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A NEW AXIOMATIZATION OF THE MIXED ASSOCIATIVE NEWMAN ALGEBRAS

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In [1] M. H. A. Newman constracted, formalized and investigated two relatively complemented algebraic systems which he called "mixed non-associative algebras" and "mixed associative algebras." In [2][1] and in the present paper only the latter system is investigated and it is called "mixed associative Newman algebras." In [2] I have proved that this system can be axiomatized equationally in the following way:

(**C**) *Any algebraic system*

$$\mathfrak{B} = \langle B, =, +, \times, \div \rangle$$

with one binary relation = *and three binary operations* +, ×, *and* ÷, *is a relatively complemented mixed associative Newman algebra if, and only if, it satisfies the following postulates:*

(i) *The closure postulates:*

P1 $[\exists a] . a \epsilon B$
P2 $[a] : a \epsilon B . \supset . a = a$

1. An acquaintance with the paper [2] is presupposed. In the deductions presented in this paper the postulates *P1-P11* and *R1-R10* will be used mostly tacitly. An enumeration of the algebraic tables, *cf.* section **2.2** below, is a continuation of the enumeration of such tables given in [2], pp. 421-422, section **5**. It should be noticed that in [2], p. 145, lines 7-9, the proof line 11 which appears there must be substituted by:

$$
\begin{aligned}
11.\quad (b \times a) + (b \times c) &= b \times (a + c) = (b \times (a + c)) \times (a + c) && [1; C1; F1]\\
&= ((b \times a) + (b \times c)) \times (a + c) && [C1]\\
&= ((b \times a) \times (a + c)) + ((b \times c) \times (a + c)) && [C2]\\
&= (((b \times a) \times a) + ((b \times a) \times c))\\
&\quad + (((b \times c) \times a) + ((b \times c) \times c)) && [C1; C1]\\
&= ((b \times a) + ((b \times a) \times c))\\
&\quad + (((b \times c) \times a) + (b \times c)). && [F1; F1]
\end{aligned}
$$

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P3 $[ab]:a, b \epsilon B . a = b .\supset. b = a$
P4 $[abc]:a, b, c \epsilon B . a = b . b = c .\supset. a = c$
P5 $[ab]:a, b \epsilon B .\supset. a + b \epsilon B$
P6 $[ab]:a, b \epsilon B .\supset. a \div b \epsilon B$
P7 $[abc]:a, b, c \epsilon B . a = c .\supset. a + b = c + b$
P8 $[abc]:a, b, c \epsilon B . b = c .\supset. a + b = a + c$
P9 $[ab]:a, b \epsilon B .\supset. a \times b \epsilon B$
P10 $[abc]:a, b, c \epsilon B . a = c .\supset. a \times b = c \times b$
P11 $[abc]:a, b, c \epsilon B . b = c .\supset. a \times b = a \times c$

(ii) *The mutually independent algebraic postulates:*

G1 $[abc]:a, b, c \epsilon B .\supset. a \times (b + c) = (a \times b) + (a \times c)$
G2 $[ab]:a, b \epsilon B .\supset. a \times b = b \times a$
G3 $[ab]:a, b \epsilon B .\supset. (a \div b) + (b \times a) = a$
G4 $[abc]:a, b, c \epsilon B .\supset. (a \div b) \times (b \times a) = c \div c$[2]

Remark: Although the binary operation $\div$ is a primitive notion in the system 𝔅, the formulas concerning the extensionality of the relation = with respect to this operation are not accepted axiomatically in 𝔅, since they are provable in this system, *cf.* the theses *G22* and *G24* in section **1** below.

Only some time after [2] was published I observed casually that in the system 𝔅 its primitive operations are not mutually independent. Namely, in the field of the postulate-system of 𝔅 we are able to define $\times$ by the operation $\div$ alone. In section **1** of this note I shall justify this fact and prove some theses which will be needed later. In section **2** an equational axiomatization of the mixed associative Newman algebras based only on two primitive operations + and $\div$ will be presented.

