \models \hat{\Pi}_1 \vee \hat{\Pi}_2$ (correspondingly, $\mathfrak{A} \not\models \hat{\hat{\Pi}}_1 \vee \hat{\hat{\Pi}}_2$).

If Π_1, Π_2 are $\mathfrak{A}$-resolvable, then $\mathbf{R}(\Pi_1, \Pi_2)$ is called the $\mathfrak{A}$-*resolvent* of Π_1, Π_2 and is denoted by $\mathbf{R}_{\mathfrak{A}}(\Pi_1, \Pi_2)$. Let S be a set of conjuncts (disjuncts). Let $\mathfrak{A} \models \hat{S}$, (correspondingly, $\mathfrak{A} \models \hat{\hat{S}}$). Let $\mathbf{R}^0_{\mathfrak{A}}(S) \stackrel{d}{=} S$ and $\mathbf{R}^{i+1}_{\mathfrak{A}}(S) = \{\Gamma |$ there exist $\Gamma_1, \Gamma_2 \in \mathbf{R}^i_{\mathfrak{A}}(S)$ such that Γ_1, Γ_2 are $\mathfrak{A}$-resolvable and $\Gamma = \mathbf{R}_{\mathfrak{A}}(\Gamma_1, \Gamma_2)\}$. Let us suppose, furthermore, that $\mathbf{R}_{\mathfrak{A}}(S) = \bigcup_{i<\omega} \mathbf{R}^i_{\mathfrak{A}}(S)$.

Definition 7. An $\mathfrak{A}$-*clash* is a set of conjuncts (disjuncts) $\{\Gamma, \Gamma_1, \ldots, \Gamma_n\}$ such that

(i) $\Gamma = \Gamma' \cup \{\pi_1, \ldots, \pi_n\}$ is non-valid (valid) in $\mathfrak{A}$ while Γ' is valid (non-valid) in $\mathfrak{A}$;

(ii) $\Gamma_i = \Gamma_i' \cup \{\overline{\pi}_i\}$ $(i = 1, \ldots, n)$ is valid (non-valid) in $\mathfrak{A}$;

(iii) $\left(\bigcup_{i=1}^{n} \Gamma_i'\right) \cap (\{\pi_1, \ldots, \pi_n\} \cup \{\overline{\pi}_1, \ldots, \overline{\pi}_n\}) = \emptyset.$ *(clash condition)*

Here Γ is called the *nucleus* of the clash, $\Gamma_1, \ldots, \Gamma_n$ the *satellites* of the clash and $\Gamma' \cup \Gamma_1' \cup \ldots \cup \Gamma_n'$ the *resolvent* of the clash (symbolically: $\mathbf{R}_{\mathfrak{A}}(\Gamma'; \Gamma_1, \ldots, \Gamma_n)$. The literals $\pi_1, \ldots, \pi_n, \overline{\pi}_1, \ldots, \overline{\pi}_n$ are said to be *resolvable* by the literals of the clash.

Let us consider the proof of Theorem 14 again. Let $\mathfrak{M} \in \mathbf{M}^t$ and Γ a consistent set of literals such that $\mathfrak{M} \models \hat{\Gamma}$. Let us observe that, while investigating Case 4 in the proof of Theorem 14, we established the existence of an element in $\mathbf{R}_{i+1}$. This element is a resolvent of two conjuncts, one of which is valid in a fixed model $\mathfrak{A}$. Let us consider the model $\mathfrak{M}$ in the place of $\mathfrak{A}$, then the following statement can easily be proved:

Theorem 16 *Let* $\varphi \in {}_0\mathsf{F}^t$ *be valid. Then:*

$$\Gamma \models \varphi \Leftrightarrow \textit{there exists } \Pi \in \mathbf{R}_{\mathfrak{M}}(\Pi^{\varphi}) \textit{ such that } \Pi \subseteq \Gamma.$$

The dual equivalent of this theorem is also true:

