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AN AMBIGUITY IN MODAL LOGIC

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1 *Introduction* It is true, for any substitution made on p or q, that either pq or $p \sim q$ or $\sim pq$ or $\sim p \sim q$. Where for some reason explainable in particular cases a substitution is such that pq, so substituted, is not true at any time, the substituents will here be said to be contrary propositions. The remainder of the disjunction is therefore true, and is equivalent to $(\sim p \vee \sim q)$, so that contrariety may be defined:

p Con q (i.e., p is contrary to q) iff $(\sim p \vee \sim q)$ is explainably true at all times.

By 'explainably true' I mean that the proposition is true, and its generality is explainable. Any truth may be explainable, either empirically by reference to the nature of the world, or non-empirically, by, for example, explaining something about language. If 'explainably' were deleted, the result would be a version of the Humean approach to modality, which defines modal terms by way of generality and truth. As it is, the definition preserves one advantage of Humeanism, that the extensionality of modal logic is explainable. Modal logic may be based on extensional propositional logic plus the formally primitive notion of contrariety. Contrariety is extensional because its extra-systematic definiens is so. If a proposition is true, so is anything logically equivalent to it. If a truth is explained, any logically equivalent truth is thereby explained. The account avoids one objection to Humeanism, which is that if the conjunction pq were consistent, it would be sometimes true, which is not a consequence of the account suggested above.

Two questions may be raised. The first is whether the account just given is viable, or needs amending or abandoning. The second is the subject of this paper. It is; what systematized relations contrariety has with other modal concepts such as necessity and entailment. For these purposes, contrariety may be assumed to be primitive. Sub-contrariety, written 'p Sub q', is defined as $(\sim p$ Con $\sim q)$. It is therefore equivalent, given the earlier definition, to '$(p \vee q)$ is explainably true at all times',

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since the extensionality of contrariety allows elimination of double negations. This fits the traditional picture clearly put, e.g., by Arnauld in '*Art of Thinking*', Pt. II, §4, of contrariety and sub-contrariety as, among other things, 'never simultaneously true' and 'never simultaneously untrue', the only added element being that the universality is explainable. A parallel account of contradiction is 'never either simultaneously true or simultaneously untrue'. Contradiction, written 'p Kon q', is accordingly defined as ((p Con q) & (p Sub q)), equivalent to '$\sim(p \equiv q)$ is explainably always true'. These modal relations, Con, Sub, and Kon, are symmetrical, non-reflexive and non-transitive. Being also distinct, it is a consequence of the definitions that a proposition has each of two relations to its own negation. They are contrariety and contradiction. In that special case they are equivalent, but not in other cases. The equivalence is shown as follows. $\sim(p \equiv q)$ is equivalent to $(p \sim q \vee \sim pq)$. Putting $\sim p$ for q gives what is equivalent to $(p \vee \sim p)$. Both p Con $\sim p$ and p Kon $\sim p$ are each equivalent to '$(p \vee \sim p)$ is explainably always true', and so to each other.

Inconsistency is sometimes explained as that relation which holds between any proposition and its own negation, but since there are two distinct relations, that makes it ambiguous, unless it is understood disjunctively as either of the two. In that case, 'p is inconsistent with q' becomes 'either p Con q or p Kon q'. Consistent propositions are therefore those neither contrary nor contradictory. The systematic features of this disjunctive account are discussed in the last section of this paper. Extra-systematically, 'p is consistent with q' is equivalent to 'neither $(\sim p \vee \sim q)$ nor $\sim(p \equiv q)$ is explainably true at all times', or alternatively '(either $(\sim p \vee \sim q)$ or $\sim(p \equiv q)$) is not explainably true at all times'. The disjunction is equivalent to $(\sim p \vee \sim q)$, so that a roughly reasonable reorganizing of the proposition is: 'it is not contrary to explanations if pq is sometimes true'. Without pursuing the interpretation of this, herein lies a broad difference from the Humean account, which by similar steps would boil down to 'pq is sometimes true'. It is not perhaps decisive in itself, but it is an objection, that it should so boil down, simply because a pair of mutually consistent propositions might chance not ever to be simultaneously true. For example, it is a consistent thought, that a vulture should be feeding on the liver of Prometheus, and Prometheus simultaneously be thinking about Shakespeare, but the two may never chance to be both true simultaneously, either because no vulture ever ate his liver, or because when a vulture did feed on his liver, Prometheus did not think of Shakespeare at that time. Necessity is traditionally conceived as belonging to a proposition whose negation is inconsistent with itself, but here it is defined in terms of contrariety. To accommodate the connection with inconsistency, and its dual nature, two forms of necessity are initially defined, and correspondingly two forms each of possibility and entailment. The latter is the relation between p and q when $p \sim q$ is inconsistent, and initially it is distinguished from strict implication, which also has two forms. It will be seen that these initial distinctions between four kinds of implication collapse, and only that which is initially called 'weak entailment' is an entailment relation.

