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SELF-CONJUGATE FUNCTIONS ON BOOLEAN ALGEBRAS

THOMAS A. SUDKAMP

In section **1** we investigate the properties of a self-conjugate function on a Boolean algebra (**BA**). In section **2**, the notion of self-conjugacy will then be extended to functions of more than one variable and Boolean algebras with operators in which each of the additional operations has this property will be examined. In particular, we extend the definition of ideal element to every structure of this type. The notation of [3] will be used and a familiarity with that paper is assumed. In addition, f^n will represent the composition of f with itself n times and id will denote the identity function on a **BA**.

1 *Self-conjugate functions* Throughout this section let

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1\rangle$$

be a fixed **BA**. Recall the following definition and theorem from [3]:

Definition 1.1: Let f and g be functions on A to A. We say that g is conjugate of f if, for any $x, y \in A$, we have

$$f(x) \cdot y = 0 \text{ if, and only if, } g(y) \cdot x = 0.$$

If, in particular, a function f is conjugate of itself, then we call f self-conjugate.

Theorem 1.2 *If $f: A \rightarrow A$, then the following are equivalent:*

(i) *f is self-conjugate.*
(ii) *f is additive, and $f(\overline{f(\overline{x})}) \cdot x = 0$ for all $x \in A$.*
(iii) *f is normal, and $f(x) \cdot y \leq f(x \cdot f(y))$ for all $x, y \in A$.*

Lemma 1.3 *If $f: A \rightarrow A$ is additive and $x \leq y$, then $f(x) \leq f(y)$ for all $x, y \in A$.*

Proof: Obvious.

Theorem 1.4 *If $f: A \rightarrow A$ is self-conjugate and $f(1) = 1$, then $f(x) \not< x$ for any $x \in A$.*

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Proof: Assume $f(x) \leqslant x$, then $f(x) \cdot \overline{x} = 0$. Hence, by 1.1, $f(\overline{x}) \cdot x = 0$, or equivalently $f(\overline{x}) \leqslant \overline{x}$. But

$$1 = f(1) = f(x + \overline{x}) = f(x) + f(\overline{x})$$

which yields $f(x) = x$ and $f(\overline{x}) = \overline{x}$.

Corollary 1.5 *If $f: A \to A$ is self-conjugate and $f(1) = 1$, then $f(x) = 0$ iff $x = 0$.*

Proof: By 1.2(iii) and 1.4.

Corollary 1.6 *If $f: A \to A$ is self-conjugate and $f(1) = 1$, then $f(x) = x$ iff $f(\overline{x}) = \overline{x}$.*

Lemma 1.7 *If $f: A \to A$ is self-conjugate and $f(1) = a$ for some $a \in A$, then $f(a) = a$ and $f(\overline{a}) = 0$.*

Proof: Since $f(1) = a$, $f(1) \cdot \overline{a} = 0$. So by 1.1 $f(\overline{a}) \cdot 1 = f(\overline{a}) = 0$. Now, by 1.2(ii), we conclude

$$a = f(1) = f(a + \overline{a}) = f(a) + f(\overline{a}) = f(a).$$

Lemma 1.8 *If $f: A \to A$ is self-conjugate, $f(a) \leqslant a$ and $f(\overline{a}) \leqslant \overline{a}$ for some $a \in A$, then $a \cdot f(x) = f(a \cdot x)$ for all $x \in A$.*

Proof: By 1.2(ii) and the remark following Theorem 2.18 in [1], p. 356.

Theorem 1.9 *$f: A \to A$ is self-conjugate and $f(1) = a$ for some $a \in A$ if, and only if, there exists a function $\hat{f}: A \to A$ which satisfies the following conditions:*

(i) *$\hat{f}$ is self-conjugate,*
(ii) *$\hat{f}(1) = 1$, and $\hat{f}(a) = a$,*
(iii) *$f(x) = a \cdot \hat{f}(x)$ for all $x \in A$.*

Proof: Given f as above, define $\hat{f}$ as follows:

$$\hat{f}(x) = f(x) + x \cdot \overline{a}.$$

f clearly satisfies (ii). By 1.3 and 1.7

$$a \cdot \hat{f}(x) = a \cdot (f(x) + x \cdot \overline{a}) = a \cdot f(x) + a \cdot x \cdot \overline{a} = a \cdot f(x) = f(x) \text{ for all } x \in A.$$

$\hat{f}$ is self-conjugate since it is the sum of two self-conjugate functions. Now let f satisfy (i), (ii), and (iii), by 1.1, 1.6, and 1.8

$$f(x) \cdot y = 0 \leftrightarrow \hat{f}(x) \cdot a \cdot y = 0 \leftrightarrow \hat{f}(a \cdot y) \cdot x = 0 \leftrightarrow \hat{f}(y) \cdot a \cdot x = 0 \leftrightarrow f(y) \cdot x = 0,$$

so we conclude by 1.1 that f is self-conjugate. Clearly $f(1) = a$.