Theorem 17 *Let* $\varphi \in {}_0\mathsf{F}^t$ *be non-tautologic. Then:*

$$\varphi \models \Gamma \Leftrightarrow \textit{there exists } \Sigma \in \mathbf{R}_{\mathfrak{M}}(\Sigma^{\varphi}) \textit{ such that } \Sigma \subseteq \Gamma.$$

Remark: It is easy to see that each element of $\mathbf{R}_{\mathfrak{M}}(\Pi^{\varphi})(\mathbf{R}_{\mathfrak{M}}(\Sigma^{\varphi}))$ which is valid (non-valid, respectively) in $\mathfrak{M}$ can be obtained not only by sequential binary resolutions but as a resolvent of an $\mathfrak{M}$-clash. The nucleus and satellites of the $\mathfrak{M}$-clash are easy to construct, by considering the derivation of the above-mentioned element of $\mathbf{R}_{\mathfrak{M}}(\Pi^{\varphi})(\mathbf{R}_{\mathfrak{M}}(\Sigma^{\varphi}))$, while the nucleus always is the "initial" conjunct (disjunct).

Now, let us generalize the concepts of conjunct and disjunct introduced above. Let $\mathsf{K}_1, \ldots, \mathsf{K}_n$ be conjuncts and $\Delta_1, \ldots, \Delta_m$ disjuncts. Then $\bigvee_{i=1}^{n} \mathsf{K}_i$ is called K-*disjunct*, and $\bigwedge_{i=1}^{m} \Delta_i$ D-*conjunct*. It is obvious that a K-*disjunct* (a D-conjunct) is a formula in DNF (CNF). Let $\varphi, \psi \in {}_0\mathsf{F}^t$. φ and ψ are *compatible* if the formula $\varphi \wedge \psi$ is valid.

Definition 8. Let Γ_1 and Γ_2 be K-disjuncts (D-conjuncts). Γ_1 and Γ_2 form a *contrary pair* if

(1) they can be represented in the forms

$$\Gamma_1 = \Gamma_1' \cup \{\mathsf{K}' \cup \{\pi\}\}$$

and

$$\Gamma_2 = \Gamma_2' \cup \{\mathsf{K}'' \cup \{\ \}\},$$

i.e., if Γ_1 contains a conjunct (disjunct) for which there exists a conjunct (disjunct) from Γ_2 such that the pair formed by them is a contrary pair;

(2) Γ_1' and Γ_2' are compatible.

And, if the K-disjuncts (D-conjuncts) Γ_1 and Γ_2 form a contrary pair then the K-disjunct (D-conjunct) $\Gamma_1' \cup \Gamma_2'$ will be called their *resolvent* and denoted by $\mathbf{R}^*(\Gamma_1, \Gamma_2)$.

We introduce the definition of all resolvents of a set S of K-disjuncts (D-conjuncts). Let

$$\mathbf{R}^0(S) \stackrel{d}{=} S$$

$\mathbf{R}^{n+1}(S) \stackrel{d}{=} \mathbf{R}^n(S) \cup \{\Gamma \mid$ there exist $\Gamma_1, \Gamma_2 \in \mathbf{R}^n(S)$ forming a contrary pair and $\Gamma = \mathbf{R}(\Gamma_1, \Gamma_2)\}$.

Let furthermore $\mathbf{R}(S) \stackrel{d}{=} \bigcup\limits_{n<\omega} \mathbf{R}^n(S)$. If S is finite then $\mathbf{R}(S)$ is finite too.