1 Let us assume the axioms *P1-P11* and *G1-G4*. Then:

1.1 We have at our disposal the theses *G5-G24* which are already proven in [2], pp. 419-420. Namely:

G5 $[abc]:a, b, c \epsilon B .\supset. (a + b) \times c = (a \times c) + (b \times c)$ [*G1*; *G2*]
G6 $[a]:a \epsilon B .\supset. a \times a = a \times (a \times a)$ [*G3*; *G4*; *G5*]
G7 $[a]:a \epsilon B .\supset. a \times a = ((a \times a) \div a) + (a \times a)$ [*G3*; *G6*]
G8 $[a]:a \epsilon B .\supset. a \times a = (a \times a) \times (a \times a)$ [*G3*; *G4*; *G6*; *G5*; *G7*]
G9 $[a]:a \epsilon B .\supset. a = a \times a$ [*G3*; *G4*; *G8*; *G5*; *G6*]
G10 $[a]:a \epsilon B .\supset. (a \div a) + a = a$ [*G3*; *G9*]
G11 $[ab]:a, b \epsilon B .\supset. (a \times b) \times b = a \times b$ [*G2*; *G3*; *G5*; *G4*; *G9*; *G10*]
G12 $[ab]:a, b \epsilon B .\supset. a \div a = b \div b$ [*G4*]
D1 $[a]:a \epsilon B .\supset. 0 = a \div a$ [*G12*]
G13 $0 \epsilon B$ [*P1*; *P8*; *D1*]
G14 $[ab]:a, b \epsilon B . a \times b = a .\supset. [\exists c] . c \epsilon B . c + a = b . c \times a = 0$ [*P8*; *G3*; *G4*; *D1*]

2. The mutual independence of the axioms *G1*, *G2*, *G3* and *G4* is established in [2], pp. 421-422, section **5**.

G15 $[ab]:a, b \epsilon B . b \times a = a .\supset. [\exists c]. c \epsilon B . c + a = b . c \times a = 0$ [*G14*; *G2*]

G16 $[a]:a \epsilon B .\supset. 0 + a = a$ [*G10*; *D1*]

G17 $[abc]:a, b, c \epsilon B .\supset. (b \div b) \times a = c \div c$ [*G12*; *G9*; *G4*]

G18 $[a]:a \epsilon B .\supset. a = a + 0$ [*G3*; *G17*; *G4*; *G1*; *G9*; *G5*; *D1*]

G19 $[ab]:a, b \epsilon B .\supset. (b \div a) \times b = b \div a$ [*G3*; *G1*; *G9*; *G4*; *D1*; *G18*]

G20 $[abx]:a, b, x \epsilon b . x + (a \times b) = b, x \times (a \times b) = 0 .\supset. b + a = x$
[*G19*; *G2*; *G5*; *G2*; *G4*; *D1*; *G1*; *G3*; *G9*; *G19*]

G21 $[abx]\therefore a, b, x \epsilon B .\supset: b \div a = x .\equiv. x + (a \times b) = b . x \times (a \times b) = 0$
[*G3*; *G4*; *D1*; *G20*]

G22 $[abc]:a, b, c \epsilon B . b = c .\supset. b \div a = c \div a$ [*P11*; *G3*; *G4*; *P8*; *G20*]

G23 $[abcd]:a, b, c, d \epsilon B . c + a = b . c \times a = 0 . d + a = b . d \times a = 0 .\supset. c = d$
[*G1*; *G2*; *G9*; *G16*; *G20*]

G24 $[abc]:a, b, c \epsilon B . a = c .\supset. b \div a = b \div c$ [*P10*; *G3*; *G4*; *D1*; *G23*]

Thus, the extensional formulas *G22* and *G24*, see Remark above, are the consequences of the postulate-system of $\mathfrak{B}$.