Conjunction is shown by contiguity for variables and negations of variables, otherwise by '&'. The ambiguous $\sim p$ Con q is understood as $(\sim p)$ Con q, not as $\sim(p$ Con $q)$: and $\sim pq$ as $(\sim p)q$, not as $\sim(pq)$.

2 *The system* **C** The axioms are:

A1 p Con $\sim p$
A2 $(p$ Con $q)$ Con pq
A3 $(p$ Con $q)$ Con $\sim(q$ Con $p)$
A4 $(pq$ Con $r)$ Con $\sim(p$ Con $qr)$
A5 $((p \vee q)$ Con $r)$ Con $\sim(p$ Con $r)$
A6 $((p$ Con $\sim q)$ & $(q$ Con $r))$ Con $\sim(p$ Con $r)$
A7 $\sim(p$ Con $q)$ Con $\sim((p$ Con $q)$ Con $(p$ Con $q))$

The primitive vocabulary of **C** is that of **PC** (the propositional calculus) plus 'Con'. The syntax is that of **PC** plus the rule that if α, β are wffs, then α Con β is a wff. The following terms are defined:

D1 p Sub $q \leftrightarrow \sim p$ Con $\sim q$
D2 p Kon $q \leftrightarrow (p$ Con $q)$ & $(p$ Sub $q)$
D3 $L^{w}p \leftrightarrow \sim p$ Con $\sim p$
D4 $M^{w}p \leftrightarrow \sim(p$ Con $p)$
D5 $L^{s}p \leftrightarrow \sim p$ Kon $\sim p$
D6 $M^{s}p \leftrightarrow \sim(p$ Kon $p)$
D7 $(p \rightarrow q) \leftrightarrow p$ Con $\sim q$
D8 $(p \leftrightarrow q) \leftrightarrow (p \rightarrow q)$ & $(q \rightarrow p)$
D9 $(p \Rightarrow q) \leftrightarrow p$ Kon $\sim q$
D10 $(p \Leftrightarrow q) \leftrightarrow (p \Rightarrow q)$ & $(q \Rightarrow p)$

L^{w} and M^{w} are referred to as weak necessity and possibility operators, L^{s} and M^{s} as strong necessity and possibility. $\rightarrow$ and $\leftrightarrow$ are called weak entailment and weak equivalence, while $\Rightarrow$ and $\Leftrightarrow$ are called strong entailment and strong equivalence. Strong strict implication and equivalence are defined by $L^{s}(p \supset q)$ and $L^{s}(p \equiv q)$. Weak strict implication and equivalence are defined by $L^{w}(p \supset q)$ and $L^{w}(p \equiv q)$.

The derivation, or deduction, rules of **C** are as follows:

R1 Uniform substitution on variables (**Sub**).
R2 Adjunction (**Adj**).
R3 If $\alpha \rightarrow \beta$ is a thesis in **C** and α is a thesis in **C**, then β is a thesis in **C** (**Det**).
R4 If $\alpha \equiv \beta$ is a thesis in **PC** or **C**, or if $\alpha \leftrightarrow \beta$ is a thesis in **C**; and F^{α} is a thesis in **C**; and F^{β} differs from F^{α} only in containing β in some place where F^{α} contains α; then F^{β} is a thesis (**Ext**).

3 *Validity and consistency* Validity will here be understood in the sense of S5 validity. The consistency of **C** may be shown by a semantic method in the sense that the axioms are valid, and the derivation rules transmit validity, so that every thesis is valid. If a thesis is valid, its negation cannot be valid, and so cannot be a thesis, so that **C** is consistent.