Throughout the rest of this paper by self-conjugate function we mean a self-conjugate function in which $f(1) = 1$.

Lemma 1.20 *If $f: A \to A$ is additive and $f^2 = \mathrm{id}$, then $f(1) = 1$ and $f(x) = 0$ iff $x = 0$.*

Proof: $f(0) = 0$ and $f(1) = 1$ follow from 1.3. Assume $f(x) = 0$, then by 1.3

$$f(f(x)) = x \leqslant f(x) = 0.$$

Lemma 1.21 *If $f\colon A \to A$ is additive and $f^2 = \mathrm{id}$, then $f(x) \not< x$ for any $x \in A$.*

Proof: $f(x) \leqslant x$ implies, by 1.3, $x = f(f(x)) \leqslant f(x) \leqslant x$.

Lemma 1.22 *If $f\colon A \to A$ is additive and $f^2 = \mathrm{id}$, then $f(\bar{x}) = \overline{f(x)}$ for all $x \in A$.*

Proof: By 1.20, $1 = f(x) + f(\bar{x})$, hence $f(\bar{x}) = \overline{f(x)} + b$ where $b \leqslant f(x)$. Now

$$\bar{x} = f(f(\bar{x})) = f(\overline{f(x)} + b) = f(\overline{f(x)}) + f(b),$$

but by 1.3, $f(b) \leqslant f(f(x)) = x$. Hence $f(b) = 0$ which, by 1.20, implies $b = 0$.

Lemma 1.23 *If $f\colon A \to A$ is self-conjugate, then $f^n\colon A \to A$ is self-conjugate.*

Proof: By 1.1.

Lemma 1.24 *If $f\colon A \to A$ is self-conjugate and $f^2(x) \not> x$ for any $x \in A$, then $f(x) \not> x$ for any $x \in A$.*

Proof: Let $f(x) = x + b$, $b \leqslant \bar{x}$, then $f(f(x)) = f(x + b) = f(x) + f(b) = x + b + f(b)$. Hence $b = 0$.

Theorem 1.25 *If $f\colon A \to A$, then the following are equivalent:*

(i) *f is additive, and $f^2 = \mathrm{id}$.*
(ii) *f is self-conjugate, and $f^2(x) \not> x$ for any $x \in A$.*

Proof: (i) implies (ii). By 1.22 and (i)

$$f(\overline{f(x)}) \cdot x = f(f(\bar{x})) \cdot x = \bar{x} \cdot x = 0.$$

By 1.20 and 1.2(ii) we conclude that f is self-conjugate. Trivially, $f^2(x) > x$.

(ii) implies (i). Additivity follows from 1.2(ii). Let

$$x = x \cdot f(x) + x \cdot \overline{f(x)}.$$

By 1.2(iii), $x \cdot f(x) \leqslant f(x \cdot f(x))$ and therefore, by 1.24,

$$x \cdot f(x) = f(x \cdot f(x)). \tag{1}$$

Now let $b = x \cdot \overline{f(x)}$, $c = b - f(f(b))$, then $f(f(b)) \cdot c = 0$. By 1.1 $f(b) \cdot f(c) = 0$, so 1.3 implies $f(c) = 0$ and 1.5 yields $c = 0$. Hence $f(f(b)) \geqslant b$, and by (ii) we obtain

$$f(f(x \cdot \overline{f(x)})) = x \cdot \overline{f(x)}. \tag{2}$$

(i) follows from (1), (2), and additivity.

Lemma 1.26 *If $f\colon A \to A$ is self-conjugate, then $f(x) \leqslant f^n(x)$ for all $x \in A$ and* $n > 2$, for n odd.

Proof: By 1.2(iii)

$$f(1) \cdot f^{n-2}(x) \leqslant f(1 \cdot f^{n-1}(x)) = f^n(x).$$

Theorem 1.27 *If* $f: A \to A$ *is self-conjugate and* $f^n = \mathrm{id}$, *then* $f = \mathrm{id}$ *or* $n = 2$.