Definition 9. Let Γ_1 and Γ_2 be K-disjuncts (D-conjuncts) such that

$$\Gamma_1 \equiv \bigvee_{i=1}^{n_1} K_{i1} \left(\Gamma_1 \equiv \bigwedge_{i=1}^{m_1} \Delta_{i1}\right) \text{ and } \Gamma_2 \equiv \bigvee_{i=1}^{n_2} K_{i2} \left(\Gamma_2 \equiv \bigwedge_{i=1}^{m_2} \Delta_{i2}\right).$$

We shall say that Γ_1 *absorbs* Γ_2 (symbolically: $\Gamma_1 \prec \Gamma_2$) iff for all $i (i = 1, \ldots, n_1)$ there exists such a $j (j = 1, \ldots, m_2)$ that $K_{i1} \supseteq K_{j2}$ (for all $j \in \{1, \ldots, m_2\}$ there exists such an $i \in \{1, \ldots, m_1\}$ that $\Delta_{i1} \supseteq \Delta_{j2}$).

It is easy to show that any formula $\varphi \in {}_0F^t$ can be represented using D-conjuncts (K-disjuncts). For this it is enough to represent φ in the form of a disjunction (conjunction) of its subformulas and then all the subformulas in CNF (DNF).

Let $\varphi \in {}_0F^t$ be valid and representable in the form $\varphi \equiv \bigvee\limits_{i=1}^{n} \Pi_i$, where $\Pi_i (i = 1, \ldots, n)$ are D-conjuncts. The set of D-conjuncts $\{\Pi_1, \ldots, \Pi_n\}$ is denoted by Π^φ. For this case Theorem 14 can be rewritten in the following way:

Theorem 18 *Let Γ be a finite consistent set of literals. Then:*

$$\Gamma \models \varphi \Leftrightarrow \textit{there exists } \Pi \in \mathbf{R}(\Pi^\varphi) \textit{ such that } \Pi \succ \Gamma.$$

Proof: It is carried out simultaneously to the proof of Theorem 14 with the difference that for each $i (i = 1, \ldots, n)$ the set of resolvents $\mathbf{R}_i$ can be written in the form:

$$\mathbf{R}_i = \mathbf{R}_{i+1}^+ \cup \mathbf{R}_{i+1}^- \cup \mathbf{R}_{i+1}$$

where in this case

$$\mathbf{R}_{i+1}^+ = \{\Pi \mid \Pi \in \mathbf{R}_i \text{ and } \Pi = \Pi' \cup \{K' \cup \{\rho_{i+1}\}\}\}$$
$$\mathbf{R}_{i+1}^- = \{\Pi \mid \Pi \in \mathbf{R}_i \text{ and } \Pi = \Pi' \cup \{K' \cup \{\neg\rho_{i+1}\}\}\} \qquad \text{Q.E.D.}$$

Now, let $\varphi \in {}_0F^t$ be non-tautologic and let it be represented in the form $\varphi \equiv \bigwedge\limits_{i=1}^{m} \Sigma_i$ where $\Sigma_i (i = 1, \ldots, m)$ are K-disjuncts. Let $\Sigma^\varphi \stackrel{d}{=} \{\Sigma_1, \ldots, \Sigma_m\}$. From Theorem 15 it follows

Theorem 19 *Let Γ be a finite consistent set of literals. Then:*

$$\varphi \models \hat{\Gamma} \Leftrightarrow \textit{there exists } \Sigma \in \mathbf{R}(\Sigma^\varphi) \textit{ such that } \Sigma \succ \Gamma.$$

Proof is analogous with that of Theorem 15.

It has to be noted that the above introduced concepts of the $\mathfrak{A}$-clash and of the resolvent of the $\mathfrak{A}$-clash (see Definition 7) can easily be extended to K-disjuncts (D-conjuncts) (*see* [3]) and the corresponding theorems will be true in this case as well.

Now, let us consider Gentzen classical propositional calculus (*see*, e.g., [4]) and let us denote it by ${}_0\mathbf{G}$. The most important fact related to ${}_0\mathbf{G}$ is the Gentzen's theorem about the elimination of cut rule. Using this fact, we shall show that a sequence $\Gamma \to \Sigma$ (where $\Gamma, \Sigma \subseteq {}_0\mathsf{F}^t$) can be regarded as the statement $\Gamma \models \Sigma$. More exactly, the following theorem holds:

Theorem 20 $\vdash_{{}_0\mathrm{G}} \Gamma \to \Sigma \Leftrightarrow \Gamma \models \Sigma$.