1.2 Now, we proceed as follows:

G25 $[ab]:a, b \epsilon B .\supset. (b \div a) \times (a \times b) = 0$ [*G4*; *D1*]

G26 $[abx]:a, b, x \epsilon B . (a \times b) + x = b . (a \times b) \times x = 0 .\supset. b \div a = x$

PR $[abx]:\mathrm{Hp}(3) .\supset.$

$b \div a = (b \div a) \times b = (b \div a) \times ((a \times b) + x)$ [1; *G19*; 2]

$= ((b \div a) \times (a \times b)) + ((b \div a) \times x)$ [*G1*]

$= 0 + ((b \div a) \times x) = (b \div a) \times x$ [*G25*; *G16*]

$= ((b \div a) \times x) + 0 = ((b \div a) \times x) + ((a \times b) \times x)$
[*G18*; 3]

$= ((b \div a) + (a \times b)) \times x = b \times x$ [*G5*; *G3*]

$= ((a \times b) + x) \times x = ((a \times b) \times x) + (x \times x)$ [2; *G5*]

$= 0 + x = x$ [3; *G9*; *G16*]

G27 $[ab]:a, b \epsilon B .\supset. a \times b = a \div (a \div b)$

PR $[ab]:\mathrm{Hp}(1) .\supset.$

2. $((a \div b) \times a) + (a \times b) = (a \div b) + (b \times a) = a$. [1; *G19*; *G2*; *G3*]

3. $((a \div b) \times a) \times (a \times b) = (a \div b) \times (b \times a) = 0$. [1; *G19*; *G2*; *G25*]

$a \times b = a \div (a \div b)$ [1; *G26*; 2; 3]

Thesis *G27* shows that in the system $\mathfrak{B}$ the operations $+$, $\times$, and $\div$ are not mutually independent.

G28 $[ab]:a, b \epsilon B .\supset. a = (b \div b) + a$ [*G10*; *G12*]

G29 $[ab]:a, b \epsilon B .\supset. a \div (a \div b) = b \div (b \div a)$ [*G27*; *G2*]

G30 $[abc]:a, b, c \epsilon B .\supset. a \div (a \div (b + c)) = (a \div a) + ((b \div (b \div a)) + (c \div (c \div a)))$

PR $[abc]:\mathrm{Hp}(1) .\supset.$

$a \div (a \div (b + c)) = a \times (b + c) = (b + c) \times a$ [1; *G27*; *G2*]

$= (b \times a) + (c \times a) = (b \div (b \div a)) + (c \div (c \div a))$ [*G5*; *G27*]

$= (a \div a) + ((b \div (b \div a)) + (c \div (c \div a)))$ [*G28*]

G31 $[ab]:a, b \epsilon B .\supset. (a \div b) + (b \div (b \div a)) = a$ [*G3*; *G27*]

G32 $[abc]:a, b, c \epsilon B .\supset. (a \div b) \div ((a \div b) \div (b \div (b \div a))) = c \div c$ [*G27*; *G4*]

G33 $[abx]: a, b, x \epsilon B . (a \div b) + x = a . (a \div b) \div ((a \div b) \div x) = a \div a .$
$\supset . a \times b = x$

PR $[abx]: \mathrm{Hp}(3) . \supset .$

4. $(a \div b) \times x = a \div a .$ [1; 3; *G27*]

$a \times b = b \times a = (b \times a) \times a = (b \times a) \times ((a \div b) + x)$ [1; *G2*; *G11*; 2]

$= ((b \times a) \times (a \div b)) + ((b \times a) \times x)$ [*G1*]

$= ((a \div b) \times (b \times a)) + ((b \times a) \times x)$ [*G2*]

$= (a \div a) + ((b \times a) \times x)$ [*G4*]

$= ((a \div b) \times x) + ((b \times a) \times x)$ [4]

$= ((a \div b) + (b \times a)) \times x = a \times x$ [*G5*; *G3*]

$= ((a \div b) + x) \times x = ((a \div b) \times x) + (x \times x)$ [2; *G5*]

$= (a \div a) + x = x$ [4; *G9*; *G28*]

G34 $[abx] \therefore a, b, x \epsilon B . a \times b = x . \supset : (a \div b) + x = a . (a \div b) \div ((a \div b) \div x) =$
$a \div a$

PR $[abx] \therefore \mathrm{Hp}(2) . \supset :$

3. $b \div (b \div a) = b \times a = a \times b = x :$ [1; *G27*; *G2*; 2]

$(a \div b) + x = a . (a \div b) \div ((a \div b) \div x) = a \div a$ [1; 3; *G31*; *G32*]