Proof: Assume $f \neq \mathrm{id}$. For all $x \in A$, $f(x) \not> x$, for if $f(x) > x$, then $f^{n-1}(f(x)) < f(x)$, contradicting 1.4. Also by 1.4, $f(x) \not< x$ for any $x \in A$. This implies that there is a $b \in A$ such that $f(b) - b = c \neq 0$. By 1.26 $c \leq f^n(b)$ if $n > 2$, for n odd. If n is even, result follows from 1.25(ii).

Theorem 1.28 *If* $f: A \to A$ *is self-conjugate and* $f^2 = \mathrm{id}$, *then* $f(x \cdot y) = f(x) \cdot f(y)$ *for all* $x, y \in A$.

Proof: By 1.3, $f(x \cdot y) \leq f(x)$ and $f(x \cdot y) \leq f(y)$, so

$$f(x \cdot y) \leq f(x) \cdot f(y).$$

By 1.2(iii) and $f^2 = \mathrm{id}$ we get

$$f(x) \cdot f(y) \leq f(x \cdot f(f(y))) = f(x \cdot y).$$

Lemma 1.29 *If* $f: A \to A$ *is self-conjugate and* $f^2 = f$, *then* $f(x) \geq x$ *for all* $x \in A$.

Proof: Let $b = x - f(x)$. By 1.1, 1.3, and hypothesis $f(x) \cdot b = 0$, so

$$f(f(x)) \cdot b = 0 \leftrightarrow f(x) \cdot f(b) = f(b) = 0.$$

Hence, by 1.5, $b = 0$.

Theorem 1.30 *If* $f: A \to A$, *then the following are equivalent:*

(i) $f(0) = 0$, $f(x) \geq x$, *and* $f(x \cdot f(y)) = f(x) \cdot f(y)$ *for all* $x, y \in A$.
(ii) f *is self-conjugate, and* $f^2 = f$.

Proof: (i) implies (ii). $f(x) \geq x$ implies $f(1) = 1$. Substituting 1 for x in the equation in (i) we get

$$f(f(y)) = f(1 \cdot f(y)) = f(1) \cdot f(y) = f(y) \text{ for all } y \in A.$$

Since $x \leq f(x)$ and $f(f(x)) = f(x)$,

$$y \cdot f(x) \leq f(x) \cdot f(y) = f(x \cdot f(f(y))) = f(x \cdot f(y)) \text{ for all } x, y \in A.$$

Hence, by 1.2(iii), f is self-conjugate.

(ii) implies (i). Additivity follows from 1.2(ii) and $x \leq f(x)$ by 1.29. Let $b = f(x) \cdot f(y)$, then $b \leq f(x)$ and $b \leq f(y)$. By 1.3 and 1.27

$$f(b) \leq f(f(x)) = f(x)$$

and

$$f(b) \leq f(f(y)) = f(y).$$

Hence $f(b) = f(f(x) \cdot f(y)) \leq f(x) \cdot f(y)$. Equality follows by 1.29, so we obtain

$$(1) \qquad f(x) \cdot f(y) = f(f(x) \cdot f(y)).$$

Again by 1.29, $x \cdot f(y) \leq f(x) \cdot f(y)$, so by (1)

$$f(x \cdot f(y)) \leq f(f(x) \cdot f(y)) = f(y) \cdot f(x).$$

On the other hand, 1.2(iii) and (ii) yield

$$f(x)\cdot f(y) \leq f(x\cdot f(f(y))) = f(x\cdot f(y)).$$

These two inequalities finish the proof.

The class of self-conjugate functions described in 1.30 is important in the field of algebraic logic. Any function on a **BA** which attempts to express an existential quantifier on a Boolean algebra of formulas must satisfy 1.30(ii), since these conditions translate to the valid formulas

$$(\exists x\varphi)\wedge\psi \leftrightarrow \varphi\wedge(\exists x\psi) \text{ and } \exists x(\exists x\varphi)\leftrightarrow\exists x\varphi.$$

Lemma 1.31 *If $f\colon A\to A$ and $g\colon A\to A$ are self-conjugate, then the following are equivalent:*

(i) $f\circ g = g\circ f$.
(ii) *$f\circ g$ is self-conjugate.*

Proof: (i) implies (ii). By 1.1 and (i)

$f(g(x))\cdot y = 0 \leftrightarrow g(f(x))\cdot y = 0 \leftrightarrow f(x)\cdot g(y) = 0 \leftrightarrow f(g(y))\cdot x = 0$ for all $x, y\in A$.