Let an S5 model be an ordered triple $\langle w_j, W, R\rangle$ and a function $V(p_j, w_j)$ called a value assignment for p_j for an interpretation w_j. W is a set of worlds; $w_j \in W$; w_j is a non-empty set; and R is a dyadic reflexive transitive symmetrical relation defined over W. The values of V range over $\{1,0\}$. The arguments of V are any wff p_j in **C**, and any set w_j supplying an interpretation of that wff. A wff p_j is valid iff on every S5 model, for every $w_j \in W$, $V(p_j, w_j) = 1$. V satisfies the conditions:

(1) For any propositional variable p_j and for any $w_j \in W$, either $V(p_j, w_j) = 1$ or $V(p_j, w_j) = 0$.
(2) ($V \sim$) For any wff p_j and any $w_j \in W$, if $V(p_j, w_j) = 0$, then $V(\sim p_j, w_j) = 1$; otherwise $V(\sim p, w_j) = 0$.
(3) (V &) For any wffs p_j, p_k, and every $w_j \in W$, if $V(p_j, w_j) = 1$ and $V(p_k, w_j) = 1$, then $V(p_j \,\&\, p_k, w_j) = 1$; otherwise $V(p_j \,\&\, p_k, w_j) = 0$.
(4) (V v) For any wffs p_j, p_k, and every $w_j \in W$, if $V(p_j, w_j) = 1$ or $V(p_k, w_j) = 1$, then $V(p_j \vee p_k, w_j) = 1$; otherwise $V(p_j \vee p_k, w_j) = 0$.
(5) (V Con) For any wffs p_j, p_k, and every $w_i \in W$, if for every $w_j \in W$ such that $w_i R w_j$, $V(\sim p_j \vee \sim p_k, w_j) = 1$, then $V(p_j \text{ Con } p_k, w_i) = 1$; otherwise $V(p_j \text{ Con } p_k, w_i) = 0$.

The validity of the axioms may be shown by reductio proofs, as follows:

A1 (i) Suppose for some $w_i \in W$ that $V(\text{A1}, w_i) = 0$. Then for some $w_j \in W$:
(ii) $V(\sim p \vee \sim\sim p, w_j) = 0$ [(i), V Con]
(iii) $V(p, w_j) = 1$ and $V(p, w_j) = 0$ [(ii), V v]
(iii) is inconsistent, so (i) is impossible.

Similar proofs may be found for the other axioms. It then remains to show that the derivation rules transmit validity.

R1 If p_i is valid, then $V(p_i, w_i) = 1$ for every $w_i \in W$. Thus if some propositional variable which is part of p_i is replaced by some component whose possible interpretations are limited to subsets of W, that will not alter the value of $V(p_i, w_i)$. Since R1 limits substitutions to wffs in **C**, whose interpretations are subsets of W, that limitation applies.
R2 V& is enough to secure that R2 transmits validity.
R3 If p_j is valid, and $p_j \rightarrow p_k$ is valid, then $V(p_j, w_i) = 1$ and $V(p_j \rightarrow p_k, w_i) = 1$ for every w_i. By definition of $\rightarrow$, V Con, V v, and $V \sim$ it follows that for every $w_j \in W$, $V(p_j, w_j) = 0$ or $V(p_k, w_j) = 1$. By disjunctive syllogism, $V(p_k, w_k) = 1$. Thus p_k is valid.
R4 If p_j is a thesis in **PC**, p_j is valid because it is a tautology.

If $p_j \equiv p_k$ is valid, then by definition of $\equiv$, V&, Vv, $V\sim$ it follows that $V(p_j, w_i) = V(p_k, w_i)$ for every $w_i \in W$. If F_j containing p_j is valid, then for every $w_i \in W$, $V(F_j, w_i) = 1$. Let F_k be the formula resulting from the replacement of some occurrence of p_j in F_j by p_k. Every formula in **C** is reducible by definitions either to a truth-functional formula (**tff**) or to a formula containing only 'Con' and elements of **PC**. If F_j is a **tff**, then $V(F_j, w_i)$ is unaffected by any feature of p_j except $V(p_j, w_i)$. Since that remains unchanged upon uniform replacement by p_k, $V(F_j, w_i) = V(F_k, w_i)$. If F_j contains 'Con', but p_j does not fall within the scope of any occurrence of 'Con', the same applies. If p_j falls within the scope of an occurrence of 'Con', thus: A_j Con B, where A_j is a component of F_j containing p_j, then the same does not directly apply.