G35 $[abx] :: a, b, x \epsilon B . \supset \therefore a \times b = x . \equiv : (a \div b) + x = a .$
$(a \div b) \div ((a \div b) \div x) = a \div a$ [*G33*; *G34*]

2 In this section we shall prove the valitity of the following formalization of mixed associative Newman algebras:

(D) *Any algebraic system*

$$\mathfrak{C} = \langle C, =, +, \div \rangle$$

with one binary relation = and two binary operations + and ÷, is a relatively complemented mixed associative Newman algebra if, and only if, it satisfies the following postulates:

(i) *The closure postulates*:

R1 $[\exists a] . a \epsilon C$

R2 $[a]: a \epsilon C . \supset . a = a$

R3 $[ab]: a, b \epsilon C . a = b . \supset . b = a$

R4 $[abc]: a, b, c \epsilon C . a = b . b = c . \supset . a = c$

R5 $[ab]: a, b \epsilon C . \supset . a + b \epsilon C$

R6 $[ab]: a, b \epsilon C . \supset . a \div b \epsilon C$

R7 $[abc]: a, b, c \epsilon C . a = c . \supset . a \div b = c + b$

R8 $[abc]: a, b, c \epsilon C . b = c . \supset . a + b = a + c$

R9 $[abc]: a, b, c \epsilon C . a = c . \supset . a \div b = c \div b$

R10 $[abc]: a, b, c \epsilon C . b = c . \supset . a \div b = a \div c$

and

(ii) *The mutually independent algebraic postulates*:

S1 $[abc]: a, b, c \epsilon C . \supset . a \div (a \div (b + c)) = (a \div a) + ((b \div (b \div a))$
$+ (c \div (c \div a)))$

S2 $[ab]: a, b \epsilon C . \supset . (a \div b) + (b \div (b \div a)) = a$

S3 $[abc]: a, b, c \epsilon C . \supset . (a \div b) \div ((a \div b) \div (b \div (b \div a))) = c \div c$

Proof:

2.1 Let us assume the postulates of the system $\mathfrak{C}$. Then:

S4 $[ab]: a, b \epsilon C .\supset. a \div a = b \div b$ $[S3, c/a; S3, c/b]$

S5 $[abcd]: a, b, c, d \epsilon C .\supset. a \div (a \div (b + c)) = (d \div d)$
$+ ((b \div (b \div a)) + (c \div (c \div a)))$ $[S1; S4, b/d]$

S6 $[abcde]: a, b, c, d, e \epsilon C .\supset. e + (a \div (a \div (b + c))) =$
$e + ((d \div d) + ((b \div (b \div a)) + (c \div (c \div a))))$
$[R8, a/e, b/a \div (a \div (b + c)), c/(d \div d) + ((b \div (b \div c)) + (c \div (c \div a))); S5]$

S7 $[a]: a \epsilon C .\supset. a \div a = (a \div a) + (a \div a)$

PR $[a]: \mathrm{Hp}(1) .\supset.$
$a \div a = ((a \div a) \div (a \div a)) + ((a \div a) \div ((a \div a) \div (a \div a)))$
$[1; S2, a/a \div a, b/a \div a]$
$= (a \div a) + ((a \div a) \div (a \div a))$ $[S4, b/a \div a]$
$= (a \div a) + (a \div a)$ $[S4, b/a \div a]$

S8 $[a]: a \epsilon C .\supset. (a \div (a \div a)) \div ((a \div (a \div a)) \div (a \div a)) = a \div a$

PR $[a]: \mathrm{Hp}(1) .\supset.$
$(a \div (a \div a)) \div ((a \div (a \div a)) \div (a \div a))$
$= (a \div (a \div a)) \div ((a \div (a \div a)) \div ((a \div a) + (a \div a)))$ $[1; S7]$
$= (a \div a) + (((a \div a) \div ((a \div a) \div (a \div (a \div a)))) + ((a \div a)$
$\div ((a \div a) \div (a \div (a \div a)))))$ $[S5, a/a \div (a \div a), b/a \div a, c/a \div a, d/a]$
$= (a \div a) + ((a \div a) + (a \div a))$ $[S3, b/a, c/a]$
$= (a \div a) + (a \div a) = a \div a$ $[S7; S7]$