Proof: Hereinafter we shall exclude the cut rule, when speaking of the derivation rules. $\Rightarrow$. If $\Gamma \to \Sigma$ is an axiom, then Γ and Σ contain the same formula and the statement is obviously true. It is easy to show that the derivation rules of ${}_0\mathbf{G}$ preserve the relation $\models$.

$\Leftarrow$. Remember that the calculus ${}_0\mathbf{G}$ does not contain the cut rule. Let us investigate the sequence $\Gamma \to \Sigma$ and build the derivation tree using by counter-application to it the derivation rules as long as it is possible. Since each counter-application reduces the logical length of a formula in this sequence, it is a finite process. Let $\Gamma_1 \to \Sigma_1, \ldots, \Gamma_k \to \Sigma_k$ and let them all be "final" sequences. It is obvious that, Γ_i and Σ_i consist of prime formulas only for any $i \in \{1, \ldots, k\}$. According to Theorem 6, $\hat{\Gamma}_i \models \check{\Sigma}_i$ holds for any i. According to Theorem 8, this is possible only if $\Sigma_i \cap \Gamma_i \neq \emptyset$, i.e., if there exists a π_i belonging to $\Gamma_i \cap \Sigma_i$. Consequently, $\Gamma_i \to \Sigma_i$ is an axiom and the constructed tree is a derivation one. Q.E.D.

7 *Some properties of the languages* ${}_1\mathcal{L}^t$ *and* ${}_2\mathcal{L}^t$ Now we introduce the following notation: Let $v \in {}_1V$, $\varphi \in {}_1\mathsf{F}^t$ and $\tau \in {}_1T^t$. $\varphi[v/\tau]$ denotes the formula obtained from φ by replacing each free occurrence of v in φ by τ.

$\varphi[v/\tau]$ can be defined in the following way:

(i) $\rho(\tau_1, \ldots, \tau_n)[v/\tau] = \rho(\tau_1[v/\tau], \ldots, \tau_n[v/\tau])$ and in $\tau_i[v/\tau]$ there is no collision
(ii) $\neg\psi[v/\tau] \stackrel{d}{=} \neg(\psi[v/\tau])$
(iii) $(\psi \wedge \chi)[v/\tau] \stackrel{d}{=} \psi[v/\tau] \wedge \chi[v/\tau]$
(iv) $(\exists w)\psi[v/\tau] = (\exists w)(\psi[v/\tau]$ if w does not occur in τ; otherwise, let $z \in V$ be a new variable not occurring in the formula and

$$(\exists w)\psi[v/\tau] \stackrel{d}{=} (\exists z)(\psi[w/z])[v/\tau]$$

$[v/\tau]$ is called a substitution. The substitution $[v_1/\tau_1, \ldots, v_n/\tau_n]$ can be defined in a similar way.

Definition 10. The formula φ is called prenex formula if $\varphi \equiv Q_1x_1 \ldots Q_nx_n\psi$, where $Q_i (i = 1, \ldots, n)$ is either $\exists$ or $\forall$, and ψ is a formula which contains no quantifier.

Theorem 21 *For any formula* $\varphi \in {}_1\mathsf{F}^t$ *there exists a prenex formula* φ' *such that* $\varphi' \equiv \varphi$.