Let $V(\sim A_j \vee \sim B, w_x) = 1$ and $V(\sim A_k \vee \sim B, w_x) = 1$ be abbreviated to Vjx and Vkx respectively. Then, by V Con, $V(A_j \text{ Con } B, w_i) = 1$ iff (Vji & . . . & Vjn), where $\{i \ldots n\}$ is given by $\{w_i \ . \ . \ . \ w_n\} = W$. Similarly $V(A_k \text{ Con } B, w_i) =$ 1 iff (Vki & . . . & Vkn). By the earlier argument, replacing p_j in A_j by p_k will not, for any x such that $i \leqslant x \leqslant n$, alter the value of Vjx. That replacement converts A_j to A_k, so that for every x, Vjx iff Vkx. Consequently (Vji & . . . & Vjn) iff (Vki & . . . & Vkn), and so $V(A_j \text{ Con } B, w_i) = V(A_k \text{ Con } B, w_i)$ for every $w_i \in W$. If A_j Con B is the only part of F_j containing an occurrence of 'Con' containing p_j, then the previous argument secures that $V(F_j, w_i) = V(F_k, w_i)$ for every $w_i \in W$. If A_j Con B lies within the scope of a further occurrence of 'Con', the same argument may be repeated mutatis mutandis to the point where $V(F_j, w_i) = V(F_k, w_i)$ for every w_i may be asserted. It then follows that $V(F_k, w_i) = 1$ for every $w_i \in W$, so that F_k is valid.

If $p_j \leftrightarrow p_k$ is valid, then by V Con, V&, $V\vee$, $V\sim$, and definitions, it follows that $V(p_j, w_i) = V(p_k, w_i)$ for every $w_i \in W$. The previous argument may then be repeated. For this argument to work, F_j must not be infinitely long, but the syntax of **C** only generates finite wffs, so the condition is satisfied.

4 *Discussion* **C** contains a fragment, $\mathbf{C}^{\text{w}}$, consisting of theses expressible using only the weak modal terms, $\rightarrow$, $\leftrightarrow$, L^{w}, and M^{w}. **C** will be said to contain a Lewisian analogue if every thesis in a Lewisian system generates a thesis in **C** when the Lewisian modal terms are replaced either by the appropriate modal terms of $\mathbf{C}^{\text{w}}$, or of $\mathbf{C}^{\text{s}}$. $\mathbf{C}^{\text{w}}$ contains an analogue of the system S5. That may be shown by showing that some set of axioms and rules adequate for S5 have analogues which are theses and rules of $\mathbf{C}^{\text{w}}$. That is done in the next section.

$\Rightarrow$, $\Leftrightarrow$, and $\leftrightarrow$ are shown to be equivalent to each other. Strong strict implication, defined by $L^{\text{s}}(p \supset q)$, has no cases, since $\sim L^{\text{s}}(p \supset q)$ is a thesis in **C**. That thesis may be understood as implying that for any p or q, that p strongly strictly implies q is not true. Strong strict implication is not equivalent either to weak strict implication, defined by $L^{\text{w}}(p \supset q)$, or to strict or strong entailment. Neither $\sim(p \rightarrow q)$ nor $\sim(p \Rightarrow q)$ are theses.

A fragment called $\mathbf{C}^{\text{s}}$ is that set of theses in **C** which are expressible using only strong modal terms. Since strong entailment and equivalence are equivalent to weak equivalence, that amounts to those theses expressible using L^{s}, M^{s}, $\leftrightarrow$, and $\rightarrow$. $\mathbf{C}^{\text{s}}$ contains no Lewisian analogue at all. $M^{\text{s}}p$ is a thesis, and also $\sim L^{\text{s}}M^{\text{s}}p$, so that, since **C** is consistent, $M^{\text{s}}p \rightarrow L^{\text{s}}M^{\text{s}}p$ is not a thesis, so that $\mathbf{C}^{\text{s}}$ differs from S5. $L^{\text{s}}p \rightarrow q$ is a thesis, so

$L^s \rightarrow L^sL^sp$ follows from that, though so also does $L^sp \rightarrow \sim L^sL^sp$. $\sim L^sp$ is a thesis, and so are $\sim L^sL^sp$, M^sM^sp, and $\sim L^sM^sM^sp$.

$\mathbf{C}^s$ differs from S5 and S8 in not containing the analogue of the axioms distinctive of those systems. It differs from S4 in that the presence in the system of the theses characteristic of S6 and S7 does not result in inconsistency. It contains all the axioms of S4, but not all the definitions. The inference rule **Det** for strong strict implication is valid in **C**, for $L^sp \rightarrow L^wp$ is valid, so that if p and $L^s(p \supset q)$ are theses, so are p and $L^w(p \supset q)$. Since $L^w(p \supset q)$ is equivalent to $p \rightarrow q$, the rule of **Det** in **C** yields q as a derived thesis. But the rule can have no application, since there are no strong strict implication theses. $\mathbf{C}^s$ differs from all Lewisian systems in lacking the Lewis relationship between the necessity operator and the implication function, and also in containing no theses of the form L^sp.