S9 $[a]: a \epsilon C .\supset. (a \div a) \div ((a \div a) \div a) = a \div a$

PR $[a]: \mathrm{Hp}(1) .\supset.$
$(a \div a) \div ((a \div a) \div a) = (a \div a) \div ((a \div a) \div ((a \div a) + (a \div (a \div a))))$
$[1; S2, b/a]$
$= (a \div a) + (((a \div a) \div ((a \div a) \div (a \div a))) + ((a \div (a \div a)) \div ((a \div (a \div a))$
$\div (a \div a))))$ $[S5, a/a \div a, b/a \div a, c/a \div (a \div a), d/a]$
$= (a \div a) + (((a \div a) \div (a \div a)) + (a \div a))$ $[S4, b/a \div a; S8]$
$= (a \div a) + ((a \div a) + (a \div a))$ $[S4, b/a \div a]$
$= (a \div a) + (a \div a) = a \div a$ $[S7; S7]$

S10 $[a]: a \epsilon C .\supset. a \div (a \div (a \div a)) = a \div a$
$[a]: \mathrm{Hp}(1) .\supset.$
$a \div (a \div (a \div a)) = a \div (a \div ((a \div a) + (a \div a)))$ $[1; S7]$
$= (a \div a) + (((a \div a) \div ((a \div a) \div a)) + ((a \div a) \div ((a \div a) \div a)))$
$[S5, b/a \div a, c/a \div a, d/a]$
$= (a \div a) + ((a \div a) + (a \div a)) = (a \div a) + (a \div a) = a \div a$ $[S9; S7; S7]$

S11 $[ab]: a, b \epsilon C .\supset. a = (b \div b) + a$

PR $[ab]: \mathrm{Hp}(1) .\supset.$
$a = (a \div (a \div (a \div a))) + ((a \div (a \div a)) \div (((a \div (a \div a)) \div a)))$
$[1; S2, b/a \div (a \div a)]$
$= (a \div a) + ((a \div (a \div a)) \div ((a \div (a \div a)) \div ((a \div a) + (a \div (a \div a)))))$
$[S10; S2, b/a]$
$= (a \div a) + ((a \div a) + (((a \div a) \div ((a \div a) \div (a \div (a \div a)))) + ((a \div (a \div a))$
$\div ((a \div (a \div a) \div (a \div (a \div a))))))$
$[S6, a/a \div (a \div a), b/(a \div a), a/a \div (a \div a), d/a, e/a \div a]$

$= (a \div a) + ((a \div a) + ((a \div a) + ((a \div (a \div a)) \div ((a \div (a \div a))$
$\div (a \div (a \div a))))))$ [*S3, b/a*]
$= (a \div a) + ((a \div a) + (((a \div (a \div a)) \div (a \div (a \div a))) + ((a \div (a \div a))$
$\div ((a \div (a \div a)) \div (a \div (a \div a))))))$ [*S4, b/a ÷ (a ÷ a)*]
$= (a \div a) + ((a \div a) + (a \div (a \div a)))$ [*S2, a/a ÷ (a ÷ a), b/a ÷ (a÷a)*]
$= (b \div b) + a$ [*S4; S2, b/a*]

S12 $[abc]: a, b, c \epsilon C . \supset . a \div (a \div (b + c)) = (b \div (b \div a)) + (c \div (c \div a))$
[*S1; S11, a/(b ÷ (b ÷ a)) + (c ÷ (c ÷ a)), b/a*]

S13 $[ab]: a, b \epsilon C . \supset . a \div (a \div b) = b \div (b \div a)$

PR $[ab]: \mathrm{Hp}(1) . \supset .$
$a \div (a \div b) = a \div (a \div ((a \div a) + b))$ [1, *S11, a/b, b/a*]
$= ((a \div a) \div ((a \div a) \div a)) + (b \div (b \div a)$ [*S12, b/a ÷ a, c/b*]
$= (a \div a) + (b \div (b \div a)) = b \div (b \div a)$ [*S9; S11, a/b ÷ (b ÷ a), b/a*]

S14 $[ab] \therefore a, b \epsilon C . \supset : [\exists x]: x \epsilon C . (a \div b) + x = a . (a \div b) \div ((a \div b)$
$\div x) = a \div a$