Hence, by 1.1, $f\circ g$ is self-conjugate.

(ii) implies (i). Again by 1.1

$$f(g(x))\cdot y = 0 \leftrightarrow f(y)\cdot g(x) = 0 \leftrightarrow g(f(y))\cdot x = 0 \text{ for all } x, y\in A.$$

This states that $f\circ g$ and $g\circ f$ are conjugate functions, but then, by (ii) and the fact that conjugate functions are unique (Theorem 1.13 in [3]), $f\circ g = g\circ f$.

Now using 1.30 and 1.31 we can obtain an equivalent, independent axiom system for diagonal free cylindric algebras using the notion of self-conjugacy. Recall the following definition, *cf.* [2], p. 168, Definition 1.19:

Definition 1.32: A diagonal free cylindric algebra of dimension α, where α is an ordinal number, is an algebraic structure

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathbf{c}_\kappa\rangle_{\kappa<\alpha}$$

where A is an arbitrary set closed under the binary operations $+$ and $\cdot$, the unary operation $-$, and the unary operation $\mathbf{c}_\kappa$ for $\kappa = 1, 2, 3, \ldots$, and containing constants 0 and 1 which satisfies the following postulates (for all $\kappa, \lambda < \alpha$ and $x, y\in A$):

(C_0) $\langle A, +, \cdot, -, 0, 1\rangle$ is a **BA**
(C_1) $\mathbf{c}_\kappa(0) = 0$
(C_2) $x \leq \mathbf{c}_\kappa(x)$
(C_3) $\mathbf{c}_\kappa(x\cdot\mathbf{c}_\kappa(y)) = \mathbf{c}_\kappa(x)\cdot\mathbf{c}_\kappa(y)$
(C_4) $\mathbf{c}_\kappa(\mathbf{c}_\lambda(x)) = \mathbf{c}_\lambda(\mathbf{c}_\kappa(x))$.

Theorem 1.33 *A structure*

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathbf{c}_\kappa\rangle_{\kappa<\alpha}$$

is a diagonal free cylindric algebra of dimensions α if, and only if, $\mathfrak{A}$ satisfies the following conditions:

(P_0) $\langle A, +, \cdot, -, 0, 1\rangle$ *is a* **BA**
(P_1) $\mathbf{c}_\kappa \cdot \mathbf{c}_\lambda$ *is self-conjugate for every* $\kappa, \lambda < \alpha$
(P_2) $\mathbf{c}_\kappa^2 = \mathbf{c}_\kappa$ *for every* $\kappa < \alpha$.

Proof: (P_0) and (P_1) imply $\mathbf{c}_\kappa$ is self-conjugate for every $\kappa < \alpha$. Then the result follows from 1.30 and 1.31.

2 *Structures with self-conjugate functions* In this section we generalize the notion of self-conjugate function to functions of n variables. Let

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1\rangle$$

be a fixed **BA**.

Definition 2.1: Let $f: A^n \to A$ be an additive function. By the ith projection of f, denoted f^i, we mean the unary function $f^i: A \to A$ obtained by replacing all the variables of f, except the ith, by the constant 1.

Definition 2.2: Let $f: A^n \to A$ be an additive function. We say f is totally self-conjugate if $f(1, 1, \ldots, 1) = 1$ and, for each $1 \leq i \leq n$, f^i is self-conjugate.

For the case $n = 1$, 2.2 reduces to the original definition of self-conjugate with the additional condition that $f(1) = 1$.

Definition 2.3: Let

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, f_\varepsilon\rangle_{\varepsilon<\alpha}$$

be a **BA** with operators. $\mathfrak{A}$ is called a self-conjugate **BA** (**SBA**) if each non-constant function f_ε is totally self-conjugate.

Definition 2.4: An **SBA**

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, f_\varepsilon\rangle_{\varepsilon<\alpha}$$

is called discrete if, for every non-constant $f_\varepsilon: A^n \to A$,

$$f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i \text{ for all } x \in A.$$

Theorem 2.5 *If* $f: A^n \to A$ *is totally self-conjugate,* $f(a, c) \leq a$, *and* $f(\bar{a}, c) \leq \bar{a}$ *for* $a \in A$, $c \in A^{n-1}$, *then* $a \cdot f(x, c) = f(a \cdot x, c)$ *for all* $x \in A$.