5 *The weak system* $\mathbf{C}^w$ The first step to show that $\mathbf{C}^w$ contains an analogue of S5 is to show that the definitions in S5 correspond to valid theses in $\mathbf{C}^w$. Some theorems in $\mathbf{C}^w$ will now be given, and their proofs sketched, which lead up to that conclusion.

T1	$p \text{ Con } q \leftrightarrow q \text{ Con } p$	A3, D7, D8, **Sub**, **Adj**, **Ext**
T2	$p \leftrightarrow p$	A1, D7, D8, **Adj**, **Ext**
T3	$pq \text{ Con } r \leftrightarrow p \text{ Con } qr$	A4, D7, D8, T1, **Sub**, **Ext**, **Adj**
T4	$pq \text{ Con } pq \leftrightarrow p \text{ Con } q$	

Proof:

$pq \text{ Con } pq \leftrightarrow p \text{ Con } qpq$	T3, **Sub**
$\leftrightarrow qp \text{ Con } p$	T1, **Ext**
$\leftrightarrow q \text{ Con } pp$	T3, **Ext**
$\leftrightarrow p \text{ Con } q$	T1, **Ext**

T5 $L^w \sim (pq) \leftrightarrow p \text{ Con } q$

Proof:

$L^w \sim (pq) \leftrightarrow \sim\sim (pq) \text{ Con } \sim\sim (pq)$	D3, T2
$\leftrightarrow p \text{ Con } q$	T4, **Ext**

T6	$L^w(p \supset q) \leftrightarrow (p \rightarrow q)$	T2, T5, D7, **Sub**, **Ext**
T7	$L^w(p \equiv q) \leftrightarrow (p \leftrightarrow q)$	T6, D8, **Sub**, **Adj**
T8	$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$	

Proof:

$p \text{ Con } \sim q \leftrightarrow \sim q \text{ Con } p$	T1, **Sub**
$\leftrightarrow \sim q \text{ Con } \sim\sim p$	**Ext**
$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$	D7

T9	$(p \leftrightarrow q) \leftrightarrow (\sim q \leftrightarrow \sim p)$	T8, D8, **Sub**, **Ext**, **Adj**
T10	$L^wp \leftrightarrow \sim M^w \sim p$	T2, T9, D3, D4, **Sub**, **Ext**
T11	$M^wp \leftrightarrow \sim L^w \sim p$	T9, T10, **Ext**, **Sub**

Theorems T10 and T11 are the analogues of the Lewis relationships between necessity and possibility. T6 shows that weak entailment and weak strict implication are equivalent in $\mathbf{C}^w$, while T7 shows that weak equivalence and weak strict equivalence are also equivalent. The definitions in $\mathbf{C}^w$ have accordingly been given in the form of weak equivalences.

The system S5 may be defined by the following rules and axioms:

Rules

S5.1 **Sub**
S5.2 **Adj**
S5.3 **Det** with respect to strict implication
S5.4 **Sub** of strict equivalents

Axioms

AS1 $L(pq \supset qp)$
AS2 $L(pq \supset p)$
AS3 $L(p \supset pp)$
AS4 $L((pq)r \supset p(qr))$
AS5 $L(L(p \supset q) \,\&\, L(q \supset r) \supset L(p \supset r))$
AS6 $L(p \,\&\, L(p \supset q) \supset q)$
AS7 $L(Mp \supset LMp)$

The rules called **Sub** and **Adj** are the same in S5 and in $\mathbf{C}^{\mathrm{w}}$. The rule S5.3 is derivable from R3 in $\mathbf{C}^{\mathrm{w}}$, because if $L^{\mathrm{w}}(\alpha \supset \beta)$ is a thesis, it follows by T6 that $\alpha \rightarrow \beta$ is a thesis, so β follows by R3 if α is a thesis. Likewise, it follows that if α and β are weak strict equivalents, it follows by T7 that they are weak equivalents, and vice versa, so that S5.4 follows from R4. Thus every rule in S5 is a rule in $\mathbf{C}^{\mathrm{w}}$.