Proof: It can be carried out by induction on the length of the formula φ with the help of

(i) the following equalities:

$$\begin{aligned}
\varphi \wedge \psi &\equiv \psi \wedge \varphi \\
\neg \exists x \varphi &\equiv \forall x \neg \varphi \\
\neg \forall x \varphi &\equiv \exists x \neg \varphi \\
(\exists x \psi) \wedge \chi &\equiv \exists z(\psi[x/z] \wedge \chi) \\
(\forall x \psi) \wedge \chi &\equiv \forall z(\psi[x/z] \wedge \chi) \text{ where } z \in V \text{ does not occur in } \psi \wedge \chi;
\end{aligned}$$

and

(ii) the following fact: for any $\varphi \in {}_1\mathsf{F}^t$ and $x \in V$:

$$\mathfrak{A} \models \varphi \Longleftrightarrow \mathfrak{A} \models \forall x \varphi \qquad \text{Q.E.D.}$$

We introduce the following notations:

$$\exists x_1 \ldots \exists x_n \overset{\mathrm{d}}{=} \exists x_1 \ldots x_n \text{ and } \forall x_1 \ldots \forall x_n \overset{\mathrm{d}}{=} \forall x_1 \ldots x_n.$$

Theorem 22 (Skolem's Existential Normal Form Theorem for ${}_2\mathcal{L}^t$)

$\exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \psi \equiv \forall f_1 \ldots f_n \exists x_1 \ldots x_n \psi[y_1/f_1(x_1) \ldots y_n/f_n(x_1 \ldots x_n)]$ where $\psi \in {}_1\mathsf{F}^t$ and for any $i \in \{1, \ldots, n\}$ $x_i, y_i \in V$, $f_i \in V_i^F$ and f_i does not occur in ψ.

Proof: It is enough to prove the statement for the formula $\exists x \forall y \chi$, where χ still may contain quantifiers. We shall make use of the definition of the relation $\models$. If $\mathfrak{A} \models \exists x \forall y \chi$, then there exists an element ${}_1k(x) \in A$ for which χ is valid for any element ${}_1k(y) \in A$. Thus, χ is valid for ${}_1k(f(x)) \in A$ as well, i.e.,

$$\mathfrak{A}_2 \not\models^t \forall f \exists x \chi[y/f(x)].$$

Otherwise, if $\mathfrak{A} \not\models \exists x \forall y \chi$, then there is no ${}_1k(x) \in A$ for which any $k(y)$ is "good". This defines a function which renders the corresponding "bad" ${}_2k(y)$ to each $k_1(x)$. Let us note that here we made use of the axiom of choice in an intuitive manner. Thus

$$\mathfrak{A}_2 \not\models^t \forall f \exists x \chi[y/f(x)].$$

To complete the proof, the statement has to be shown for the formulas $\exists x_1, \ldots, x_n \forall y \chi$ and $\exists x_1, \ldots, x_n \forall y_1, \ldots, y_n \psi$. This can easily be done by induction. Q.E.D.

The following theorem is the dual pair of the preceding one.

Theorem 23 (Skolem's Universal Normal Form Theorem for ${}_2\mathcal{L}^t$) *Let* $\psi \in {}_1\mathsf{F}^t$ *and* $x_i, y_i \in V, f_i \in V_i^F$ *for any* $i \in \{1, \ldots, n\}$ *while* f_i *does not occur in* ψ. *Then:*

$$\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \psi \equiv \forall x_1 \ldots x_n \exists f_1 \ldots f_n \psi[y_1/f_1(x_1), \ldots, y_n/f_n(x_1, \ldots, x_n)]$$

Proof: Without restricting generalities, we investigate the formula $\forall x \exists y \chi$. In the proof, we make use of the definition of $\models$. Let $\mathfrak{A}_1 \models^t \forall x \exists y \chi$. Then, for any ${}_1k(x) \in A$, there exists such an element ${}_1k(y) \in A$ that χ is valid. The axiom of choice permits the determination of such a function $\mathfrak{A}_f$, for which $\mathfrak{A}_f(k(x)) = k(y)$. Therefore:

$$\mathfrak{A}_2 \models^t \forall x \exists f \chi[y/f(x)].$$

Now, let $\mathfrak{A}_2 \models^t \forall x \exists f \chi[y/f(x)]$. This means that for any ${}_2k(x) \in A$ there exists a functional variable f such that χ is valid for the element ${}_2\overline{k}(f(x)) \in A$. Consequently, there exists an element ${}_1k(y) \in A$, determined by f, for which χ is valid. Hence, $\mathfrak{A}_1 \models^t \forall x \exists y \chi$. Q.E.D.