Proof: By 2.2 and the remark following Theorem 2.18 in [1], p. 356.

Now we let

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, f_\varepsilon, \mathbf{c}_\delta\rangle_{\varepsilon<\alpha, \delta<\beta}$$

be a fixed **SBA** in which f_ε is totally self-conjugate, for each $\varepsilon < \alpha$, and $\mathbf{c}_\delta$ is constant, for each $\delta < \beta$. We let f denote an arbitrary f_ε and f^i the ith projection. Also, we let n_ε denote the rank of f_ε.

Definition 2.6: An element a, $a \in A$, is said to be an ideal element if $f_\varepsilon^i(a) = a$ for each $\varepsilon < \alpha$, $1 \leq i \leq n_\varepsilon$.

Theorem 2.7 *If a is an ideal element, $a \in A$, then* $a \cdot f(x_1, x_2, \ldots, x_n) = f(a \cdot x_1, a \cdot x_2, \ldots, a \cdot x_n)$ *for all* $x_i \in A$.

Proof: By 1.6 $f^1(a) = a$ and $f^1(\bar{a}) = \bar{a}$, so by additivity $f(a, x_2, x_3, \ldots, x_n) \leq a$ and $f(\bar{a}, x_2, x_3, \ldots, x_n) \leq \bar{a}$ for all $x_2, x_3, \ldots, x_n \in A$. Hence 2.6 yields

$$a \cdot f(x_1, x_2, \ldots, x_n) = f(a \cdot x_1, x_2, \ldots, x_n) \text{ for all } x_i \in A.$$

Repeating this for each argument completes the proof.

Theorem 2.8 *If a and b are ideal elements, then*

(i) $\bar{a}$ *is an ideal element,*
(ii) $a + b$ *is an ideal element,*
(iii) $a \cdot b$ *is an ideal element,*
(iv) 0 *and* 1 *are ideals elements.*

Proof: (i). By 2.2, 2.6, and 1.5.

(ii) By 2.2, 2.6, and additivity.
(iii) By 1.8 and 2.6,

$$b \cdot a = b \cdot f^i(a) = f^i(b \cdot a).$$

(iv) 0 is an ideal element by 1.2(iii) and 1 by 2.2.

Now we show that the set of ideal elements is closed under the operations f_ε. Note that if f is an n-ary totally self-conjugate function, then the function f' obtained from f by replacing $m < n$ of the arguments by 1 is totally self-conjugate. Moreover, ideal elements of f act as ideal elements of f'.

Theorem 2.9 *If g: $A^n \to A$ is totally self-conjugate and $a_1, a_2, \ldots, a_n$ act as ideal elements for g (that is, $g^i(a_j) = a_j$ for $i, j < n$), then*

$$g(a_1, a_2, \ldots, a_n) = \prod_{i=1}^{n} a_i.$$

Proof: By induction on the number of variables of g. If $n = 1$, trivial. Assume 2.9 holds for $n = k - 1$ variables. Let g be a totally self-conjugate function of k variables, $b_1, b_2, \ldots, b_k$ ideal elements of g. By induction and previous remark

$$g(b_1, b_2, \ldots, b_{k-1}, 1) = \prod_{i=1}^{k-1} b_i \tag{1}$$

By 2.2

$$g(b_1, b_2, \ldots, b_{k-1}, b_k) \leq g^k(b_k) = b_k$$

and

$$g(b_1, b_2, \ldots, b_{k-1}, \overline{b_k}) \leq g^k(\overline{b_k}) = \overline{b_k}.$$

So, by 2.5 and (1),

$$b \cdot \prod_{i=1}^{k-1} b_i = b_k \cdot g(b_1, b_2, \ldots, b_{k-1}, 1) = g(b_1, b_2, \ldots, b_k)$$

Theorem 2.10 *If*

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, f_\varepsilon \rangle_{\varepsilon<\alpha}$$

is an **SBA** *and each* f_ε *is non-constant, then*

$$\mathfrak{A}' = \langle I, +, \cdot, -, 0, 1, f_\varepsilon \rangle_{\varepsilon<\alpha}$$

is a disrete **SBA**, *where* I *is the set of ideal elements of* $\mathfrak{A}$.

Proof: By 2.4, 2.8, and 2.9.