T12	$L^{\mathrm{w}}(pq \supset qp)$	A1, T6, D7, **Ext**, **Sub**
T13	$L^{\mathrm{w}}(pq \supset p)$	A1, A5, T1, D7, **Sub**, **Ext**, **Det**
T14	$L^{\mathrm{w}}(p \supset pp)$	A1, T6, D7, **Ext**
T15	$L^{\mathrm{w}}((pq)r \supset p(qr))$	A1, D7, **Ext**
T16	$L^{\mathrm{w}}(L^{\mathrm{w}}(p \supset q) \,\&\, L^{\mathrm{w}}(q \supset r) \supset L^{\mathrm{w}}(p \supset r))$	A6, T6, **Sub**, **Ext**
T17	$L^{\mathrm{w}}(p \,\&\, L^{\mathrm{w}}(p \supset q) \supset q)$	

Proof:

$(p \text{ Con } q) \text{ Con } pq$	A2
$q \text{ Con } (p \,\&\, (p \text{ Con } q))$	T1, T3, **Ext**
$(p \,\&\, (p \text{ Con } {\sim} q)) \text{ Con } {\sim} q$	T1, **Sub**
$L^{\mathrm{w}}(p \,\&\, L^{\mathrm{w}}(p \supset q) \supset q)$	D7, T6, **Ext**

T18 $L^{\mathrm{w}}p \rightarrow p$

Proof:

${\sim} p \text{ Con } {\sim} p \rightarrow {\sim}({\sim} p \sim p)$	A2, D7, **Sub**
$L^{\mathrm{w}}p \rightarrow p$	D3, **Ext**

T19 $L^{\mathrm{w}}(M^{\mathrm{w}}p \supset L^{\mathrm{w}}M^{\mathrm{w}}p)$

Proof:

${\sim}(p \text{ Con } p) \rightarrow (p \text{ Con } p) \text{ Con } (p \text{ Con } p)$	A7, D7, **Sub**
$M^{\mathrm{w}}p \rightarrow L^{\mathrm{w}}{\sim}(p \text{ Con } p)$	T5, D3, D4, **Ext** (replace $(p \text{ Con } p)$ by ${\sim}{\sim}(p \text{ Con } p)$)
$L^{\mathrm{w}}(M^{\mathrm{w}}p \supset L^{\mathrm{w}}M^{\mathrm{w}}p)$	D4, T6

The theorems T12, T13, T14, T15, T16, T17, and T19 are the analogues of AS1-AS7, hence $\mathbf{C}^{\mathrm{w}}$ contains the analogue of S5.

6 *The strong system* $\mathbf{C}^s$ The two strong modal operators are interchangeable in the same way as the weak ones.

T20	$L^s p \leftrightarrow \sim M^s \sim p$	T2, T9, D5, D6, **Ext**, **Sub**
T21	$M^s p \leftrightarrow \sim L^s \sim p$	T9, T20, **Ext**, **Sub**

A resemblance to Lewis systems S6 and S7 is shown by the following theorems.

T22 $L^w \sim (p \text{ Kon } p)$

Proof:

$p \text{ Kon } p \leftrightarrow p \text{ Con } p \;\&\; \sim p \text{ Con } \sim p$	T2, D1, D2
$\leftrightarrow \sim (pp) \;\&\; \sim (\sim p \sim p)$	T5, T18, **Sub**
(a) $\rightarrow p \;\&\; \sim p$	**Ext**
(b) $(p \rightarrow q) \rightarrow (M^w p \rightarrow M^w q)$	thesis in S5, hence in $\mathbf{C}^w$
(c) $M^w(p \text{ Kon } p) \rightarrow M^w(p \;\&\; \sim p)$	(a), (b), **Sub**, **Det**
(d) $\sim M^w(p \;\&\; \sim p)$	T2, T7, T10, **Ext**
$\sim M^w(p \text{ Kon } p)$	(c), (d), T8, **Det**
$L^w \sim (p \text{ Kon } p)$	T10, **Sub**, **Ext**

T23	$\sim (p \text{ Kon } p)$	T22, T18, **Sub**, **Det**
T24	$M^s p$	T23, D6, **Ext**
T25	$M^s M^s p$	T24, **Sub**

The non-existence of any cases of strong necessity or of strong strict implication then follows directly.