Definition 11. We say that a type t has enough function symbols if and only if t contains infinitely many function symbols of each arity, i.e., for any $n < \omega$, there are infinitely many symbols $f_n^1, f_n^2, \ldots \in \mathrm{Do}\, t''$ such that $t''(f_n^i) = n$ $(i < \omega)$.

Corollary 22.1 *If the type t has enough function symbols then, for any such formula* $\varphi \in {}_1\mathrm{F}^t$ *that* $\varphi \equiv \exists x_1 \forall y_1 \ldots \exists x_n \forall y\, \psi$, *there exist terms* $\tau_1, \ldots, \tau_n \in {}_1T^t$ *such that*

$$\models \exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \psi \Leftrightarrow \exists x_1 \ldots x_n \psi[y_1/\tau_1, \ldots, y_n/\tau_n].$$

Proof: From Theorem 22 we have

$$\models \exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \psi \Leftrightarrow \models \forall f_1 \ldots f_n \exists x_1 \ldots x_n \psi[y_1/f_1(x_1), \ldots, y_n/f_n(x_1 \ldots x_n)]$$

Now, let us choose, for each functional variable $f_i (i = 1, \ldots, n)$ a corresponding functional symbol f_i' from $\mathrm{Do}\, t''$ such that $t''(f_i') = i$ and f_i' does not occur in ψ. This can be done because t has enough function symbols. From the definition of $\mathbf{M}^t$ and from that of $\models$ we have

$$\begin{aligned} &{}_2\models^t \forall f_1 \ldots f_n \exists x_1 \ldots, x_n \psi[y_1/f_1(x_1), \ldots, y_n/f_n(x_1, \ldots, x_n)] \\ &\Leftrightarrow {}_1\models^t \exists x_1 \ldots x_n \psi[y_1/f_1'(x_1), \ldots, y_n/f_n'(x_1, \ldots, x_n)]. \end{aligned}$$

$\Rightarrow$. In fact, we assumed, without restricting generalities, that ${}_2\models^t \forall f \exists x \psi[y/f(x)]$ where $f \in V_1^F$ and $x \in V$. This means that, for any model $\mathfrak{A} \in \mathbf{M}^t$, there exists an element ${}_2k(x) \in A$ such that χ is valid for any ${}_2\overline{k}(f(x))$. Let f be an arbitrary fixed function variable and f' the function symbol corresponding to it. Then there exists a model $\mathfrak{B} \in \mathbf{M}^t$ such that $\mathfrak{B}_1 \models^t \exists x \psi[y/f'(x)]$. Since this is true for any f then we can conclude that

$${}_1\models^t \exists x \psi[y/f'(x)].$$

$\Leftarrow$. Let ${}_2\not\models^t \forall f \exists x \psi[y/f(x)]$. This means that there exists a model $\mathfrak{A}$ such that

$$\mathfrak{A}_2 \not\models^t \forall f \exists x \psi[y/f(x)],$$

which, in turn, means that for any ${}_2k(x)$ there exists such a ${}_2k(f)$ which ψ is not valid. Let us consider now the functional symbol $f' \in \mathrm{Do}\, t''$ corresponding to this f. Then there exists a model $\mathfrak{B}$ such that

$$\mathfrak{B}_1 \not\models^t \exists x \psi[y/f'(x)]$$

Consequently,

$${}_1\not\models^t \exists x \psi[y/f'(x)]$$

Q.E.D.