Definition 2.11: Let $\mathfrak{A}$ be an **SBA**. Let $a \in A$ and B be the set of $x \in A$ such that $x \leqslant a$. Then the system

$$\langle B, +, \cdot, -', 0, a, f_\varepsilon, \mathbf{c}_\delta \cdot a \rangle_{\varepsilon<\alpha, \delta<\beta}$$

where $-'$ denotes complementation with respect to a, will be denoted $\mathfrak{A}(a)$.

Theorem 2.12 *If* $\mathfrak{A}$ *is an* **SBA** *and* a *is an ideal element,* $a \in A$, *then* $\mathfrak{A}(a)$ *is an* **SBA** *and* $\varphi: A \to B$ *defined by*

$$\varphi(x) = a \cdot x \textit{ for all } x \in A$$

map $\mathfrak{A}$ *homomorphically onto* $\mathfrak{A}(a)$.

Proof: By Boolean algebra it is known that φ maps $\langle A, +, \cdot, -, 0, 1\rangle$ homomorphically onto $\langle B, +, \cdot, -', 0, a\rangle$. By 2.7

$$\varphi(f(x_1, \ldots, x_n)) = a \cdot f(x_1, \ldots, x_n) = f(a \cdot x_1, \ldots, a \cdot x_n) = f(\varphi(x_1), \ldots, \varphi(x_n)).$$

Clearly $\varphi(\mathbf{c}_\delta) = \mathbf{c}_\delta \cdot a$. The total self-conjugacy of f follows from 1.9.

Definition 2.13: An algebra $\mathfrak{A} = \langle A, g_\varepsilon \rangle_{\varepsilon<\alpha}$ is simple if $|A| > 1$ and every non-constant homomorphism on $\mathfrak{A}$ is an isomorphism.

For the next result we let

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, f_i, \mathbf{c}_\delta \rangle_{i<k, \delta<\beta, k<\omega}$$

be an **SBA** such that $\mathfrak{A}$ is a member of an equational class. Theorem 2.15 is a generalization of the result presented in [3] for relation algebras, and follows the same argument.

Definition 2.14: $\Phi(x) = f_1^1(f_1^2(\ldots (f_1^{n_1}(f_2^1) \ldots (f_2^{n_2}(\ldots (f_{k-1}^{n_{k-1}}(x)) \ldots)$ for every $x \in A$.

Theorem 2.15 *If* f_i^n, f_j^m *satisfy* 1.30(ii) *and* 1.31(ii) *for all* $i, j < k$ *and* $n < n_i$, $m < m_i$, *then the following are equivalent:*

(i) $\mathfrak{A}$ *is simple.*
(ii) $|A| > 1$ *and* $\mathfrak{A}$ *has no ideal elements other than* 0 *and* 1.
(iii) *For every* $x \neq 0$, $\Phi(x) = 1$.

Proof: (i) implies (ii). By 2.12.

(ii) implies (iii). By 1.30 and 1.31, $f_i^n(\Phi(x)) = \Phi(x)$ for all $i < k$, $n < n_i$, hence $\Phi(x) = 1$ for all $x \neq 0$.

(iii) implies (i). Let $\varphi: A \rightarrow B$ be a homomorphism mapping $\mathfrak{A}$ onto $\mathfrak{B}$

$$\mathfrak{B} = \langle B, +, \cdot, -, 0, 1, f_i, \mathbf{c}_\delta \rangle_{i<k, \delta<\beta}.$$

Assume $\varphi(x) = \varphi(y)$ for $x, y \in A$. Let $z = x \cdot \overline{y} + y \cdot \overline{x}$, then $\varphi(z) = 0$. By 1.2(iii) and φ being a homomorphism of a member of an equational class $\varphi(\Phi(z)) = 0$. Hence $\varphi(\Phi(z)) \neq 1 = \Phi(1)$, so $\Phi(z) \neq 1$. Hence $z = 0$ and $x = y$.

Theorem 2.15 shows that it is essentially the self-conjugacy of the operations in the systems of algebraic logic which allow us to establish the simplicity results. From the above we immediately obtain a simplicity theorem for quantifier algebras, as well as the result for finite dimensional or, in fact, locally finite cylindric algebras (*see* Theorem 2.3.14 in [2]). We obtain the result for relation algebras by using an axiom system which omits converse $\smile$ as an undefined operation (*see* p. 354 in [1]).

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Seminar in Symbolic Logic
University of Notre Dame
Notre Dame, Indiana