PR $[ab] \therefore \mathrm{Hp}(1) . \supset :$
2. $b \div (b \div a) \epsilon C :$ [1; *R6, a/b, b/a; R6, a/b, b/b ÷ a*]
$[\exists c]: c \epsilon C . (a \div b) + x = a . (a \div b) \div ((a \div b) \div x) = a \div a$
[1; 2; *S2*; *S3, c/a*]

S15 $[abxy]: a, b, x, y \epsilon C . (a \div b) + x = a . (a \div b) \div ((a \div b) \div x)$
$= a \div a . (a \div b) + y = a . (a \div b) \div ((a \div b) \div y) = a \div a . \supset . x = y$

PR $[abxy]: \mathrm{Hp}(5) . \supset .$
6. $x \div (x \div a) = x \div (x \div (((a \div b) + y))$ [1; 4]
$= ((a \div b) \div ((a \div b) \div x)) + (y \div (y \div x))$ [*S12, a/x, b/a ÷ b, c/y*]
$= (a \div a) + (y \div (y \div x))$ [3]
$= ((a \div b) \div ((a \div b) \div y)) + (x \div (x \div y))$ [5; *S13, a/y, b/x*]
$= y \div (y \div ((a \div b) + x)) = y \div (y \div a) .$ [*S12, a/y, b/a ÷ b, c/x*; 2]
7. $x \div (x \div a) = x \div ((x \div ((a \div b) + x))$ [1; 2]
$= ((a \div b) \div ((a \div b) \div x)) + (x \div (x \div x))$ [*S12, a/x, b/a ÷ b, c/x*]
$= (a \div a) + (x \div (x \div x))$ [3]
$= (x \div x) + (x \div (x \div x)) = x .$ [*S4, b/x; S2, a/x, b/x*]
8. $x \div (x \div a) = y \div (y \div a) = y \div (y \div ((a \div b) + y))$ [1; 6; 4]
$= ((a \div b) \div ((a \div b) \div y)) + (y \div (y \div y))$ [*S12, a/y, b/a ÷ b, c/y*]
$= (a \div a) + (y \div (y \div y))$ [5]
$= (y \div y) + (y \div (y \div y)) = y.$ [*S4, b/y; S2, a/y, b/y*]
$x = y$ [7; 8]

Since we have *S14* and *S15*, we can introduce into the system the following definition:

D1 $[abx] \therefore a, b, x \epsilon C . \supset : a \times b = x . \equiv . (a \div b) + x = a . (a \div b)$
$\div ((a \div b) \div x) = a \div a$ [*S14; S15*]

S16 $[ab]: a, b \epsilon C . \supset . a \times b = a \div (a \div b)$

PR $[ab]: \mathrm{Hp}(1) . \supset .$
$a \times b = b \div (b \div a) = a \div (a \div b)$ [1; *D1, x/b ÷ (b ÷ a); S2; S3, c/a; S13*]

S17 $[abc]: a, b, c \epsilon C . \supset a \times (b + c) = (a \times b) + (a \times c)$

PR $[abc]: \mathrm{Hp}(1) . \supset .$

$a \times (b + c) = a \div (a \div (b + c))$ [1; *S16*, $b/b + c$]

$= (b \div (b \div a)) + (c \div (c \div a))$ [*S12*]

$= (a \div (a \div b)) + (a \div (a \div c))$ [*S13*; *S13*, b/c]

$= (a \times b) + (a \times c)$ [*S16*; *S16*, b/c]

S18 $[ab]: a, b \in C .\supset. (a \times b) = (b \times a)$ [*S16*; *S13*; *S16*, a/b, b/a]

S19 $[ab]: a, b \in C .\supset. (a \div b) + (b \times a) = a$ [*S16*, a/b, b/a; *S2*]

S20 $[abc]: a, b, c \in C .\supset. (a \div b) \times (b \times a) = c \div c$

PR $[abc]: \mathrm{Hp}(1) .\supset.$

$(a \div b) \times (b \times a) = (a \div b) + ((a \div b) + (b \div (b + a)))$

[*S16*, $a/a \div b$, $b/b \times a$; *S16*, a/b, b/a]