Remark: The above corollary says that for any formula $\varphi \in {}_1\mathsf{F}^t$ there exists its existential representation while φ is tautological if and only if its existential representation is tautologic. The latter statement is weaker than that of Theorem 22. This is connected with the fact that the interpretation of functional variables differs from that of functional symbols (see Definition 5). Accordingly, the statement that, if $\mathfrak{A}$ is a model of φ, then $\mathfrak{A}$ also provides a model for the existential representation of φ, is not true. We can only say that there exists a model $\mathfrak{B} \in \mathbf{M}^t$ in which the existential representation of the formula φ is valid.

Corollary 23.1 (Skolem's Normal Form for Validity for ${}_1\mathcal{L}^t$) *Let the type t have enough function symbols. Then for any formula $\varphi \in {}_1\mathsf{F}^t$ of the form $\varphi \rightleftharpoons \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \psi$ there exist such terms $\tau_1, \ldots, \tau_n \in {}_1T^t$ that φ is valid if and only if the formula $\forall x_1, \ldots, x_n \psi[y_1/\tau_1, \ldots, y_n/\tau_n]$ is valid.*

Proof: Without restricting generality, we suppose that

$$\varphi \rightleftharpoons \forall x \exists y \chi$$

$\Rightarrow$. From Theorem 23 it follows that $\forall x \exists y \chi$ is valid if and only if $\forall x \exists f \chi[y/f(x)]$ is valid. Let $\forall x \exists f \chi[y/f(x)]$ be valid. This means that there exists a model $\mathfrak{A} \in \mathbf{M}^t$ such that for any element ${}_2k(x) \in A$ there is such a function variable f that χ is valid for ${}_2k(f(x))$. Let $f' \in \mathrm{Dot}''$ correspond to the function variable f. As the type t has enough function symbols, it follows from the axiom of choice that there exists a model $\mathfrak{B}$ in which the interpretations $\mathfrak{B}_{f'}$ and ${}_2k(f)$ are equivalent. Consequently,

$$\mathfrak{B}_1 \models^t \forall x \chi[y/f'(x)].$$

$\Leftarrow$. Let $\mathfrak{A}_1 \models^t \forall x \chi[y/f'(x)]$ and let us choose a functional variable f corresponding to the functional symbol f'. Then we can find a model $\mathfrak{B}$ in which the formula χ is valid by ${}_2k(f)$ any element ${}_2k(x) \in B$, i.e.,

$$\mathfrak{B}_2 \models^t \forall x \exists f \chi[y/f(x)]. \qquad \text{Q.E.D.}$$

When establishing the semantic properties of formulas from ${}_1\mathsf{F}^t$, it would be convenient to use the methods available for the language ${}_0\mathcal{L}^t$. This is possible according to Herbrand's Theorem (*see*, e.g., [1]). Two theorems semantically analogous with Herbrand's Theorem will be formulated: the first one relates to tautology and the other one to validity. Remember that ${}_1\mathsf{S}^t$ denotes the set of first order closed formulas.

Theorem 24 *Let t have enough functional symbols. Then for any formula $\varphi \in {}_1\mathsf{S}^t$ there exists a finite set of formulas $\Sigma \subseteq {}_0\mathsf{F}^t$ such that φ is tautological if and only if $\check{\Sigma}$ is tautological.*

Proof: For any formula $\varphi \in {}_1\mathsf{F}^t$, according to Corollary 22.1, there exists an existential representation. We assume, without restricting generality, that it has the form $\exists x \psi$. Then $\models \varphi \Leftrightarrow \exists x \psi$.