T26	$\sim L^s p$	T24, T21, **Sub**, **Ext**, **Det**
T27	$\sim L^s L^s p$	T26, **Sub**
T28	$\sim L^s(p \supset q)$	T26, **Sub**
T29	$L^s p \rightarrow q$	

Proof:

(a) $L^w \sim p \rightarrow L^w \sim (pq)$	thesis in S5, hence in $\mathbf{C}^w$
$L^w \sim ((p \text{ Kon } p) \;\&\; \sim q)$	(a), T22, **Sub**, **Det**
$(p \text{ Kon } p) \text{ Con } \sim q$	T5, **Sub**, **Det**
$(\sim p \text{ Kon } \sim p) \rightarrow q$	**Sub**, D7
$L^s p \rightarrow q$	D6, **Ext**

The eliminability of $\Rightarrow$ is shown by the following theorems.

T30	$(p \Rightarrow q) \leftrightarrow (p \leftrightarrow q)$	T1, D1, D2, D7, D8, D9, **Ext**
T31	$(p \Rightarrow q) \leftrightarrow (q \Rightarrow p)$	T1, D1, D2, D9, **Ext**
T32	$(p \Rightarrow q) \leftrightarrow (p \Leftrightarrow q)$	T2, T31, D10, **Ext**
T33	$(p \Rightarrow q) \rightarrow (p \rightarrow q)$	T6, T13, T16, D2, D7, D8, D9, **Sub**, **Det**

A difference between Con and Kon is shown by T35 and T36, while their equivalence in one important special case is shown by T34.

T34 $p \text{ Con } \sim p \leftrightarrow p \text{ Kon } \sim p$

Proof:

$p \text{ Con } \sim p \rightarrow p \text{ Con } \sim p \;\&\; p \text{ Con } \sim p$	T14, T6, **Sub**, **Ext**
$\rightarrow p \text{ Con } \sim p \;\&\; \sim p \text{ Con } p$	T1, **Sub**, **Ext**
$\rightarrow p \text{ Con } \sim p \;\&\; p \text{ } \mathbf{Sub} \sim p$	D1, **Sub**, **Ext**
$\rightarrow p \text{ Kon } \sim p$	D2

The converse follows from D2.

T35 $(p \sim p)$ Con $(p \sim p)$ — A1, T4, by derived rule: if $\vdash \alpha$ and $\vdash \alpha \leftrightarrow \beta$, then $\vdash \beta$

T36 $\sim((p \sim p)$ Kon $(p \sim p))$ — T23, **Sub**

T37 $L^s p \rightarrow L^s L^s p$ — T29, **Sub**

The addition of T37 to T12-T17 gives a mixed analogue of the system S4, but it differs in that the implicative function used to formulate the theses is not equivalent to strong strict implication $L^s(p \supset q)$. If it were, **C** would be inconsistent, since the conjunction of T28 with, for example, T2, would yield a contradiction. Consequently the rule of substitution of strong strict equivalents is not justified by R4, and $\mathbf{C}^s$ does not contain S4, or any other Lewisian analogue. The rule of detachment for strong strict implication is valid, since $L^s p \rightarrow L^w p$ follows from T29, but T28 shows that if there were any premises in **C** for **Det** to be used to derive theses from, **C** would have to be inconsistent. T29 implies the validity of $L^s(p \equiv q) \rightarrow (p \leftrightarrow q)$. The converse is not valid, since if it were, T9 and T26 would imply that $\sim(p \leftrightarrow q)$ was a thesis, from which $\sim(p \rightarrow p)$ follows, contradicting A1. By a similar argument, $L^s(p \supset q) \rightarrow (p \rightarrow q)$ is valid, but its converse would render **C** inconsistent.

Every Lewis system contains at least one strict implication thesis. If L^s were the analogue of Lewisian necessity, T28 would render any of those systems inconsistent. Thus no Lewis system is analogous to $\mathbf{C}^s$.

7 *Absolute necessity* Necessity need not be treated as ambiguous, by defining 'absolute necessity' as 'necessary in at least one of the two previously defined senses', thus:

$$Lp \leftrightarrow L^s p \vee L^w p$$

Possibility is similarly made unambiguous by defining it as:

$$Mp \leftrightarrow M^s p \;\&\; M^w p$$

Both new forms collapse into the weak modal operators, since $Mp \leftrightarrow M^w p$ and $Lp \leftrightarrow L^w p$ are theorems in **C** when these definitions are added. The transposed form of T13 yields $p \rightarrow (p \vee q)$, so that $L^w p \rightarrow Lp$ and $Mp \rightarrow M^w p$ are easily proved. Since $L^s p \rightarrow L^w p$ is demonstrable from T29, and $(q \rightarrow r) \rightarrow (p \vee q \rightarrow p \vee r)$ is demonstrable, $Lp \rightarrow L^w p \vee L^w p$ follows from the definition of L, thus giving $Lp \rightarrow L^w p$ and $Lp \leftrightarrow L^w p$. $M^w p \rightarrow M^s p$ follows from T11 and T20, and $(p \rightarrow q) \rightarrow (p \rightarrow pq)$ is demonstrable, giving $M^w p \rightarrow M^w p \;\&\; M^s p$, and so $M^w p \rightarrow Mp$ and $M^w p \leftrightarrow Mp$.

In that case, accepting the definition D3, **C** turns out to be an alternative axiomatization of S5, as well as an analogue of it. Since it has been shown that p Con q is equivalent to $L\sim(pq)$ (T5), contrariety can be defined in terms of L, which may then become primitive. From the axioms are derivable the following laws of necessity:

$$p \rightarrow p$$
$$Lp \rightarrow p$$

$$L(p \vee q) \rightarrow L(q \vee p)$$
$$L((p \vee q) \vee r) \rightarrow L(p \vee (q \vee r))$$
$$L(pq \vee r) \rightarrow L(p \vee r)$$
$$(p \rightarrow q) \; \& \; (q \rightarrow r) \rightarrow (p \rightarrow r)$$
$$Mp \rightarrow LMp$$

These are theses in S5, and A1-A7 are derivable back again by the derivation rules of S5. R1-R4 are all rules of S5. Consequently, given the definitional relationships, and provided nothing more is demanded of a definition than equivalence, **C** and S5 are equivalent systems. The significance of **C** at the level of unquantified logic is therefore simply in showing that the family of interdefinable modal terms is wider then perhaps it seems, and that there is a wider choice of primitive terms than has been commonly used. One result of introducing extra definitions is to produce more theorems than are usually noticed (e.g., T1, T3, T4, etc.) though the neglected theorems concerning $p \circ q$ in Lewis and Langford's '*Symbolic Logic*' may be construed as theorems concerning p Con q. Another result is that of the two ancient forms of 'opposition', contrariety and contradiction, the first is of primary importance. Strong necessity and contradiction are not involved in formulating the notable common laws of modal logic.

$L^s p$ is equivalent to $(\sim p$ Con $\sim p)$ & $(\sim p$ Sub $\sim p)$ and so to Lp & $L \sim p$. $L^s \sim p$ is equivalent to the same thing, so $\sim L^s \sim p$ is equivalent to $\sim Lp \vee \sim L \sim p$ and so to Mp. T24, apparently a peculiar law of S6 if L^s is not distinguished from L^w and from L, is simply equivalent to the square of opposition laws of S1. T25 is a special case of the same thing. T29 is also a thesis of S1, detachable from the square of opposition laws plus one of the 'paradox' laws. Thus the laws of S6 and S7 allow of an interpretation of necessity, according to which both systems are fragments of S1. Accepting the basic idea in all this, represented by D3, it follows that we may take any one of L, M, Con, Sub, or $\rightarrow$ as the sole primitive modal term for systems S1-S5. If $\rightarrow$ is primitive, for example, in the primitive language then resulting, p Con q becomes $p \rightarrow \sim q$; p Sub q becomes $\sim p \rightarrow q$; Lp becomes $\sim p \rightarrow p$; and Mp becomes $\sim (p \rightarrow \sim p)$. The paradoxes of strict implication with Con as primitive appear as two substitution instances of the generalization principle, that if p is contrary to itself, it is contrary to everything, or p Con $p \rightarrow p$ Con q.

That Kantian account of analyticity, according to which a proposition A is analytic if and only if its negation is self-contradictory, is built on ambiguity, depending on what is made of the pre-analytical notion of self-contradiction. If it means self-contrariety, the definition is harmless as far as it goes. If it is identified with reflexive contradictoriness in the sense of $\sim A$ Kon $\sim A$, which customary terminology superficially favours, it follows that A is analytic if and only if both A and $\sim A$ are each necessary. Modal systems from S1 to S5 and beyond agree in ruling that that is not possible. The moral is not profound. It is that 'self-contradiction' as used unsystematically, is interpretable as the reflexive case of either of two different logical relations, which both hold in some cases (T34), not in others, and occasionally are mutually incompatible (T35-36).

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