$= c \div c$ [*S3*]

S21 $[ab]: a, b \in C .\supset. a \times b \in C$ [*R6*; *R6*, $b/a \div b$; *S16*]

S22 $[abc]: a, b, c \in C . a = c .\supset a \times b = c \times b$

PR $[abc]: \mathrm{Hp}(2) .\supset.$

$a \times b = a \div (a \div b) = c \div (c \div b) = c \times b$ [1; *S16*; 2; *S16*, a/c]

S23 $[abc]: a, b, c \in C . b = c .\supset. a \times b = a \times c$

PR $[abc]: \mathrm{Hp}(2) .\supset.$

$a \times b = b \times a = c \times a = a \times c$ [1; *S18*; *S22*, a/b, b/a, *S18*, a/c, b/a]

2.2 The mutual independence of the axioms *S1*, *S2*, and *S3* is established by using the following algebraic tables (matrices):

$\mathfrak{M}13$

$+$	O	α	β	γ
O	O	α	β	γ
α	α	α	β	γ
β	β	β	β	α
γ	γ	γ	γ	γ

$\div$	O	α	β	γ
O	O	O	O	O
α	α	O	α	α
β	β	β	O	β
γ	γ	γ	γ	O

$\mathfrak{M}14$

$+$	α	β
α	α	β
β	β	α

$\div$	α	β
α	α	α
β	α	α

$\mathfrak{M}15$

$+$	α	β
α	α	α
β	β	β

$\div$	α	β
α	α	α
β	β	β

Concerning the matrix $\mathfrak{M}14$ *cf.* $\mathfrak{M}11$ in [2], p. 422. Since:

(a) matrix $\mathfrak{M}13$ verifies *S2* and *S3*, but falsifies *S1* for a/α, b/β, and c/γ: (i) $\alpha \div (\alpha \div (\beta + \gamma)) = \alpha \div (\alpha \div \alpha) = \alpha \div O = \alpha$, and (ii) $(\alpha \div \alpha) + ((\beta \div (\beta \div \alpha)) + (\gamma \div (\gamma \div \alpha))) = O + ((\beta \div \beta) + (\gamma \div \gamma)) = O + (O + O) = O + O = O$,
(b) matrix $\mathfrak{M}14$ verifies *S1* and *S3*, but falsifies *S2* for a/β and b/β: (i) $(\beta \div \beta) + (\beta \div (\beta \div \beta)) = \alpha + (\beta \div \alpha) = \alpha + \alpha = \alpha$, and (ii) $\beta = \beta$,
(c) matrix $\mathfrak{M}15$ verifies *S1* and *S2*, but falsifies *S3* for a/α, b/α, and c/β: (i) $(\alpha \div \alpha) \div ((\alpha \div \alpha) \div (\alpha \div (\alpha \div \alpha))) = \alpha \div (\alpha \div (\alpha \div \alpha)) = \alpha \div (\alpha \div \alpha) = \alpha \div \alpha = \alpha$, and (ii) $\beta \div \beta = \beta$,

we know that the axioms *S1*, *S2*, and *S3* are mutually independent.

2.3 An inspection of the deductions presented in sections **1** and **2.1** shows that

(i) The theses *P1, P2, P3, P4, P5, P6, P7, P8, P9, P10, P11, G1, G2, G3, G4, G22, G24, G30, G31,* and *G32* of $\mathfrak{B}$ correspond synonymously and respectively to the theses *R1, R2, R3, R4, R5, R6, R7, R8, S21, S22, S23, S17, S18, S19, S20, R9, R10, S1, S2,* and *S3* of $\mathfrak{C}$;

and, moreover, that

(ii) Definition *D1* which is introduced into the system $\mathfrak{C}$ corresponds synonymously to the thesis *G35* which is provable in the field of the system $\mathfrak{B}$.

Therefore, it follows from the points (i) and (ii) that the systems $\mathfrak{B}$ and $\mathfrak{C}$ are inferentially equivalent up to isomorphism. Thus, the proof that the system $\mathfrak{C}$ is a correct axiomatization of the mixed associative Newman algebras is complete.

REFERENCES

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