$\Rightarrow$. As it is known from Theorem 1, any model $\mathfrak{A} \in \mathbf{M}^t$ has a smallest C-submodel $\mathfrak{B}$. Since $\models \exists x \psi$, then

$$\mathfrak{B} \models \exists x \psi.$$

Hence, according to the corollary of Theorem 2 and to Definition 5, there exists a term $\tau \in {}_0T^t$ such that $\mathfrak{B} \models \psi[x/\tau]$. As $\psi[x/\tau] \in {}_0\mathrm{F}^t$ and $\mathfrak{B} \subseteq \mathfrak{A}$, then $\mathfrak{A} \models \psi[x/\tau]$ (see Theorem 2). Consequently,

$$\models \bigvee_{\tau \in {}_0T^t} \psi[x/\tau].$$

From the Compactness Theorem (Theorem 3) it follows that there exists a finite subset of terms $T \subseteq {}_0T^t$ such that

$$\models \bigvee_{\tau \in T} \psi[x/\tau]$$

$\Leftarrow$. From $\models \bigvee_{\tau \in T} \psi[x/\tau]$ it is obvious that $\models \exists x \psi$. Q.E.D.

Theorem 25 *Let t have enough functional symbols. Then, for any formula* $\varphi \in {}_1\mathrm{S}^t$, *there exists a set of formulas* $\Sigma \subseteq {}_0\mathrm{F}^t$ *such that* φ *is valid if and only if the formula* $\hat{\Gamma}$ *is valid for any finite subset* $\Gamma \subseteq \Sigma$.

Proof: $\Rightarrow$. Let $\varphi \in {}_1\mathrm{S}^t$ be valid. Then, according to Corollary 23.1, its universal representation is also true. Let us assume that the latter has the form $\forall x \psi$ where ψ is a quantifier free formula. Let $\mathfrak{A} \models \forall x \psi$ and let $\mathfrak{B}$ be the C-submodel of $\mathfrak{A}$. Then $\mathfrak{B} \models \forall x \psi$ (see Theorem 2). Hence, according to the corollary of Theorem 2 and to Definition 5 $\mathfrak{B} \models \psi[x/\tau]$ for any term $\tau \in {}_0T^t$. Consequently, $\mathfrak{A} \models \bigwedge_{\tau \in {}_0T^t} \psi[x/\tau]$. From the Compactness Theorem (Theorem 3′) it follows that $\mathfrak{A} \models \bigwedge_{\tau \in T} \varphi[x/\tau]$ for any finite subset of terms $T \subseteq {}_0T^t$.

$\Leftarrow$. Let $\mathfrak{A} \models \bigwedge_{\tau \in T} \psi[x/\tau]$ for any finite subset of terms $T \subseteq {}_0T^t$. Then, according to Theorem 3′, $\mathfrak{A} \models \bigwedge_{\tau \in {}_0T^t} \psi[x/\tau]$. Consequently, $\mathfrak{A} \models \forall x \psi$. Then, according to Corollary 23.1 it can be concluded that φ is valid. Q.E.D.

With the help of Theorems 24 and 25 any formula of the language ${}_1\mathcal{L}^t$ can be reduced to a set of formulas of ${}_0\mathcal{L}^t$. Thus, all the theorem proving methods developed for formulas from ${}_0\mathrm{F}^t$ are also applicable to the language ${}_1\mathcal{L}^t$. However, it has to be noted that there exist syntactical methods which, in order to increase the effectivity of establishing the properties of the formulas of ${}_1\mathcal{L}^t$, transform the formulas into those of ${}_0\mathcal{L}^t$ after performing all the possible simplifications.

REFERENCES

[1] Andréka, H., T. Gergely, and I. Németi, "Easily comprehensible mathematical logic and its model theory," *KFKI*-75-24, Budapest (1975).

[2] Church, A., *Introduction to Mathematical Logic,* Princeton University Press, Princeton (1956).

[3] Leleski, A. V., and A. I. Malashonoc, *Calculus of K-clauses*, Mathematical questions of the theory of intelligent machines, Kiev (1975). (pp. 3-33 in Russian).

[4] Gentzen, G., "Untersuchungen über das logishe Schliessen, I.II," *Mathematische Zeitschrift*, B.39 (1935), pp. 176-210 and 405-443.

T. Gergely:

Research Institute of Applied Computer Science
Budapest, Hungary

and

K. P. Vershinin:

Institute of Cybernetics
Kiev, U.S.S.R.